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Asymptotic Solutions of Second Order Equations with Holomorphic Coefficients with Degeneracies and Laplace's Equations on A Manifold with A Cuspidal Singularity

By M. V. Korovina

Abstract- In this paper, we construct the asymptotics for second order linear differential equations with higher-order singularity for the case where the principle symbol has multiple roots. In addition, we solve the problem of constructing asymptotic solutions of Laplace's equation on a manifold with a second order cuspidal singularity.

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Ref

1. Kondratev V.A. Boundary Value Problems for Elliptic Equations in Domains with Conical or Angular Points, Tr. Mosk. Mat. Obs.,1967, vol,16, pp 209-292.

Asymptotic Solutions of Second Order Equations with Holomorphic Coefficients with Degeneracies and Laplace's Equations on a Manifold with a Cuspidal Singularity

M. V. Korovina

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I. Introduction

This paper is devoted to asymptotic expansions for solutions to equations with higher-order degeneracies, namely, to equations of the form

$$H\left(r,-\frac{1}{k}r^{k+1}\frac{d}{dr},x,-i\frac{\partial}{\partial x}\right)u=0, \tag{1}$$

where

$$H\left(r,r^{k+1}\frac{d}{dr},x,-i\frac{\partial}{\partial x}\right)=\sum_{j=0}^{2}\sum_{l=1}^{2}a_{jl}(x,r)\left(-i\frac{\partial}{\partial x}\right)^{j}\left(r^{k+1}\frac{d}{dr}\right)^{l}$$

$k-$integer non-negative number, $a_{jl}(x,r)$ holomorphic coefficients in the neighbourhood of zero in variable r. Here $r\in C$ and x belongs to a compact manifold without edge. Such equations are referred to as equations with cuspidal singularitus of order *k+1, for* $k=0$, such singularitus are said to be conical. The case of conical singularitus was studied dy Kondratev in [1]. Here we consider the case of cuspidal singularitus. Note that any linear differential equations of second order with holomorphic coefficients with singularitus in one of the variables is representable in the form (1). Laplace's equation on a manifold with cuspidal singularity is a typical example of such an equation.

In the first part of the article we construct asymptotic solutions of ordinary differential equations of second order with coefficients $a_i(r), i=0,1,2$ which is holomorphic in some neighborhood of the point $r=0$

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$$a_2(r)\left(\frac{d}{dr}\right)^2 u(r)+a_1(r)\left(\frac{d}{dr}\right)u(r)+a_0(r)u(r)=0. \qquad (2)$$

$a_i(r), i=0,1,2$ are is holomorphic in some neighborhood of the point $r=0$
In paper [5] received asymptotic solutions of equation

$$H\left(r,-r^2\frac{d}{dr}\right)u=0, \qquad (3)$$

where $H(r,p)=\sum_{i=0}^{n}a_i(r)p^i$, with the roots of principle operator symbol $H_0(p)=H(0,p)$ being simple. The asymptotic solutions of the above equation have the form

$$u=\sum_{i=1}^{n}e^{\alpha_i/r}r^{\sigma_i}\sum_{k=0}^{\infty}a_i^k r^k \qquad (4)$$

where $\alpha_i, i=1,\ldots,n$-are the roots of $H(0,p)$ and σ_i and a_i^k–are some complex numbers. However, if the asymptotic expansion has at last two terms corresponding to values α_1 and α_2 with distinct real parts (to be definite, we assume that $\mathrm{Re}\,\alpha_1>\mathrm{Re}\,\alpha_2$), then it becomes quite difficult to interpret the rights hand part of (4). The point is that all terms of the first element corresponding to the value α_1 (the dominant component) have a higher order as $r\to 0$ then any term of the second (the recessive element). If the argument r moves in the complex plane, then the role of the components can be changed. Therefore, to interpret the expansion (4), one should sum the (not necessarily convergent) series (3), the analysis of asymptotic expansions of solutions of equations (1) requires the introduction of regular summation method for divergent series for the construction of uniform asympto tic expansions of solutions with respect to the variable r.

In paper [2] and [5] author examined the conditions of infinite continuable for Laplace-Borel k-transforms of solutions to these equations and proved their continuability along any path on the Riemann surface not passing through a certain discrete set of points depending on the function, the exact definition of resurgent function is given in below.

Based on the concept of resurgent function first introduced by J. Ecalle [3], apparatus for summing expressions of the form (4), based on the Borel-Laplace transformation is called resurgent analysis. The fundamentals of resurgent analysis and of the Borel-Laplace transform are based on can be found in [4].
In articles [7], [8] asymptotic solutions of equations

$$\left(\frac{d}{dx}\right)^2 u(x)+a_1(x)\left(\frac{d}{dx}\right)u(x)+a_0(x)u(x)=0$$

are constructed in the neighborhood of infinity, provided that the coefficients $a_i(x)$ are holomorphic in the neighborhood of infinity. This equation is reduced to the second order equations with cuspidal singularitus in the neighborhood of the point $r=0$, by substituting $x=\frac{1}{r}$, which is a private case of tasks which are considering in that paper, that is equations of any order singularitus.

Ref

2. Korovina M. V., Existence of Resurgent Solutions for Equatons with Higher –Order Degenerations, Differ. Uravn, 2011, vol. 47, no.3, pp. 349-357.

In the second part of the paper we consider partial differential equations with cuspidal singularitus. As an example we construct asymptotic solutions of Laplace's equation on a manifold with a second order caspidal singularity.

The paper continues the research into asymptotic behaviour of solutions to equations with singularit carried out in a series of articles [2], [5], [6] and so on.

In [5], asymptotic expansions for solutions to equations of type (1) are constructed for $k=1$, and, in [6], they are constructed for $k>1$ if the roots of the principle operator symbol are simple. The problem of asymptotic solutions in the case of multiple roots is much more complicated and is still an open problem. This paper proposes a method for obtaining asymptotic solutions of second order equations with higher-order singularity in the case of multiple roots. The method is also applicable to some types of higher-order ordinary or partial differential equations.

II. Basic Definitions

In this section we introduce some notions of resurgent analysis for further use.

Let $S_{R,\varepsilon}$ denote the sector $S_{R,\varepsilon}=\{r \,|\, -\varepsilon<\arg r<\varepsilon, |r|<R\}$. We say that the function f analytic at $S_{R,\varepsilon}$ has at most k-exponential growth if there exist nonnegative constants C and α such that the inequality

$$|f|<Ce^{a\frac{1}{|r|^k}}$$

holds in the sector $S_{R,\varepsilon}$.

Let $E_k(S_{R,\varepsilon})$ denote the space of holomorphic functions of k-exponential growth, and let $E(\tilde{\Omega}_{R,\varepsilon})$ denote the space of holomorphic functions of exponential growth in $\tilde{\Omega}_{R,\varepsilon}$.The domain $\tilde{\Omega}_{R,\varepsilon}$ is shown in Fig1. $E(C)$ will denote the space of entire functions of exponential growth.

The Laplace-Borel k-transform of the function $f(r)\in E_k(S_{R,\varepsilon})$ is given by

$$B_k f=\int_0^{r_0} e^{-p/r^k} f(r)\frac{dr}{r^{k+1}}.$$

We can show that $B_k: E_k(S_{R,\varepsilon})\to \mathrm{E}(\tilde{\Omega}_{R,\varepsilon})/\mathrm{E(C)}$. The inverse -transform is defined by

$$B_k^{-1}\tilde{f}=\frac{k}{2\pi i}\int_{\tilde{\gamma}} e^{p/r^k}\tilde{f}(p)\,dp.$$

where $\tilde{\gamma}$ is shown in Fig. 1.

Ref

5. Korovina M. V. and Shatalov V. E., Differential Equations with Degeneration and Resurgent Analysis, Differ. Uravn, 2010, vol. 46, no 9, p 1259-1277.

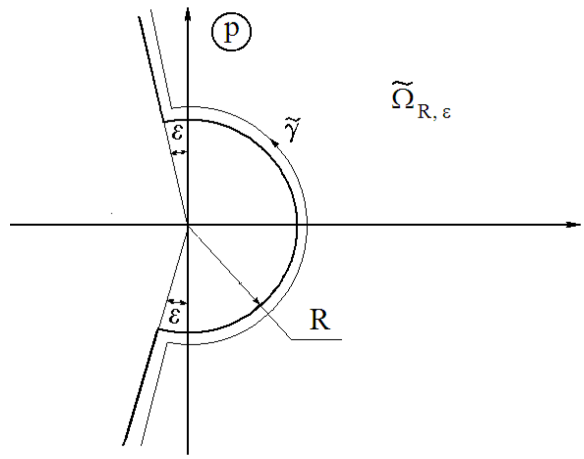

Fig. 1: The Laplace-Borel transform domain of holomorphism and the reverse transform calculation

We can now give the definition of a k-resurgent function.

Definition 1. The function $\tilde{f}$ is called k-endlessly continuable, if for any R there exists a discrete set of points Z_R in C such that the function $\tilde{f}$ can be analytically continued from the initial domain along any path of length < R not passing through Z_R.

Definition 2. The element f of the space $E_k(S_{R,\varepsilon})$ is called a k-resurgent function, if its Borel k-transform $\tilde{f} = B_k f$ is endlessly continuable.

III. Asymptotic Solutions of Ordinary Differential Equations

In this section we consider second-order homogeneous ordinary differential equations with cuspidal singularitus, that is the equations of the form

$$H(r, \frac{1}{n} r^{n+1} \frac{d}{dr}) = 0,$$

where the symbol $H(r,p)$ is second-order polynomial in p with holomorphic coefficients. In [2] it is proved that solutions of these equations are resurgent functions as formulated in Definition 2. Here we assume that the principle symbol $H_0(p) = H(0,p)$ has a multiple root. In the case of simple roots of a principle symbol such equations are discussed in [6]. In other words, we consider the equations of the form

$$\left(\frac{1}{n} r^{n+1} \frac{d}{dr}\right)^2 u + a_1(r)\left(\frac{1}{n} r^{n+1} \frac{d}{dr}\right) u + a_0(r) u + r v(r)\left(\frac{1}{n} r^{n+1} \frac{d}{dr}\right)^2 u = 0, \qquad (5)$$

where $a_i(r)$ are holomorphic coefficients.

$$a_1(r) = br^k + r^{k+1} b_1(r),$$
$$a_0(r) = cr^p + r^{p+1} c_1(r)$$

Here $b_1(r), c_1(r)$ are holomorphic functions, with non-zero c and b.

Thus we rewrite equation (5) as

$$\left(\frac{1}{n} r^{n+1} \frac{d}{dr}\right)^2 u + br^k\left(\frac{1}{n} r^{n+1} \frac{d}{dr}\right) u + cr^p u + r^{k+1} b_1(r)\left(\frac{1}{n} r^{n+1} \frac{d}{dr}\right) u +$$

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6. Korovina M. V. Asymptotic Solutions for Equatons with Higher – Order Degenerations, Doklady Mathematics Vol. 437, No. 3, 2011 p. 302-304.

$$+r^{p+1}c_1(r)u+rv(r)\left(\frac{1}{n}r^{n+1}\frac{d}{dr}\right)^2u=0. \tag{6}$$

Since for any $m:0<m<n+1$ the relation‘

$$\left(r^{n+1}\frac{d}{dr}\right)^2=mr^{n+m}\left(r^{n-m+1}\frac{d}{dr}\right)+r^{2m}\left(r^{n-m+1}\frac{d}{dr}\right)^2,$$

holds, then equation (6) can be rewritten in the form

$$\begin{gathered}r^{2m}\left(\frac{1}{n}r^{n-m+1}\frac{d}{dr}\right)^2u+\frac{m}{n}r^{n+m}\left(\frac{1}{n}r^{n-m+1}\frac{d}{dr}\right)u+br^{k+m}\left(\frac{1}{n}r^{n-m+1}\frac{d}{dr}\right)u+cr^pu+\\+r^{k+1+m}b_1(r)\left(\frac{1}{n}r^{n-m+1}\frac{d}{dr}\right)u+r^{p+1}c_1(r)u+r^{2m+1}v(r)\left(\frac{1}{n}r^{n-m+1}\frac{d}{dr}\right)^2u+\\+\frac{m}{n}r^{m+n+1}v(r)\left(\frac{1}{n}r^{n-m+1}\frac{d}{dr}\right)u=0\end{gathered} \tag{7}$$

The following two cases are considered.

1. $\frac{p}{2}\ge n$ and $k\ge n$.

This is the simplest case. We set $m=n$, then equation (7) has the form

$$\begin{gathered}r^{2n}\left(\frac{1}{n}r\frac{d}{dr}\right)^2u+r^{2n}\left(\frac{1}{n}r\frac{d}{dr}\right)u+br^{k+n}\left(\frac{1}{n}r\frac{d}{dr}\right)u+cr^pu+\\+r^{k+n+1}b_1(r)\left(\frac{1}{n}r\frac{d}{dr}\right)u+r^{p+1}c_1(r)u+r^{2n+1}v(r)\left(\frac{1}{n}r\frac{d}{dr}\right)^2u+\\+r^{2n+1}v(r)\left(\frac{1}{n}r\frac{d}{dr}\right)u=0\end{gathered} \tag{8}$$

Dividing (8) by r^{2n}, we obtain the equation

$$\begin{gathered}\left(\frac{1}{n}r\frac{d}{dr}\right)^2u+\left(\frac{1}{n}r\frac{d}{dr}\right)u+br^{k-n}\left(\frac{1}{n}r\frac{d}{dr}\right)u+cr^{p-2n}u+\\+r^{k-n+1}b_1(r)\left(\frac{1}{n}r\frac{d}{dr}\right)u+r^{p-2n+1}c_1(r)u+rv(r)\left(\frac{1}{n}r\frac{d}{dr}\right)^2u+\\+rv(r)\left(\frac{1}{n}r\frac{d}{dr}\right)u=0\end{gathered}$$

This is an equation with a conic singularity. In other words, it is not the case of a cuspidal singularity. This case is have been studied extensively (e.g. [1]). It is reasonable to search for the solutions of these equations in weighted Sobolev spaces $H^{k,\sigma}(0,\infty)$. For nonhomogeneous equations a solution can be represented in the form of conormal asymptotics. For homogeneous equations, it is identically equal to zero.

Ref

1. Kondratev V.A. Boundary Value Problems for Elliptic Equations in Domains with Conical or Angular Points, Tr. Mosk. Mat. Obs.,1967, vol,16, pp 209-292.

2. Let $\frac{p}{2} < n$ or $k < n$, and if $k < \frac{p}{2}$, then, similarly to the conic case, we set $m = k$ in (7). Divide the equation (7) by r^{2k} and multiplying by $\left(\frac{n}{n-k}\right)^2$ we obtain the equation

$$\left(\frac{1}{n-k} r^{n-k+1} \frac{d}{dr}\right)^2 u + \frac{n}{n-k} r^{n-k} \left(\frac{1}{n-k} r^{n-k+1} \frac{d}{dr}\right) u + b \frac{n}{n-k} \left(\frac{1}{n-k} r^{n-k+1} \frac{d}{dr}\right) u +$$

$$+ c \frac{n^2}{(n-k)^2} r^{p-2k} u + r b_1(r) \frac{n}{n-k} \left(\frac{1}{n-k} r^{n-k+1} \frac{d}{dr}\right) u + r^{p+1-2k} \left(\frac{n}{n-k}\right)^2 c_1(r) u +$$

$$+ r v(r) \left(\frac{1}{n-k} r^{n-k+1} \frac{d}{dr}\right)^2 u + \frac{n}{n-k} r v(r) \left(\frac{1}{n-k} r^{n-k+1} \frac{d}{dr}\right)^2 u = 0$$

The principle symbol in this case is equal to $H_0(p) = p^2 + b \frac{n}{n-k} p$, and has simple roots. In [6], the solution of this equation is shown to belong to the space $E_{n-k}(S_{R,\varepsilon})$. Asymptotic solutions for this equation has the form

$$r^{\sigma_0} v_0(r) \exp\left(\sum_{i=1}^{n-k-1} \frac{\alpha^0{}_{n-k-1}}{r^{n-k-i}}\right) + r^{\sigma_1} v_1(r) \exp\left(-\frac{bn}{(n-k) r^{n-k}} + \sum_{i=1}^{n-k-1} \frac{\alpha^1{}_{k-i}}{r^{n-k-i}}\right).$$

In this case $v_i(r), i = 0,1$ denotes, in general, divergent series

$$v_i(r) = \sum_{j=1}^{\infty} v_j^i r^j, i = 0,1$$

and $\sigma_i(r), i = 0,1$ and $\alpha^j{}_i$ denote corresponding numbers. The method to calculate them is outlined in [6].

Now assume that $\frac{p}{2} < k$ then, we set $m = \frac{p}{2}$, in equation (7) and divide it by r^p, thus obtain the equation

$$\left(\frac{1}{n} r^{n-\frac{p}{2}+1} \frac{d}{dr}\right)^2 u + \frac{p}{2n} r^{n-\frac{p}{2}} \left(\frac{1}{n} r^{n-\frac{p}{2}+1} \frac{d}{dr}\right) u + b r^{k-\frac{p}{2}} \left(\frac{1}{n} r^{n-\frac{p}{2}+1} \frac{d}{dr}\right) u + cu +$$

$$+ r^{k+1-\frac{p}{2}} b_1(r) \left(\frac{1}{n} r^{n-\frac{p}{2}+1} \frac{d}{dr}\right) u + r\, c_1(r) u + r v(r) \left(\frac{1}{n} r^{n-\frac{p}{2}+1} \frac{d}{dr}\right)^2 u + \tag{9}$$

$$+ \frac{p}{2n} r^{n+1-q} v(r) \left(\frac{1}{n-q} r^{n-\frac{p}{2}+1} \frac{d}{dr}\right) u = 0$$

If $p = 2q$ is an even number, then the equation takes the form

$$\left(\frac{1}{n-q} r^{n-q+1} \frac{d}{dr}\right)^2 u + \frac{q}{n-q} r^{n-q} \left(\frac{1}{n-q} r^{n-q+1} \frac{d}{dr}\right) u +$$

$$+ b r^{k-q} \frac{n}{n-q} \left(\frac{1}{n-q} r^{n-q+1} \frac{d}{dr}\right) u + c \left(\frac{n}{n-q}\right)^2 u +$$

Ref

6. Korovina M. V. Asymptotic Solutions for Equations with Higher – Order Degenerations, Doklady Mathematics Vol. 437, No. 3, 2011 p. 302-304.

$$+r^{k+1-q}b_1(r)\frac{n}{n-q}\left(\frac{1}{n-q}r^{n-q+1}\frac{d}{dr}\right)u+r\left(\frac{n}{n-q}\right)^2c_1(r)u+$$

$$+rv(r)\left(\frac{1}{n-q}r^{n-q+1}\frac{d}{dr}\right)^2u+$$

$$+\frac{q}{n-q}r^{n-q+1}v(r)\left(\frac{1}{n-q}r^{n-q+1}\frac{d}{dr}\right)u=0$$

Then the symbol is equal to $H_0(p)=p^2+\left(\frac{n}{n-q}\right)^2c$. Its roots are simple. The solution of this equation belongs to the space $E_{n-q}(S_{R,\varepsilon})$. In this case, as shown in the article [6], the asymptotics has the form

$$r^{\sigma_1}v_1(r)\exp\left(-\frac{i\frac{n}{n-q}\sqrt{c}}{r^{n-q}}+\sum_{i=1}^{n-q-1}\frac{\alpha^1{}_{n-q-i}}{r^{n-q-i}}\right)+r^{\sigma_2}v_0(r)\exp\left(\frac{i\frac{n}{n-q}\sqrt{c}}{r^{n-q}}+\sum_{i=1}^{n-q-1}\frac{\alpha^2{}_{n-q-i}}{r^{n-q-i}}\right).$$

If p is odd, we make the change (of) $x=r^{\frac{1}{2}}$ and by substituting

$$r^{n-\frac{p}{2}+1}\frac{d}{dr}=\frac{1}{2}\left(x^{2n-p+1}\frac{du}{dx}\right)$$

into equation (9) we obtain

$$\left(\frac{1}{2n-p}x^{2n+1-p}\frac{d}{dx}\right)^2u+\frac{p}{2n-p}x^{2n-p}\left(\frac{1}{2n-p}x^{2n+1-p}\frac{d}{dx}\right)u+bx^{2k-p}\frac{2n}{2n-p}\left(\frac{1}{2n-p}x^{2n+1-p}\frac{d}{dx}\right)u+$$

$$+\left(\frac{2n}{2n-p}\right)^2cu+x^{2k+2-p}b_1(x^2)\frac{2n}{2n-p}\left(\frac{1}{2n-p}x^{2n+1-p}\frac{d}{dx}\right)u+$$

$$+x^2\left(\frac{2n}{2n-p}\right)^2c_1(x^2)u+x^2v(x^2)\left(\frac{1}{2n-p}x^{2n+1-p}\frac{d}{dx}\right)^2u+$$

$$+\frac{2p}{2n-p}x^{2n+2-p}v(x^2)\left(\frac{1}{2n-p}x^{2n+1-p}\frac{d}{dx}\right)^2u=0$$

In this case the principle symbol is equal to $H_0(p)=p^2+\left(\frac{2n}{2n-p}\right)^2c$. The asymptotic solution of the last equation has the form

$$\exp\left(-\frac{2\pi i\sqrt{c}}{(2n-p)x^{2n-p}}+\sum_{i=1}^{2n-p-1}\frac{\alpha^1{}_i}{x^{2n-p-i}}\right)x^{\sigma_1}v_0(x)+\exp\left(\frac{2\pi i\sqrt{c}}{(2n-p)x^{2n-p}}+\sum_{i=1}^{2n-p-1}\frac{\alpha^2{}_{ii}}{x^{2n-p-i}}\right)x^{\sigma_2}v_1(x)=$$

$$=\exp\left(-\frac{2\pi i\sqrt{c}}{(2n-p)r^{n-\frac{p}{2}}}+\sum_{i=1}^{2n-p-1}\frac{\alpha^1{}_i}{r^{n-\frac{p}{2}-\frac{i}{2}}}\right)r^{\frac{\sigma_1}{2}}v_0\left(r^{\frac{1}{2}}\right)+\exp\left(\frac{2\pi i\sqrt{c}}{(2n-p)r^{n-\frac{p}{2}}}+\sum_{i=1}^{2n-p-1}\frac{\alpha^2{}_i}{r^{n-\frac{p}{2}-\frac{i}{2}}}\right)r^{\frac{\sigma_2}{2}}v_1\left(r^{\frac{1}{2}}\right)$$

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6. Korovina M. V. Asymptotic Solutions for Equatons with Higher – Order Degenerations, Doklady Mathematics Vol. 437, No. 3, 2011 p. 302-304.

Since

$$v_1\left(r^{\frac{1}{2}}\right)=\sum_{i=1}^{\infty} v_i^1 r^{\frac{i}{2}}=\sum_{i=1}^{\infty} v_{2i}^1\ r^i+\sum_{i=0}^{\infty} v_{2i+1}^1\ r^{i+\frac{1}{2}}=\sum_{i=1}^{\infty} v^1{}_{2i} r^i+r^{\frac{1}{2}}\sum_{i=0}^{\infty} v^1{}_{2i+1} r^i,$$

then the asymptotic solution can be rewritten in the form

$$\exp\left(-\frac{2\pi i\sqrt{c}}{(2n-p)r^{n-\frac{p}{2}}}+\sum_{i=1}^{2n-p-1}\frac{\alpha^1{}_i}{r^{n-\frac{p}{2}-\frac{i}{2}}}\right)r^{\frac{\sigma_1}{2}}\left(\tilde{v}_1^1(r)+r^{\frac{1}{2}}\tilde{v}_1^2(r)\right)+$$

$$+\exp\left(\frac{2\pi i\sqrt{c}}{(2n-p)r^{n-\frac{p}{2}}}+\sum_{i=1}^{2n-p-1}\frac{\alpha^2{}_i}{r^{n-\frac{p}{2}-\frac{i}{2}}}\right)r^{\frac{\sigma_2}{2}}\left(\tilde{v}_2^1(r)+r^{\frac{1}{2}}\tilde{v}_2^2(r)\right).$$

Here $\tilde{v}_j^1=\sum_{i=1}^{\infty} v_{2i}^j r^i, \tilde{v}_j^2=\sum_{i=1}^{\infty} v_{2i+1}^j r^i,\ j=1,2.$ In this case we have obtained a new asymptotic type, namely the asymptotics with nonintegral degrees r in their exponents.

Let us consider the last case. Suppose $k=\frac{p}{2}<n$. We set $m=k=\frac{p}{2}$ and dividing by r^{2m} we obtain and multiplying on $\left(\frac{n}{n-k}\right)^2$ then

$$\left(\frac{1}{n-k}r^{n-k+1}\frac{d}{dr}\right)^2 u+\frac{n}{n-k}r^{n-k}\left(\frac{1}{n-k}r^{n-k+1}\frac{d}{dr}\right)u+b\frac{n}{n-k}\left(\frac{1}{n-k}r^{n-k+1}\frac{d}{dr}\right)u+\left(\frac{n}{n-k}\right)^2 cu+$$

$$+rb_1(r)\frac{n}{n-k}\left(\frac{1}{n-k}r^{n-k+1}\frac{d}{dr}\right)u+r\left(\frac{n}{n-k}\right)^2 c_1(r)u+$$

$$+rv(r)\left(\frac{1}{n-k}r^{n-k+1}\frac{d}{dr}\right)^2 u+\frac{n}{n-k}r^{n-k+1}v(r)\left(\frac{1}{n-k}r^{n-k+1}\frac{d}{dr}\right)u=0$$

The principle symbol in this case has the form $H_0(p)=p^2+b\frac{n}{n-k}p+c\left(\frac{n}{n-k}\right)^2$. If $b\neq 2\sqrt{c}$, then the polynomial $p^2+b\frac{n}{n-k}p+c\left(\frac{n}{n-k}\right)^2$ has two roots c_1, c_2 and the asymptotics has the form

$$r^{\sigma_0}v_0(r)\exp\left(\frac{c_1}{r^{n-k}}+\sum_{i=1}^{n-k-1}\frac{\alpha^0{}_i}{r^{n-k-i}}\right)+r^{\sigma_1}v_1(r)\exp\left(\frac{c_2}{r^{n-k}}+\sum_{i=1}^{n-k-1}\frac{\alpha^1{}_i}{r^{n-k-i}}\right)$$

If $b=2\sqrt{c}$, namely $p^2+b\frac{n}{n-k}p+c\left(\frac{n}{n-k}\right)^2$, then by replacing $\tilde{u}=e^{\frac{n}{n-k}\frac{\sqrt{c}}{r}}u$ we obtain the equation with a multiple root at zero. The asymptotic solution of this equation is constructed in the same way as is shown above and depends on the degrees of degeneracy of the functions $b_1(r), c_1(r)$.

Thus we have cuspidal degeneracies in the case where $\frac{p}{2}<n$ or $k<n$. The obtained results are written in the table

Notes

Notes

Theorem 1: Let $\frac{p}{2} < n$ or $k < n$ then

Conditions	Space	Asymptotics
$k < \frac{p}{2}$	$u \in E_{n-k}(S_{R,\varepsilon})$	$r^{\sigma_0} v_0(r) \exp\left(\sum_{i=1}^{n-k-1} \frac{\alpha^0{}_{n-k-1}}{r^{n-k-i}}\right) +$ $+ r^{\sigma_1} v_1(r) \exp\left(-\frac{bn}{r^{n-k}(n-k)} + \sum_{i=1}^{n-k-1} \frac{\alpha^1{}_{k-i}}{r^{n-k-i}}\right)$
$k > \frac{p}{2}$, $p = 2q, q \in N$	$u \in E_{n-q}(S_{R,\varepsilon})$	$r^{\sigma_1} v_0(r) \exp\left(-\frac{i\frac{n}{n-q}\sqrt{c}}{r^{n-q}} + \sum_{i=1}^{n-q-1} \frac{\alpha^1{}_{n-q-i}}{r^{n-q-i}}\right) +$ $+ r^{\sigma_2} v_1(r) \exp\left(\frac{i\frac{n}{n-q}\sqrt{c}}{r^{n-q}} + \sum_{i=1}^{n-q-1} \frac{\alpha^2{}_{n-q-i}}{r^{n-q-i}}\right)$
$k > \frac{p}{2}$, p is odd	$u \in E_{n-\frac{p}{2}-1}(S_{R,\varepsilon})$	$\exp\left(-\frac{2\pi i\sqrt{c}}{(2n-p)r^{n-\frac{p}{2}}} + \sum_{i=1}^{2n-p-1} \frac{\alpha^1{}_i}{r^{n-\frac{p}{2}-\frac{i}{2}}}\right) r^{\frac{\sigma_1}{2}} \left(\tilde{v}_1^1(r) + r^{\frac{1}{2}} \tilde{v}_1^2\right.$ $+ \exp\left(\frac{2\pi i\sqrt{c}}{(2n-p)r^{n-\frac{p}{2}}} + \sum_{i=1}^{2n-p-1} \frac{\alpha^2{}_i}{r^{n-\frac{p}{2}-\frac{i}{2}}}\right) r^{\frac{\sigma_2}{2}} \left(\tilde{v}_2^1(r) + r^{\frac{1}{2}} \tilde{v}_2^2\right.$
$k = \frac{p}{2}$ and $b \neq 2\sqrt{c}$	$u \in E_{n-k-1}(S_{R,\varepsilon})$	$r^{\sigma_0} v_0(r) \exp\left(\frac{c_1}{r^{n-k}} + \sum_{i=}^{n-k-1} \frac{\alpha^0{}_{n-k-1}}{r^{n-k-i}}\right) +$ $+ r^{\sigma_1} v_1(r) \exp\left(\frac{c_2}{r^{n-k}} + \sum_{i=2}^{n-k-1} \frac{\alpha^1{}_{k-i}}{r^{n-k-i}}\right)$

Now we proceed to examine the higher-order equations

$$\left(\frac{1}{n}r^{n+1}\frac{d}{dr}\right)^{k}u+b_{k-1}(r)\left(\frac{1}{n}r^{n+1}\frac{d}{dr}\right)^{k-1}u+...+b_{1}(r)\left(\frac{1}{n}r^{n+1}\frac{d}{dr}\right)u+b_{0}(r)u=0$$

where $b_i(r)=\sum_{j=m_i}^{\infty}c_i^j r^j, i=0,...,k-1$ are entire functions. We assume that $c_i^{m_i}\neq 0$. The number m_i will be called a degree of degeneracy of the coefficient $b_i(r)$. Suppose that zero is the root of this equation principle symbol and the degree of degeneracy of the coefficients $b_1(r)$ or $b_0(r)$ is no more than the degree of degeneracy for any of the coefficients $b_i(r),\ i=2,...,k-1$. The above method is also applicable to this case. Specifically, the equation is to be divided by r^p, where $p=\min(m_0,m_1)$, then we have the equation with a symbol of the form $H\left(r, r^{n-\frac{p}{k}}\frac{d}{dr}\right)$, which is solved similarly to the previous one.

IV. Partial Differential Equations

We consider a second-order partial differential equation with holomorphic coefficients. It can be represented in the form

$$H\left(r,-\frac{1}{k}r^{k+1}\frac{d}{dr},x,-i\frac{\partial}{\partial x}\right)u=0,$$

where x varies on some compact manifold without boundary. These equations can be interpreted as equations with respect to the functions with values in Banach spaces, namely

$$\hat{H}\left(r,-\frac{1}{k}r^{k+1}\frac{d}{dr}\right)u=0, \tag{10}$$

where $\hat{H}:E_k\left(S_{R,\varepsilon},B_1\right)\to E_k\left(S_{R,\varepsilon},B_2\right)$. Here $B_i,\ i=1{,}2$ denote some Banach spaces (e.g. the space $H^s(\Omega)$). The degree k of the function u grows exponentially for $r\to 0$, for fixed r and p

$$\hat{H}(r,p):B_1\to B_2$$

is a bounded operator acting in Banach spaces which is polynomial dependent on p and holomorphic in the neighbourhood of zero.

In what follows, we will assume that the operator family $\hat{H}_0(p)=\hat{H}(0,p)$ is a *Fredholm* family. This implies that the operator $\hat{H}_0(p)$ is a Fredholm operator for each fixed p, and there exists $p_0\in C$ such that the operator $\hat{H}_0(p_0)$ is invertible.

In addition to the requirement of the Fredholm property for the family $\hat{H}_0(p)$ we assume that there exists a cone in C containing the imaginary axis and does not contain the points of the spectrum of $\hat{H}_0(p)$ for a sufficiently large $|p|$. The spectrum of a Fredholm family is a set of points $p\in C$ such that $\hat{H}_0(p)$ is irreversible.

Suppose that the point $p_1\in spec\hat{H}_0(p)$ *is such that the following conditions hold :*

Notes

1. The operator-valued function $\hat{H}^{-1}{}_0(p)$ has the second-order pole at the point p_1.
2. The dimensionality of the $\ker \hat{H}_0(p_1)$ is equal to one.

In this case theorem 1 is also true. The proof is analogous to that of Theorem 9 in [5] In this case the coefficients of the series $\sum_{j=1}^{\infty} v_j^i r^j$ are the elements of the space B_1.

V. Laplace's Equation on a Manifold with a Caspidal Singularity

We consider the Laplace equation $\Delta u = 0$ on the 2D-Riemannian manifold with a second order caspidal singularity. The Riemannian metric induced from R^3 with the help of embedding which is defined as the surface by the rotational of the parabolic branch $y = r^2$ around axis $0r$. in R^3. We choose local coordinates r, φ on the manifold in the neighbourhood of zero. The metrics on this manifold is given by

$$ds^2 = dr^2 + 4r^2dr^2 + r^4 d\varphi^2 = (1+4r^2)dr^2 + r^4 d\varphi^2,$$

hence

$$gradu = (A_1, A_2) = \left(\frac{1}{\sqrt{4r^2+1}} \frac{\partial u}{\partial r}, \frac{1}{r^2} \frac{\partial u}{\partial \varphi} \right).$$

and

$$divu = \frac{1}{h_1 h_2}\left(\frac{\partial}{\partial r}(h_2 A_1) + \frac{\partial}{\partial \varphi}(h_1 A_2) \right) = \frac{1}{r^2\sqrt{4r^2+1}} \frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r^2}\frac{\partial}{\partial \varphi} A_2.$$

Finally, for the Laplace operator we obtain

$$\Delta u = divgradu = \frac{1}{r^2\sqrt{4r^2+1}} \frac{\partial}{\partial r}\left(r^2 \frac{1}{\sqrt{4r^2+1}} \frac{\partial u}{\partial r} \right) + \frac{1}{r^4}\left(\frac{\partial}{\partial \varphi} \right)^2 u.$$

Thus Laplace's equation on a manifold has the form

$$\hat{H}u = \frac{1}{4r^2+1}\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{2r^2+1}{(4r^2+1)^2}\frac{\partial u}{\partial r} + \frac{1}{r^4}\frac{\partial^2 u}{\partial \varphi^2} = 0 \quad . \tag{11}$$

In other words

$$\hat{H} : E_k(S_{R,\varepsilon}, H^s(S_1)) \to E_k(S_{R,\varepsilon}, H^{s-2}(S_1)),$$

We rewrite equation (11) in the form

$$\left(r^2 \frac{d}{dr} \right)^2 u + \frac{\partial^2 u}{\partial \varphi^2} = -4r^2 \frac{\partial^2 u}{\partial \varphi^2} + \frac{4r^3}{(4r^2+1)}\left(r^2 \frac{\partial u}{\partial r} \right) \tag{12}$$

Obviously this case does not satisfy condition 2 who formulated in part 3, (see of the paper) Therefore, the results formulated there is not applicable. We perform the Laplace-Borel transform

Ref

5. Korovina M. V. and Shatalov V. E., Differential Equations with Degeneration and Resurgent Analysis, Differ. Uravn, 2010, vol. 46, no 9, p 1259-1277.

$$p^2\tilde{u}+\left(\frac{\partial}{\partial\varphi}\right)^2\tilde{u}=-Br^3\,a_1(r)\left(r^2\frac{\partial}{\partial r}\right)\tilde{u}-4\frac{\partial^2}{\partial\varphi^2}Br^2\tilde{u}+\tilde{f}(p,\varphi), \tag{13}$$

where $a_1(r)=\dfrac{r}{(4r^2+1)}$, and $\tilde{f}(p,\varphi)$ denotes a holomorphic function.

The principle symbol of this operator is equal $H(o,p)=p^2+\left(\frac{\partial}{\partial\varphi}\right)^2$. Therefore the corresponding operator family $\hat{H}_0(p)$ is a Fredholm one. The existence of resurgent solution of this equation follows from the results of [2], [3]. We denote $\hat{r}^n=Br^nB^{-1}$, as was done in [5]. If $a(r)$ is holomorphic in the neighbourhood of $r=0$, vanishing at the origin, and

$$a(r)=\sum_{k=1}^{\infty}a_k r^k \tag{14}$$

is its Taylor expansion, then the function

$$\tilde{a}(p)=\sum_{k=1}^{\infty}(-1)^k a_k\frac{p^{k-1}}{(k-1)!} \tag{15}$$

is called a *formal Laplace-Borel transform* of (14), then as shown in the article [5] the equation

$$B\circ a(r)\circ B^{-1}\tilde{f}=\int_{p_0}^{p}\tilde{a}(p-p')\tilde{f}(p')dp' \tag{16}$$

Equality (16) implies that the right-hand part of equation (13) can be transformed

$$-Br^2B^{-1}Bra_1(r)\left(r^2\frac{\partial}{\partial r}\right)\tilde{u}-4\frac{\partial^2}{\partial\varphi^2}Br^2\tilde{u}+\tilde{f}(p,\varphi)=$$

$$=-\hat{r}^2\int_{p_0}^{p}\tilde{a}(p-p')p'\tilde{u}(p',\varphi)dp'-4\frac{\partial^2}{\partial\varphi^2}\hat{r}^2\tilde{u}(p',\varphi)+\tilde{f}(p,\varphi)$$

Equation (13) is ultimately transformed into following

$$p^2\tilde{u}+\left(\frac{\partial}{\partial\varphi}\right)^2\tilde{u}=-\hat{r}^2\int_{p_0}^{p}\tilde{a}(p-p')p'\tilde{u}(p',\varphi)dp'-4\hat{r}^2\frac{\partial^2}{\partial\varphi^2}\tilde{u}(p,\varphi)+\tilde{f}(p,\varphi) \tag{17}$$

Decompose functions $\tilde{u}$ and $\tilde{f}$ into a series on eigen function of operator $\dfrac{\partial^2}{\partial\varphi^2}$

$$\tilde{u}(p,\varphi)=\sum_{k=-\infty}^{\infty}A_k(p)e^{ik\varphi}$$

$$\tilde{f}(p,\varphi)=\sum_{k=1}^{\infty}a_k(p)e^{ik\varphi} \tag{18}$$

Substituting (18) into equation, (17) we obtain

Ref

3. J. Ecalle. Cinq applications des fonctions resurgentes, 1984. Preprint 84T 62, Orsay.

$$p^2\sum_{k=-\infty}^{\infty}A_k(p)e^{ik\varphi}-\sum_{k=1}^{\infty}k^2A_k(p)e^{ik\varphi}=$$

$$=-\hat r^3\int_{p_0}^{p}\tilde a(p-p')\sum_{k=-\infty}^{\infty}A_k(p')e^{ik\varphi}dp'+4\hat r^2\sum_{k=-\infty}^{\infty}k^2A_k(p)e^{ik\varphi}+\sum_{k=1}^{\infty}a_k(p)e^{ik\varphi}$$

This equality is equivalent to the system of equations

$$\left(p^2-k^2\right)A_k(p)=-\hat r^2\int_{p_0}^{p}\tilde a(p-p')p'A_k(p')dp'+4\hat r^2k^2A_k(p)+a_k(p)\ ,$$

Here k is an integer. The cases $k=0$ and $k\neq 0$ should be considered separately. In the first case the singular point of the function $A_0(p)$ is $p=0$, being a pole of second order. We solve this equation by the method of successive approximations.

$$A_0(p)=-\frac{2}{p^2}\hat r^2\int_{p_0}^{p}\tilde a(p-p')p'\frac{A_0(p')}{2}dp'+\frac{1}{p^2}a_0(p) \tag{19}$$

Substitute $A_0(p)=\frac{1}{p^2}a_0(p)$ into the integral in right-hand part side of (19). Applying successive approximation method. Obtain the equality

$$\frac{1}{p^2}\hat r^2\int_{p_0}^{p}\tilde a(p-p')\frac{1}{p'}a_0(p')dp'=\frac{1}{p^2}\hat r^2\int_{p_0}^{p}\frac{G(p,p')}{p'}dp'$$

where $G(p,p')=\tilde a(p-p')a_0(p')$. We represent this function as the series $G(p,p')=\sum_{i=0}^{\infty}l_i(p)p'^i$, then similarly to [5] we obtain

$$\frac{1}{p^2}\hat r^2\int_{p_0}^{p}\frac{\sum_{i=0}^{\infty}l_i(p)p'^i}{p'}dp'=\frac{1}{p^2}\int_0^{p_2}\int_{p_0}^{p_2}\left(b_1(p_1)\ln p_1+n(p_1)\right)dp_1dp_2=$$

$$=C_{-1}\frac{1}{p}+C_0\ln p+\sum_{i=1}^{\infty}C_ip^i\ln p+g(p)$$

where $g(p)$ denotes the function holomorphic in the zero neighbourhood, and C_i denotes corresponding constants. The proof of convergence of the successive approximation method is the same as analogous to the proof in [6]. Thus we obtain that the function $A_0(p)$ with an accuracy of a holomorphic summand in the neighbourhood of the point $p=0$ has the form

$$A_0(p)=\frac{M_{-2}}{p^2}+\frac{M_{-1}^0}{p}+\sum_{i=0}^{\infty}M_i^0p^i\ln p$$

Here by M_i^0 we denote corresponding constants.

Now we consider the second case $k\neq 0$. In this case the points $p=k$ and $p=-k$ are first-order poles of the function $A_k(p)$. Here we have the equation

Ref

5. Korovina M. V. and Shatalov V. E., Differential Equations with Degeneration and Resurgent Analysis, Differ. Uravn, 2010, vol. 46, no 9, p 1259-1277.

$$A_k(p) = -\frac{1}{p^2-k^2}\hat{r}^2\int_{p_0}^{p}\tilde{a}(p-p')p'A_k(p')dp' + \frac{1}{p^2-k^2}4\hat{r}^2k^2A_k(p) + \frac{a_k(p)}{p^2-k^2}$$

Equations of this form have been studied in [5]. The asymptotics of the function $A_k(p)$ in the neighbourhood of the singular point $p=k$, with an accuracy of a holomorphic function, will take the form

$$A_k(p) = \frac{M_{-1}^k}{p-k} + \sum_{j=0}^{\infty} M_j^k (p-k)^j \ln(p-k)$$

Here by M_i^k we denote corresponding constants. Thus we have demonstrated that, with an accuracy of a holomorphic function, the solution of equation (17) has the form

$$\tilde{u}(p,\varphi) = \sum_{k=-\infty}^{\infty} A_k(p)e^{ik\varphi} = \frac{M_{-2}}{p^2} + \sum_{k=-\infty}^{\infty}\left(\frac{M_{-1}^k}{p-k} + \sum_{j=0} M_i^k (p-k)^i \ln(p-k)\right)e^{ik\varphi}$$

Hence it follows that singular points of the function $\tilde{u}(p,\varphi)$ lie in any half-plane $\operatorname{Re} p > A > 0$. In order to construct the solution of problem (17) we represent the function $\tilde{u}(p,\varphi)$ as a sum of two functions $\tilde{u}_-(p,\varphi)$ and $\tilde{u}_+(p,\varphi)$ such that the singularities of the first one lie to the left of the line $\operatorname{Re} p > A$, and the singularities of the second one lie to the right of this line $\operatorname{Re} p > A$, where $\operatorname{Re} A \notin Z$. Then the solution can be constructed as the inverse transformation of the function $\tilde{u}_-(p,\varphi)$ (see [5]). To put it differently the equality

$$u(r,\varphi) = B^{-1}\tilde{u}_-(p,\varphi) = B^{-1}\left(\frac{M_{-2}}{p^2} + \sum_{k=-\infty}^{N}\left(\frac{M_{-1}^k}{p-k} + \sum_{j=0}^{\infty} M_j^k (p-k)^j \ln(p-k)\right)e^{i\varphi k}\right)$$

holds with an accuracy of an entire function.

Hence it follows that the asymptotic solution of Laplace's equation (13) has the form

$$u(r,\varphi) = \frac{C_{-1}}{r} + \sum_{k=-\infty}^{N} e^{\frac{k}{r}} e^{i\varphi k} \sum_{i=0}^{\infty} M_i^k r^i$$

We should note that the solution depends on the representation $\tilde{u}(p,\varphi) = \tilde{u}_-(p,\varphi) + \tilde{u}_+(p,\varphi)$. One solution differs from the other solutions by an operator kernel. Thus we have proved

Assertion. Any solution of equation of exponential growth (12) can be represented as

$$u(r,\varphi) = \frac{C_{-1}}{r} + \sum_j u_j(r,\varphi) + O\left(e^{-\frac{A}{r}}\right),$$

where A is an arbitrary positive number, and the sum contains a finite number of summands, each of which corresponding to a point $p=k$ located in the half-plane $\operatorname{Re} p > -A$, and $u_j(r)$ have asymptotic expansions

Ref

5. Korovina M. V. and Shatalov V. E., Differential Equations with Degeneration and Resurgent Analysis, Differ. Uravn, 2010, vol. 46, no 9, p 1259-1277.

$$u_j(r,\varphi) = e^{\frac{j}{r}} e^{i\varphi j} \sum_{l=0}^{\infty} M_l^k r^l,$$

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Notes

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