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# Number of common sites visited by $N$ random walkers

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We compute analytically the mean number of *common* sites, $W_N(t)$, visited by $N$ independent random walkers each of length $t$ and all starting at the origin at $t = 0$ in $d$ dimensions. We show that in the $(N - d)$ plane, there are three distinct regimes for the asymptotic large-$t$ growth of $W_N(t)$. These three regimes are separated by two critical lines $d = 2$ and $d = d_c(N) = 2N/(N - 1)$ in the $(N\text{-}d)$ plane. For $d < 2$, $W_N(t) \sim t^{d/2}$ for large $t$ (the $N$ dependence is only in the prefactor). For $2 < d < d_c(N)$, $W_N(t) \sim t^{\nu}$ where the exponent $\nu = N - d(N - 1)/2$ varies with $N$ and $d$. For $d > d_c(N)$, $W_N(t) \to$ const as $t \to \infty$. Exactly at the critical dimensions there are logarithmic corrections: for $d = 2$, we get $W_N(t) \sim t/[\ln t]^N$, while for $d = d_c(N)$, $W_N(t) \sim \ln t$ for large $t$. Our analytical predictions are verified in numerical simulations.

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Computing the average number of distinct sites visited by a *single* $t$-step random walker on a $d$-dimensional lattice, denoted by $S_1(t)$, is by now a classic problem with a variety of applications ranging from the annealing of defects in crystals to the size of the territory covered by a diffusing animal during the foraging period. First posed and studied by Dvoretzky and Erdös in 1951 [1], this problem was solved exactly in a number of papers in the 1960s [2,3]. It is well established (see [4] for a review) that asymptotically for large $t$, $S_1(t) \sim t^{d/2}$ for $d < 2$, $\sim t/\ln(t)$ for $d = 2$, and $\sim t$ for $d > 2$. These results have been widely used in a number of applications in physics [5–7], chemistry [8], metallurgy [9–11], and ecology [12,13]. In 1992, Larralde and co-workers generalized this problem to the case of $N$ independent random walkers (each of $t$ steps) all starting at the origin of a $d$-dimensional lattice [14]. They computed analytically $S_N(t)$, the mean number of sites visited by *at least* one of the $N$ walkers in $d$ dimensions, and found two interesting time scales associated with the growth of $S_N(t)$. In the ecological context, $S_N(t)$ represents the mean size of the territory covered by an animal population of size $N$. The original results of Larralde *et al.* have subsequently been corrected [15], used, and generalized in a number of other applications [16–26].

In this paper, we study a complementary question: what is the average number of *common* sites, $W_N(t)$, visited by $N$ independent walkers, each of them consisting of $t$ steps and starting at the origin of a $d$-dimensional lattice? A typical realization in $d = 2$ for $N = 3$ walkers is shown in Fig. 1. Our exact results demonstrate that $W_N(t)$ exhibits a rather rich asymptotic behavior for large $t$. In the $(N\text{-}d)$ plane ($N$ being the number of walkers, or the population size in an ecological context, and $d$ the space dimension) we find an interesting phase diagram where two critical lines $d = 2$ and $d_c(N) = 2N/(N - 1)$ separate three phases with different asymptotic growth of $W_N(t)$ (see Fig. 2). For large $t$, we show that

$$\begin{aligned} W_N(t) &\sim t^{d/2} \quad \text{for} \quad d < 2, \\ &\sim t^{\nu} \quad \text{for} \quad 2 < d < d_c(N) = \frac{2N}{N-1}, \\ &\sim \text{const} \quad \text{for} \quad d > d_c(N), \end{aligned} \tag{1}$$

where the exponent $\nu = N - d(N - 1)/2$ varies with $N$ and $d$. Exactly at the two critical dimensions, there are logarithmic corrections. In particular, for large $t$, $W_N(t) \sim t/[\ln t]^N$ in $d = 2$, and $W_N(t) \sim \ln t$ in $d = d_c(N)$ (with $N > 1$). The existence of the intermediate phase $2 < d < d_c(N) = 2N/(N - 1)$, with a growth exponent $\nu$ varying with $N$ and $d$, is perhaps the most striking of our results. For instance, for $N = 2$ we have $d_c(2) = 4$ and so in $2 < d = 3 < 4$ our result predicts $\nu = 1/2$, i.e., $W_2(t) \sim t^{1/2}$, a prediction that is verified in our numerical simulations.

The statistics of the number of most popular sites, i.e., the sites visited by all the walkers, arises quite naturally in a number of contexts such as sociology, ecology, artificial networks (e.g., the internet, transport, and engineering networks), and polymer networks just to name a few. For example, in a multiple-user network such as the internet, the most popular "hub" sites visited by all the users are known to play a very important role in the dissemination of information [27]. The knowledge of how many of them are there is fundamental for many applications. In the tourism industry, it is important to know the number of the most popular sites in a given area or city that are visited by all the tourists. Motivated by this general question, in this paper we study the statistics of the number of most popular sites in perhaps the simplest model, namely, for $N$ independent random walkers in the $d$-dimensional space, and show that even in this simple model, the asymptotic temporal growth of the mean number of common sites frequented by all $N$ walkers exhibits surprisingly rich behavior. We show that our results also have close connections to the probability of nonintersection of random walks studied in the mathematics literature [28,29]. Given the abundance of random walks used as fundamental models to study numerous natural and artificial systems, and the richness of our exact results, we believe that they will be useful in more specific applications in the future.

We consider $N$ independent $t$-step walkers on a $d$-dimensional lattice, each starting at the origin. To compute the number of common sites visited by all the $N$ walkers, it is first useful to introduce a binary random variable $\sigma_{k,N}(\vec{x},t)$ associated with each site $\vec{x}$ such that $\sigma_{k,N}(\vec{x},t) = 1$ if the site

 

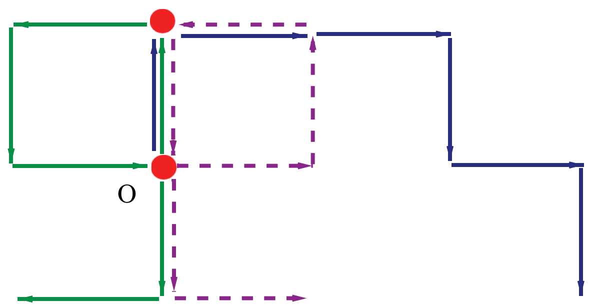


FIG. 1. (Color online) A realization of three random walks, each of six steps [denoted respectively by solid (blue), dashed (purple), and dash-dotted (green) lines] on a square lattice, all starting at the origin $O$. There are two sites [each marked by a filled (red) circle] that are visited by all three walkers.

$\vec{x}$ is visited by exactly $k$ of the $N$ walkers and $\sigma_{k,N}(\vec{x},t) = 0$ otherwise. Then the sum $V_{k,N}(t) = \sum_{\vec{x}} \sigma_{k,N}(\vec{x},t)$ represents the number of sites visited by exactly $k$ of the $N$ walkers, each of $t$ steps, in a particular realization of the walks. Clearly, $V_{k,N}(t)$ is a random variable that fluctuates from one sample to another. Taking the average gives the mean number of sites visited by exactly $k$ walkers, $\langle V_{k,N}(t)\rangle = \sum_{\vec{x}} P_{k,N}(\vec{x},t)$, where $P_{k,N}(\vec{x},t) = \langle \sigma_{k,N}(\vec{x},t)\rangle$ is the probability that the site $\vec{x}$ is visited by exactly $k$ of the $N$ walkers. Since the walkers are independent, one can write

$$P_{k,N}(\vec{x},t) = \binom{N}{k} [p(\vec{x},t)]^k [1 - p(\vec{x},t)]^{N-k}, \qquad (2)$$

where $p(\vec{x},t)$ is the probability that the site $\vec{x}$ is visited by a *single* $t$-step walker starting at the origin. Thus,

$$\langle V_{k,N}(t)\rangle = \binom{N}{k} \sum_{\vec{x}} [p(\vec{x},t)]^k [1 - p(\vec{x},t)]^{N-k}. \qquad (3)$$

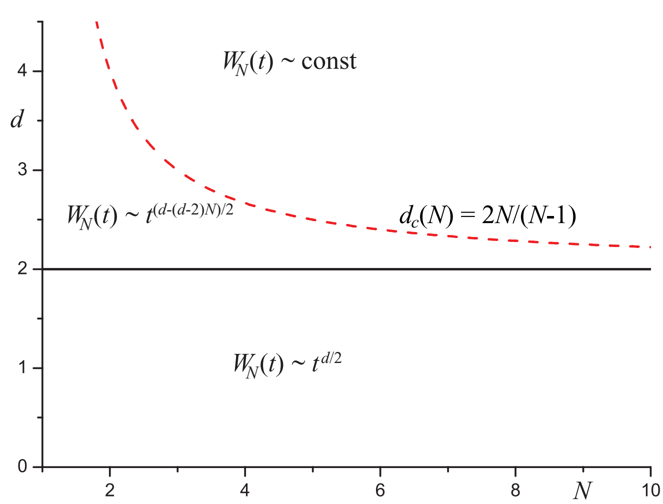


FIG. 2. (Color online) In the ($N$-$d$) plane, there are two critical lines $d = 2$ (lower horizontal line) and $d_c(N) = 2N/(N-1)$ [upper dashed (red) curve]. The mean numbers of common sites $W_N(t)$ visited by $N$ walkers, each of $t$ steps and all starting at the origin at $t = 0$, have different asymptotic behaviors for large $t$ in the three regimes $d < 2$, $2 < d < d_c(N)$, and $d > d_c(N)$.

Finally, the mean number of *common* sites visited by all the $N$ walkers is simply

$$W_N(t) = \langle V_{N,N}(t)\rangle = \sum_{\vec{x}} [p(\vec{x},t)]^N. \qquad (4)$$

Hence, once the basic quantity $p(\vec{x},t)$ for a single walker is known, we can determine $\langle V_{k,N}(t)\rangle$ and, in particular, $W_N(t)$ just by summing over all sites as in Eq. (4). Note that, by definition, $p(0,t) = 1$ for all $t \geqslant 0$ since the walker starts at the origin.

The probability $p(\vec{x},t)$ can be fully determined for a lattice walker with discrete time steps using the standard generating function technique [3]. However, since we are interested here mainly in the asymptotic large-$t$ regime, it is much easier to work directly in the continuum limit where we treat both space $\vec{x}$ and time $t$ as continuous variables. Consider then a single Brownian motion of length $t$ and diffusion constant $D$ in $d$ dimensions starting at the origin. We are interested in $p(\vec{x},t)$, the probability that the site $\vec{x}$ is visited (at least once) by the walker up to time $t$. Let $\tau$ denote the last time before $t$ that the site $\vec{x}$ was visited by the walker. Then, clearly,

$$p(\vec{x},t) = \int_0^t G(\vec{x},\tau) q(t-\tau) d\tau, \qquad (5)$$

where $G(\vec{x},\tau) = e^{-x^2/4Dt}/(4\pi D t)^{d/2}$ (where $x = |\vec{x}|$) is the standard Green's function denoting the probability that the particle is at $\vec{x}$ at time $\tau$ and $q(\tau)$ denotes the persistence, i.e., the probability that, starting at $\vec{x}$, the walker does not return to its starting point up to time $\tau$. Note that $q(\tau)$ does not depend on the starting point $\vec{x}$ and is the same as the probability of no return to the origin up to time $\tau$. Indeed, $q(\tau) = \int_\tau^\infty f(\tau')d\tau'$ where $f(\tau) = -dq/d\tau$ is the standard first-passage probability to the origin [30].

The no-return probability $q(\tau)$ for a Brownian walker has been studied extensively, and it is well known that for large $\tau$, $q(\tau) \sim \tau^{d/2-1}$ for $d < 2$, and $q(\tau) \sim 1/\ln \tau$ for $d = 2$, while it approaches a constant for $d > 2$ since the walker can escape to infinity with a finite probability for $d > 2$ [30]. One can show that to analyze the large-$t$ behavior of $p(\vec{x},t)$ in Eq. (5) in the scaling regime where $x \to \infty$, $t \to \infty$ but keeping $x/\sqrt{t}$ fixed, it suffices to substitute only the asymptotic behavior of $q(\tau)$ in Eq. (5). This gives, for large $t$,

$$p(\vec{x},t) \sim \int_0^t G(\vec{x},\tau)(t-\tau)^{d/2-1} d\tau \quad \text{for } d < 2, \qquad (6)$$

$$p(\vec{x},t) \sim \int_0^t G(\vec{x},\tau) d\tau \quad \text{for } d > 2, \qquad (7)$$

where we have dropped unimportant constants for convenience. For $d = 2$, $p(\vec{x},t) \sim \int_0^t G(\vec{x},\tau)d\tau/\ln(t-\tau)$. Substituting the exact Green's function $G(\vec{x},\tau) = e^{-x^2/4D\tau}/(4\pi D\tau)^{d/2}$ one finds that $p(\vec{x},t)$ has the following asymptotic scaling behavior:

$$p(\vec{x},t) \approx f_<\left(\frac{x}{\sqrt{4Dt}}\right) \quad \text{for } d < 2, \qquad (8)$$

$$p(\vec{x},t) \approx t^{1-d/2} f_>\left(\frac{x}{\sqrt{4Dt}}\right) \quad \text{for } d > 2, \qquad (9)$$

where the scaling functions for $d < 2$ and $d > 2$ can be expressed explicitly as

$$f_<(z) = \int_0^1 e^{-z^2/u} u^{-d/2} (1-u)^{d/2-1} du, \tag{10}$$

$$f_>(z) = \int_0^1 e^{-z^2/u} u^{-d/2} du. \tag{11}$$

Exactly at $d = 2$, one gets $p(\vec{x},t) \approx (1/\ln t) f_2(x/\sqrt{4Dt})$ where $f_2(z) = \int_0^1 du\, e^{-z^2/u}/u$.

It is easy to derive the asymptotic tails of the scaling functions. One finds

$$\begin{aligned} f_<(z) &\approx \text{const} \quad \text{as } z \to 0, \\ &\approx z^{-d} e^{-z^2} \quad \text{as } z \to \infty, \end{aligned} \tag{12}$$

and

$$\begin{aligned} f_>(z) &\approx z^{-(d-2)} \quad \text{as } z \to 0, \\ &\approx z^{-2} e^{-z^2} \quad \text{as } z \to \infty. \end{aligned} \tag{13}$$

At $d = 2$, one finds $f_2(z) \sim -2\ln(z)$ as $z \to 0$, and $f_2(z) \sim e^{-z^2}/z^2$ as $z \to \infty$. Note that the scaling forms postulated in Eqs. (8) and (9) do not, in general, hold for very small $x$. For $d < 2$, the scaling regime can actually be extended all the way to $x \to 0$, and indeed, the exact relation $p(0,t) = 1$ is actually part of the scaling regime. This is seen by taking the $x \to 0$ limit in Eq. (8) and using the asymptotic small-$z$ behavior of $f_<(z)$ in Eq. (12). In contrast, for $d > 2$, one cannot recover $p(0,t) = 1$ by taking the $x \to 0$ limit in Eq. (9). This is a manifestation of the fact that for $d > 2$ one always needs a finite lattice cutoff $a > 0$ (see, e.g., Ref. [4]). Thus, for $d > 2$, the continuum scaling result in Eq. (9) does not hold for $x < a$.

We next substitute Eqs. (8) and (9) in Eq. (4) and replace the sum by an integral over space. Note that even though we started out with $d$ and $N$ being integers, the general formula (4) can be analytically continued to real $d > 0$ and real $N > 0$. So from now on we will consider $d$ and $N$ to be continuous real positive numbers as, e.g., represented in the phase diagram in Fig. 2. Indeed, noninteger values of $d$ can be interpreted in terms of random walks on fractal manifolds with noninteger dimensions. Consider first the case $d < 2$ where we get, dropping unimportant prefactors, for large $t$

$$W_N(t) \sim t^{d/2} \int_0^\infty [f_<(z)]^N z^{d-1} dz. \tag{14}$$

From the tails of the scaling function $f_<(z)$ in Eq. (12), it is evident that the integral in Eq. (14) is convergent and is just a constant, and hence for $d < 2$, $W_N(t) \sim b_N\, t^{d/2}$ for large $t$, with only the prefactor $b_N$, but not the exponent, depending on $N$. Exactly at $d = 2$, using $p(\vec{x},t) \sim [1/\ln t] f_2(x/\sqrt{4Dt})$, and following a similar analysis we get for large $t$

$$W_N(t) \sim \frac{t}{[\ln t]^N} \int_0^\infty [f_2(z)]^N z dz. \tag{15}$$

Using the exact form of the scaling function $f_2(z)$ described before, one can check that the integral above is convergent and, hence, for $d = 2$, $W_N(t) \sim t/[\ln t]^N$ for large $t$.

For $d > 2$, a similar manipulation is a bit more delicate. We recall that the scaling result for $p(\vec{x},t)$ in Eq. (9) holds only for $x > a$ where $a$ is a lattice cutoff, while $p(0,t) = 1$ identically. Thus, in the sum in Eq. (4) we separate the $x = 0$ term and replace the rest of the sum by an integral over the scaling form,

$$W_N(t) \approx 1 + A_d\, t^{N(1-d/2)} \int_a^\infty \left[ f_> \left( \frac{x}{\sqrt{4Dt}} \right) \right]^N x^{d-1} dx, \tag{16}$$

where $A_d$ is a volume-dependent constant and $a$ is the lattice cutoff. This gives, after rescaling $z = x/\sqrt{4Dt}$,

$$W_N(t) \approx 1 + A_d t^{N-(N-1)d/2} \int_{a/\sqrt{4Dt}}^\infty [f_>(z)]^N z^{d-1} dz. \tag{17}$$

We now have to check how the integral behaves as $t \to \infty$, i.e., its lower limit approaches 0. This is controlled by the small-$z$ behavior of the integrand. From Eq. (13), we get $[f_>(z)]^N \sim z^{-N(d-2)}$ as $z \to 0$. Hence, the integrand behaves as $z^{d-(d-2)N-1}$ as $z \to 0$. Thus, two situations arise. If $d-(d-2)N > 0$, i.e., $d < d_c(N) = 2N/(N-1)$ (recall that $d > 2$ already), the integral is convergent at the lower limit, and one can safely take the $t \to \infty$ limit, and then Eq. (17) predicts that for large $t$ and $2 < d < d_c(N) = 2N/(N-1)$

$$W_N(t) \sim t^\nu, \quad \nu = N - d(N-1)/2. \tag{18}$$

In contrast, if $d-(d-2)N < 0$, i.e., $d > d_c(N) = 2N/(N-1)$, the lower limit of the integral behaves as $\sim t^{(N-1)d/2-N}$ for large $t$, which precisely cancels the power-law prefactor, and

$$W_N(t) \to \text{const}, \quad d > d_c(N) = 2N/(N-1), \tag{19}$$

where the constant evidently depends on the cutoff, i.e., on the details of the lattice, and is thus nonuniversal. Physically this means that for $d > d_c(N)$, the common sites visited by all the walkers are typically close to the origin and are visited at relatively early times. At late times, the walkers hardly overlap and hence $W_N(t)$ does not grow with time. Finally, exactly at $d = d_c(N)$, a similar analysis shows that $W_N(t) \sim \ln(t)$ for large $t$. The upper phase boundary in Fig. 2 depicts the critical line $d_c(N) = 2N/(N-1)$ as a function of $N$. Alternatively, for fixed $2 < d < d_c(N)$, this critical line can also be described as $N_c(d) = d/(d-2)$. For $1 \leqslant N \leqslant N_c(d)$, we have $W_N(t) \sim t^\nu$ with $\nu = N - d(N-1)/2$.

To check our analytical predictions, we have computed $W_N(t)$ numerically for $d = 1$, 2, 3 and for several values of $N$. In $d = 1$, our result predicts that $W_N(t) \sim b_N t^{1/2}$ for large $t$ where the exponent 1/2 is independent of $N$ and only the prefactor $b_N$ depends on $N$. The results in Fig. 3(a) are consistent with this prediction. In $d = 2$, our results predict that $W_N(t) \sim t/[\ln t]^N$ which is verified numerically in Fig. 3(b). For $d = 3$, our result predicts that there is a critical value $N_c = 3$ such that $W_N(t) \sim t^{(3-N)/2}$ for $N < 3$, $W_N(t) \sim \ln(t)$ for $N = 3$, and $W_N(t) \sim$ const for $N > 3$. The simulation results for $d = 3$ in Fig. 3(c) are consistent with these predictions.

Interestingly, the critical dimension $d_c(N) = 2N/(N-1)$ has also appeared in the probability literature [29] in the context of the probability of no intersection of $N$ random walkers up to $t$ steps all starting at the origin [28]. To make a precise connection with our work presented here, consider the random variable $V_{N,N}(t)$ that denotes the number of common sites visited by all the $N$ walkers up to $t$ steps. Since all the

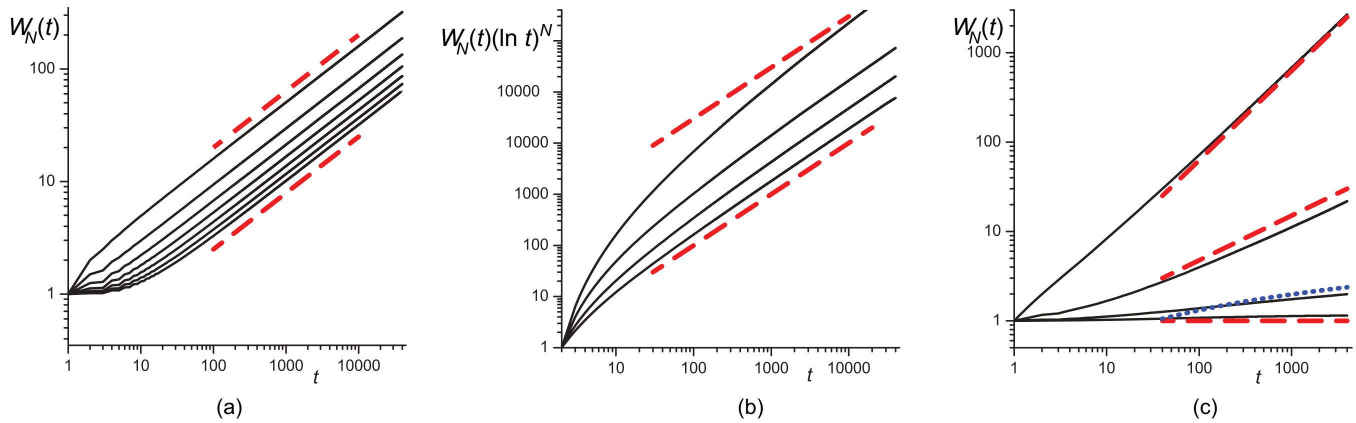


FIG. 3. (Color online) Numerical results for $W_N(t)$ vs $t$ for different values of $N$ and $d$: (a) In $d = 1$, the black lines (from top to bottom) correspond to $W_N(t)$ vs $t$ for $N = 1,2,\ldots,7$ averaged over 100 000 realizations, and the dashed (red) lines have slope 1/2. (b) In $d = 2$, we plot $a_N W_N(t)[\ln t]^N$ vs $t$ for $N = 1,2,3,4$ (from bottom to top) averaged over 100 000 realizations. The prefactor $a_N$ is chosen so that the curves start at the same point to make the visualization better. (c) In $d = 3$, the black lines correspond to $N = 1,2,3,4$ (from top to bottom) averaged over 7000 realizations. Analytical results predict $W_N(t) \sim t^{(3-N)/2}$ for $N < 3$, $W_N(t) \sim \ln t$ for $N = 3$, and $W_N(t) \sim$ const for $N = 4$. The dashed (red) lines have slopes 1 ($N = 1$), 1/2 ($N = 2$), and 0 ($N = 4$); the dotted (blue) line is proportional to $\ln t$ ($N = 3$).

walkers start at the origin, clearly the number of common sites visited must be at least 1, implying $V_{N,N}(t) \geqslant 1$. When $V_{N,N}(t) = 1$, it corresponds to the event that the walkers do not intersect further up to step $t$ and the origin at $t = 0$ remains the only site visited by all of them up to step $t$. Thus, the probability of no further intersection up to step $t$ is $F_N(t) = \Pr[V_{N,N}(t) = 1]$. Lawler studied the decay of $F_N(t)$ for large $t$ rigorously in special cases [28], and Duplantier showed [29] that $F_N(t)$ approaches a constant as $t \to \infty$ for $d > d_c(N) = 2N/(N-1)$. For $d < d_c(N)$, $F_N(t) \sim t^{-\zeta}$ and the exponent $\zeta$ was computed using an $\epsilon$ expansion around the critical dimension [29]. In contrast, in this paper we have computed the mean of the random variable $V_{N,N}(t)$, i.e., $W_N(t) = \langle V_{N,N}(t)\rangle$. Note that while $F_N(t)$ is not exactly computable in all $d$, $W_N(t)$ is, as we have shown here.

Another interesting related problem is to compute the mean number of $N$-fold self-intersections of a single ideal polymer chain of length $t$. In Ref. [31], it was stated that in $d = 3$ this grows as $t^{(3-N)/2}$, which looks similar to our result $W_N(t) \sim t^{(3-N)/2}$ in the intermediate phase in $d = 3$ and for $1 < N < 3$. However, the two problems are not exactly identical, and even the single-chain result in Ref. [31] was qualitatively argued for, not rigorously proved, and the logarithmic correction for $N = 3$ was not mentioned.

In summary, we have presented exact asymptotic results for the mean number of common sites $W_N(t)$ visited by $N$ independent random walkers in $d$ dimensions. We have shown that, as a function of $N$ and $d$ in the ($N$-$d$) plane, there are three distinct regimes for the growth of $W_N(t)$, including, in particular, an anomalous intermediate regime $2 < d < d_c(N) = 2N/(N-1)$.

We conclude with a few additional remarks. In this paper, we have computed analytically the scaling behavior of $p(\vec{x},t)$, the probability that the site $\vec{x}$ is visited by a single $t$-step walker. This result turns out to be the key ingredient to address other related questions. For instance, it would be easy to compute the mean number of sites visited exactly by $k$ walkers (out of $N$) up to time $t$ using our result in Eq. (2). Here we have restricted our attention only to the $k = N$ case for simplicity. Furthermore, it follows by putting $k = 0$ in Eq. (2) that the probability a site $\vec{x}$ is not visited by any of the $N$ walkers is simply $P_{0,N}(\vec{x},t) = [1 - p(\vec{x},t)]^N$. Hence, the probability that a site $\vec{x}$ is visited by at least one of the walkers is $1 - P_{0,N}(\vec{x},t) = 1 - [1 - p(\vec{x},t)]^N$. Summing over $\vec{x}$, one then gets the mean number of distinct sites visited by the walkers, $S_N(t) = \sum_{\vec{x}}[1 - \{1 - p(\vec{x},t)\}^N]$. Thus, knowing the behavior of $p(\vec{x},t)$, one can fully analyze $S_N(t)$ and recover rather simply the results of Ref. [14].

There are several directions in which our work can be generalized. It would be interesting to consider cases where the walkers have different step lengths or when they start at different positions [32]. Also, computing the full distribution of the number of common sites visited by all walkers remains a challenging open problem.

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