Geophys. J. Int. (1990) **102**, 273–281

Electromagnetic induction in a spherical earth with non-uniform oceans and continents in electric contact with the underlying medium—I. Theory, method and example

E. B. Fainberg, A. V. Kuvshinov and B. Sh. Singer

Institute of Terrestrial Magnetism, Ionosphere and Radiowave Propagation of the USSR Academy of Sciences, 142092 *Troitsk, Moscow region, USSR*

Accepted 1989 November 29. Received 1989 July 13; in original form 1987 November 23

SUMMARY

This paper presents basic principles of the iterative–dissipative method and main points of its numerical realization. A spherical earth model consisting of a thin non-uniform surface layer being in galvanic contact with an underlying stratified medium is considered. The current system induced by a uniform magnetic field in the model with a realistic distribution of surface conductance is calculated. It is shown that leakage currents are significant even for the earth model with a highly resistive crust and upper mantle.

Key words: electromagnetic induction, iterative–dissipative method, non-uniform earth, numerical modelling.

INTRODUCTION

Distortion effects of surface inhomogeneities is the central problem of modern geoelectrics. Beginning with the study of Price (1949), the main efforts to solve this problem were concentrated on calculations of electromagnetic fields in thin sheet models. Price's model consists of an inhomogeneous surface layer separated from the underlying laterally homogeneous medium by a thin isolating layer. A surface current in the model

$$\mathbf{j}^s = -\mathbf{n} \times \nabla_\tau \psi, \tag{1}$$

can be expressed via a single scalar (current) function ψ, satisfying Price's equation

$$\nabla_\tau \cdot (S^{-1} \nabla_\tau \psi) = -iw\mu_0 H_n, \tag{2}$$

where S is the surface conductance, $\mathbf{n}$ and H_n are the radial unit vector and the corresponding component of the magnetic field, ∇_τ is the surface part of the operator ∇ and $\exp(-iwt)$ is the time factor.

The iterative–dissipative method (IDM) was proposed by Singer & Fainberg (1979) to solve equation (2). It was used by Fainberg & Singer (1981) to calculate global currents induced by an external magnetic field in a spherical earth model with a realistic conductance of oceans and continental sediments. The resultant current distributions were used by Fainberg, Singer & Kuvshinov (1983) to estimate the effect of the world-wide surface inhomogeneities on the magnetotelluric impedances. Comparison of the numerical results with experimental data showed that modelling had predicted a greater distortion of impedances than had been actually observed. This discrepancy was explained by the fact that the authors had ignored the leakage currents arising due to the galvanic contact of the surface layer with the underlying medium. The significance of leakage currents was stressed by a number of authors, see for example the review by Cox (1980).

In Vasseur & Weidelt (1977) the theory of electromagnetic induction was generalized to include models with a non-uniform surface layer in galvanic contact with an underlying stratified medium. It was shown by Fainberg & Singer (1980) that IDM could be used to calculate the electromagnetic field in such models. Unfortunately, Price's equation could not be straightforwardly deduced from the equations of Fainberg & Singer (1980). The thin sheet matching conditions, free of this drawback, were obtained by Singer & Fainberg (1985). Below, the main aspects of the IDM realization are presented. The same notation as in Fainberg & Singer (1987) is used throughout the paper.

2 BASIC EQUATIONS

The distribution of currents in a surface layer in galvanic contact with the underlying stratified medium

$$\mathbf{j}^s = -\mathbf{n} \times \nabla_\tau \psi - \nabla_\tau W \tag{3}$$

is defined via two scalar functions $\psi(\theta, \varphi)$ and $W(r_0 - 0, \theta, \varphi)$. Here r_0 is the earth radius. Current function

$$\psi = \delta(\partial_n V) \tag{4}$$

coincides with a jump of the radial derivative of the induction potential across the surface layer. Tangential $\mathbf{H}_\tau$,

$\mathbf{E}_\tau$ and vertical H_n, E_n field components outside the thin sheet are expressed via the induction potential $V(r, \theta, \varphi)$ and the galvanic potential $W(r, \theta, \varphi)$:

$$\mathbf{H}_\tau = -\nabla_\tau \partial_n V - \mathbf{n} \times \nabla_\tau W, \tag{5}$$

$$\mathbf{E}_\tau = i w \mu_0 \mathbf{n} \times \nabla_\tau V + \nabla_\tau(\sigma^{-1})\, \partial_n W), \tag{6}$$

$$H_n = \nabla_\tau^2 V, \tag{7}$$

$$E_n = -\sigma^{-1}\, \nabla_\tau^2 W. \tag{8}$$

The potentials satisfy the equations

$$\partial_n^2 V + \nabla_\tau^2 V + i w \mu_0 \sigma V = 0, \tag{9}$$

$$\sigma\, \partial_n(\sigma^{-1}\, \partial_n W) + \nabla_\tau^2 W + i w \mu_0 \sigma W = 0. \tag{10}$$

Note that $W = 0$ in a non-conductive atmosphere. The field distribution inside and outside the earth can be found if the current function ψ and potential W on the earth's surface are known. To evaluate $\psi(\theta, \varphi)$ and $W(r_0 - 0, \theta, \varphi)$, the following system of equations has to be solved:

$$\nabla_\tau \cdot [S^{-1}(\nabla_\tau \psi - \mathbf{n} \times \nabla W)] = i w \mu_0 \nabla_\tau^2 V, \tag{11}$$

$$\nabla_\tau \cdot [S^{-1}(\mathbf{n} \times \nabla_\tau \psi + \nabla_\tau W)] = -\nabla_\tau^2(\sigma^{-1}\, \partial_n W). \tag{12}$$

If the surface layer is not in contact with the underlying medium, then $W(r_0 - 0, \theta, \varphi)$ vanishes and equation (11) reduces to Prices's equation (2).

Setting

$$S^{-1}(\theta, \varphi) = R_0 + R^*(\theta, \varphi), \tag{13}$$

where $R_0 =$ constant, in equations (11) and (12), one can obtain a system of integral equations

$$\psi(\theta, \varphi) = \psi_0(\theta, \varphi) + \int_s \mathbf{n} \times \nabla_\tau Q^{\mathrm{i}}(\cos\gamma) \cdot \frac{R^*}{R_0} \mathbf{j}^{\mathrm{s}}(\theta', \varphi')\, ds',$$

$$W(r_0 - 0, \varphi) = -\int_s \nabla_\tau Q^{\mathrm{g}}(\cos\gamma) \cdot \frac{R^*}{R_0} \mathbf{j}^{\mathrm{s}}(\theta', \varphi')\, ds', \tag{14}$$

where $\mathbf{j}^{\mathrm{s}}$ is given by (3) and γ denotes the angle between directions (θ, φ) and (θ', φ'), and the integrals are calculated along the earth's surface. The free term in (14) is

$$\psi_0(\theta, \varphi) = \sum_n \sum_m (2n+1) \frac{i\zeta_n}{1 - i\zeta_n} a_{nm}^{\mathrm{e}} S_{nm}(\theta, \varphi), \tag{15}$$

which specifies the surface current induced by the external magnetic field in the reference model. The reference model is the model obtained from the initial one by a replacement of the inhomogeneous surface layer with a homogeneous layer whose conductance is equal to R_0^{-1}. The coefficients $\{a_{nm}^{\mathrm{e}}\}$ are determined by an external magnetic field:

$$H_n^{\mathrm{e}}(r_0, \theta, \varphi) = \sum_n \sum_m \frac{n(n+1)}{r_0} a_{nm}^{\mathrm{e}}\, S_{nm}(\theta, \varphi),$$

where $S_{nm}(\theta, \varphi)$ are the spherical functions. The parameter

$$\zeta_n = \omega \mu_0 R_0^{-1} r_0 \frac{\alpha_n(r_0)}{2n+1}, \tag{16}$$

where $\alpha_n(r_0)^{-1}$, is the ratio of the internal to external parts for a vertical magnetic field at the earth's surface. It can be expressed via the induction mode impedance of the underlying medium $Z_n^i(r_0)$ or in terms of the penetration depth $\lambda_n(r_0)$:

$$\alpha_n(r) = (2n+1) \frac{\lambda_n}{r + n\lambda_n},$$

$$Z_n^i(r) = -i\omega\mu_0 \lambda_n(r). \tag{17}$$

Kernels of the system (14) are

$$4\pi Q^{\mathrm{i}}(z) = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} (1 - i\zeta_n)^{-1} P_n(z), \tag{18}$$

$$4\pi Q^{\mathrm{g}}(z) = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} (1 + \eta_n)^{-1} P_n(z). \tag{19}$$

Here P_n are the Legendre polynomials,

$$\eta_n = Z_n^{\mathrm{g}}(r_0) R_0^{-1} \tag{20}$$

and $Z_n^{\mathrm{g}}(r_0)$ is the galvanic impedance of the underlying medium.

Equations (14) can be solved using IDM, but it is convenient to first transform them to the integral equation for the surface current $\mathbf{j}^{\mathrm{s}}$. Using equations (3) and (14), one finds

$$\mathbf{j}^{\mathrm{s}}(\theta, \varphi) = \mathbf{j}^{\mathrm{s}}(\theta, \varphi) - \int_s [(\mathbf{n} \times \nabla_\tau) \bullet (\mathbf{n} \times \nabla'_\tau) Q^{\mathrm{i}}(\cos\gamma)$$

$$+ \nabla_\tau \bullet \nabla'_\tau Q^{\mathrm{g}}(\cos\gamma)] \cdot \frac{R^*}{R_0} \mathbf{j}^{\mathrm{s}}(\theta, \varphi')\, ds', \tag{21}$$

where $\mathbf{j}_0^{\mathrm{s}} = -\mathbf{n} \times \nabla_\tau \psi_0$, ∇'_τ is the differentiation operator with respect to θ', φ' and $\bullet$ is the tensor product sign.

3 THE ITERATIVE–DISSIPATIVE METHOD

We consider below tangential vector fields defined over a sphere as elements of a Hilbert space $\mathcal{M}$. Vector summation and multiplication by a complex number are defined as usual. The scalar product of vectors $\mathbf{u}$, $\mathbf{v}$ and the norm of $\mathbf{u}$ are defined as

$$(\mathbf{u}, \mathbf{v}) = \int_s \mathbf{u}(\theta, \varphi) \bar{\mathbf{v}}(\theta, \varphi)\, ds,$$

$$\|\mathbf{u}\| = (\mathbf{u}, \mathbf{u})^{1/2} = \left(\int_s |\mathbf{u}(\theta, \varphi)|^2\, ds\right)^{1/2}. \tag{22}$$

Then, equation (21) can be reduced to

$$\mathbf{j}^{\mathrm{s}} = \mathbf{j}_0^{\mathrm{s}} - \hat{G}\hat{R}\mathbf{j}^{\mathrm{s}}, \tag{23}$$

where

$$\hat{G} = \sum_n \sum_m [(1 - i\zeta_n)^{-1} \hat{P}_{nm}^{i} + (1 + \eta_n)^{-1} \hat{P}_{nm}^{\mathrm{g}}], \tag{24}$$

and $\hat{R}$ transforms $\mathbf{j}^{\mathrm{s}}(\theta, \varphi)$ into $\frac{R^*}{R_0} \cdot \mathbf{j}^{\mathrm{s}}(\theta, \varphi)$. Operators $\hat{P}_{nm}^{\mathrm{i}}$ and $\hat{P}_{nm}^{\mathrm{g}}$ in (24) are the projection operators onto vectors $\nabla_\tau S_{nm}$ and $\mathbf{n} \times \nabla_\tau S_{nm}$. The set of these vectors with $n = 1, 2, \ldots$ and $|m| = 0, 1, 2, \ldots, n$ is complete in $\mathcal{M}$.

It was shown by Fainberg & Singer (1980; see Appendix B) that spectral impedances of any laterally uniform medium satisfy inequalities

$$\mathcal{R}e\, Z_n^i(r) \geq 0, \qquad \mathcal{R}e\, Z_n^{\mathrm{g}}(r) \geq 0. \tag{25}$$

Taking into account that the first of these inequalities can be written in the form

$$\mathcal{I}m\,[\omega\mu_0\alpha_n(r)] \geq 0, \tag{26}$$

one can see that the coefficients of the series in equation (24) are restricted by unity:

$$|1 - i\zeta_n|^{-1} \leq 1, \qquad |1 + \eta_n|^{-1} \leq 1,$$

so that for any vector $\mathbf{u}$

$$\|\hat{G}\mathbf{u}\| \leq \|\mathbf{u}\|. \tag{27}$$

As to the operator $\hat{R}$, it follows from its definition that

$$\|\hat{R}\mathbf{u}\| \leq \max\left|\frac{R^*}{R_0}\right| \|\mathbf{u}\|. \tag{28}$$

Therefore, if

$$\max\left|\frac{R^*}{R_0}\right| < 1, \tag{29}$$

then the operator $\hat{G}\hat{R}$ is a contracting one and equation (23) has a unique solution which can be found by the direct iteration method. To satisfy (29) one should choose the reference level to satisfy the condition

$$R_0^{-1} < 2\min(S). \tag{30}$$

This can always be done for any positive surface conductance $S(\theta, \varphi)$. If the surface current $\mathbf{j}^s(\theta, \varphi)$ has been found, one can calculate the tangential electric field straightforwardly from Ohm's law

$$\mathbf{j}^s(\theta, \varphi) = S(\theta, \varphi) \cdot \mathbf{E}_\tau(r_0, \theta, \varphi) \tag{31}$$

and the magnetic field after the integral transformations

$$\begin{aligned} \mathbf{n}\times\mathbf{H}_\tau(r_0+0, \theta, \varphi) &= \mathbf{n}\times\mathbf{H}^0_\tau(r_0+0, \theta, \varphi) \\ &\quad - \int_s (\mathbf{n}\times\nabla_\tau)\bullet(\mathbf{n}\times\nabla'_\tau)Q^\tau(\cos\gamma)\cdot\frac{R^*}{R_0}\mathbf{j}^s(\theta', \varphi')\,ds', \\ H_n(r_0, \theta, \varphi) &= H^0_n(r_0, \theta, \varphi) - \int_s \mathbf{n}\times\nabla_\tau Q^n \\ &\quad \times(\cos\gamma)\cdot\frac{R^*}{R_0}\mathbf{j}^s(\theta', \varphi')\,ds'. \end{aligned} \tag{32}$$

Here

$$\begin{aligned} 4\pi Q^\tau(z) &= \sum_{n=1}^{\infty}\frac{\alpha_n(r_0)}{n+1}(1-i\zeta_n)^{-1}P_n(z), \\ 4\pi Q^n(z) &= \sum_{n=1}^{\infty}\frac{\alpha_n(r_0)}{n}(1-i\zeta_n)^{-1}P_n(z) \end{aligned} \tag{33}$$

and $\mathbf{H}^0$ is the magnetic field excited in the reference model.

Thus, IDM makes it possible to solve equation (21) for an arbitrary frequency ω and any distribution of surface conductance $S(\theta, \varphi)$ and underlying medium conductivity $\sigma(r)$.

4 NUMERICAL REALIZATION OF THE IDM

The integral equation (21) is solved numerically on a mesh

$$s = \bigcup_{kl} s_{kl},$$

$$s_{kl} = \{\theta_{k-1/2} \leq \theta < \theta_{k+1/2},\ \varphi_{l-1/2} \leq \varphi < \varphi_{l+1/2}\}, \tag{34}$$

where

$$\theta_k = h_\theta(k - \tfrac{1}{2}), \qquad \varphi_l = h_\varphi(l - \tfrac{1}{2}),$$

$$h_\theta = \frac{\pi}{N_\theta}, \qquad h_\varphi = 2\frac{\pi}{N_\varphi},$$

$$k = 1, 2, \ldots, N_\theta, \qquad l = 1, 2, \ldots, N_\varphi.$$

We assume that the surface conductance is constant inside each cell, i.e.

$$S(\theta, \varphi) = S(\theta_k, \varphi_l), \qquad R^*(\theta, \varphi) = R^*(\theta_k, \varphi_l). \tag{35}$$

when $(\theta, \varphi) \in s_{kl}$. To construct the integration rule, it is necessary to make certain assumptions concerning the behaviour of the unknown function within the cells s_{kl}. We assume below that $\mathbf{j}^s(\theta, \varphi)$ also remains constant in any cell. This means that solution of (23) is sought, not in the entire Hilbert space $\mathcal{M}$, but within the subspace $\mathcal{M}_c$, which consists of tangential vector fields being constant in each cell s_{kl} of the mesh (34). The fact that $\mathcal{M}_c$ is a linear subspace follows from its obvious invariance with respect to linear operations.

The subspace $\mathcal{M}_c$ is not invariant with respect to the operator $\hat{G}$. This means that $\hat{G}\mathbf{u}$ may not be a piecewise-constant function, although $\mathbf{u}$ is. The vector nearest to $\hat{G}\mathbf{u}$ that belongs to $\mathcal{M}_c$ is $\hat{P}_c\hat{G}\mathbf{u}$, where

$$\hat{P}_c = \sum_k\sum_l \hat{P}_{kl} \tag{36}$$

is the projection operator on $\mathcal{M}_c$. The operator $\hat{P}_{kl}$ transforms an arbitrary vector $\mathbf{u}$ into

$$\begin{aligned} (\hat{P}_{kl}\mathbf{u})_{\theta,\varphi} &= \int_{s_{kl}}\mathbf{u}(\theta', \varphi')\,ds' \Big/ \int_{s_{kl}} ds', \qquad (\theta, \varphi) \in s_{kl} \\ &= 0, \qquad (\theta, \varphi) \notin s_{kl}. \end{aligned} \tag{37}$$

It can be easily seen that $\hat{P}_{kl}\hat{P}_{k'l'} = \delta_{kk'}\delta_{ll'}\hat{P}_{kl}$. Definitions (36) and (37) mean that operator $\hat{P}_c$ when acting on a function $\mathbf{u}$ substitutes its value inside a cell s_{kl} with the mean value calculated for the cell. Thus, one obtains the following finite-dimensional representation of equation (23):

$$\mathbf{J}^s = \mathbf{J}^s_0 - \hat{P}_c\hat{G}\hat{R}\hat{P}_c\mathbf{J}^s, \tag{38}$$

where $\mathbf{J}^s_0 = \hat{P}_c\mathbf{j}^s_0$. As

$$\|\hat{P}_c\mathbf{u}\| \leq \|\mathbf{u}\| \tag{39}$$

for any vector $\mathbf{u}$, the $\hat{P}_c\hat{G}\hat{R}\hat{P}_c$ is a contracting operator if condition (29) is satisfied. Hence, the solution of equation (38) can be found by the simple iteration method as well as by equation (23).

It should be mentioned that the solution of (38), in general, differs from the projection of the solution of (23) onto $\mathcal{M}_c$. If

$$\mathbf{d} = \hat{P}_c\mathbf{j}^s - \mathbf{J}^s, \tag{40}$$

then $\mathbf{d}$ can be found from the equation

$$\mathbf{d} = \mathbf{d}_0 - \hat{P}_c\hat{G}\hat{R}\hat{P}_c\mathbf{d} \tag{41}$$

whose free term

$$\mathbf{d}_0 = -\hat{P}_c\hat{G}\hat{R}(\hat{I} - \hat{P}_c)\mathbf{j}^s \tag{42}$$

is determined by the 'true' solution of the problem. In practice, an additional calculation on a double mesh may be used to estimate $\mathbf{d}$.

Using equations (36) and (37), one can reduce (38) to

$$\mathbf{J}^{s}_{kl}=\mathbf{J}^{s0}_{kl}-\sum_{k}\sum_{l}\hat{K}_{k,k',l-l'}\left(\frac{R^*}{R_0}\right)_{k'l'}\mathbf{J}^{s}_{k'l'} \qquad (43)$$

where $\left(\frac{R^*}{R_0}\right)_{kl}$, $\mathbf{J}^{s}_{kl}$, $\mathbf{J}^{s0}_{kl}$ denote the values of the piecewise-constant functions $\frac{R}{R_0}(\theta,\varphi)$, $\mathbf{J}^{s}(\theta,\varphi)$, $\mathbf{J}^{s}_{0}(\theta,\varphi)$ inside s_{kl}, and the second-order tensor kernel in (43) is

$$\hat{K}_{k,k',l-l'}=A_k^{-1}\int_{s_{kl}}\left[(\mathbf{n}\times\nabla_\tau)\bullet\int_{s_{k'l'}}(\mathbf{n}\times\nabla'_\tau)Q^{i}(\cos\gamma)\,ds' + \nabla'_\tau\bullet\int_{s_{k'l'}}\nabla'_\tau Q^{g}(\cos\gamma)\,ds'\right]ds, \qquad (44)$$

$$A_k=\int_{s_{kl}}ds=2h_\varphi r_0^2\sin\theta_k\sin(h_\theta/2).$$

The part of the algorithm where the tensor is calculated should be constructed with the utmost care because we assumed earlier that the operator $\hat{G}$ was known exactly. Moreover, the higher the accuracy of the tensor calculation, the greater the contrast $\max(S)/\min(S)$ in the model. Although the approach described above is the most logical way to the IDM numerical realization, another one which does not include averaging over s_{kl} can also be used to avoid the evaluation of the four-fold integral in equation (44). Practical calculations have shown that the simplified algorithm holds convergence up to a rather high value of the contrast.

There are some technical details concerning a calculation of spectral impedances of the laterally uniform underlying medium and summation of the poorly convergent series (33). These details can be found in Appendices A and B.

5 NUMERICAL MODELLING EXAMPLE: DISCUSSION OF THE RESULTS

As an example of how the method works, we calculated global currents induced in a spherical model by a uniform external magnetic field parallel to the polar axis $\theta=0°$. The conductance of the oceans and the sedimentary cover published by Fainberg & Sidorov (1978) and the underlying medium conductivity

$$\sigma(r)=1.4\times10^{-5}\,\mathrm{S\,m^{-1}},\qquad 0\leqslant r_0-r\leqslant 10^5\,\mathrm{m},$$
$$=\sigma_0[(r_0-r)/h_0]^{\gamma},\qquad r_0-r>10^5\,\mathrm{m} \qquad (45)$$

were used. In agreement with the global sounding results (Fainberg 1983), $\sigma_0=1.4\times10^{-14}\,\mathrm{S\,m^{-1}}$, $\gamma=4.735$ and $h_0=10^3$ m. The theoretical apparent resistivity curve for the model (45) agrees well with experimental data. The transverse resistance T of the upper layer with thickness 100 km equals $7\times10^9\,\Omega\,\mathrm{m}^2$ according to the estimates of the crust and upper mantle resistance for the 'cold' section of the Baltic shield (Jamaletdinov 1982). A numerical mesh with $5°\times5°$ cells was used.

Azimuthal and meridional components of surface currents excited by an external field of unit amplitude at 1 hr periods are displayed in Figs 1–4. The radial component of currents just under the surface layer is presented in Figs 5 and 6. A current density can be obtained if one multiplies the number shown on an isoline by the coefficient C given in the figure caption. Comparison of Figs 1–4 with the results obtained by Fainberg & Singer (1981) shows that leakage currents appreciably change the induction pattern. The currents flowing around poorly conducting continents are now less pronounced, although large continental masses such as Eurasia, North and South America, Africa, Antarctica and Greenland are still obstacles for the current. The

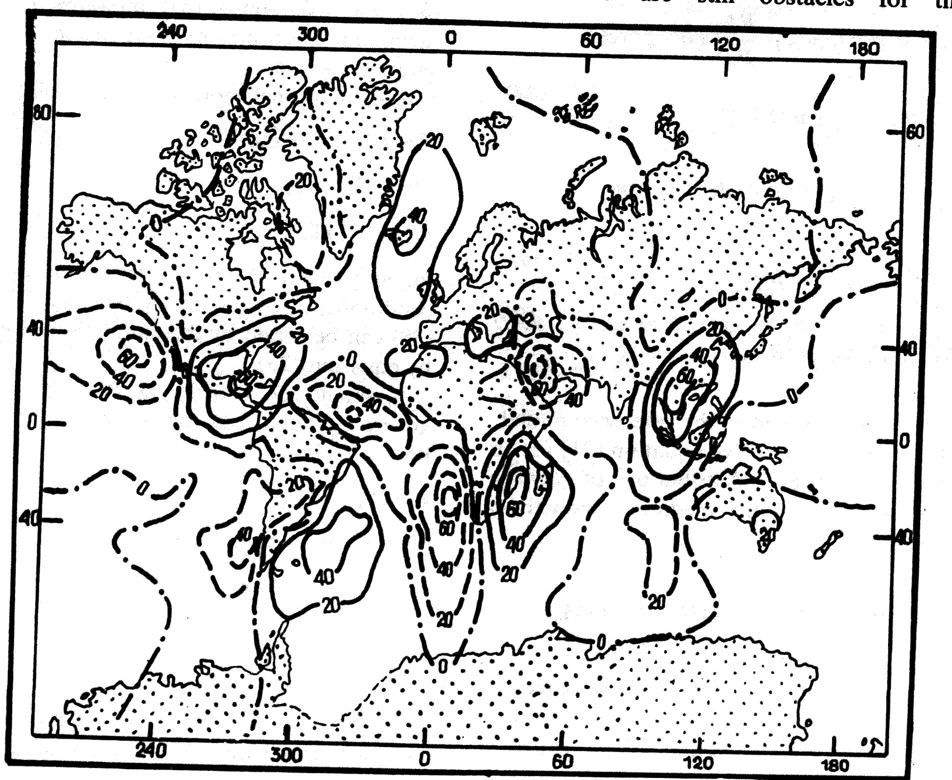

Figure 1. Real part of j_θ; $C=5.93\times10^{-3}\,\mathrm{A\,m^{-1}}$.

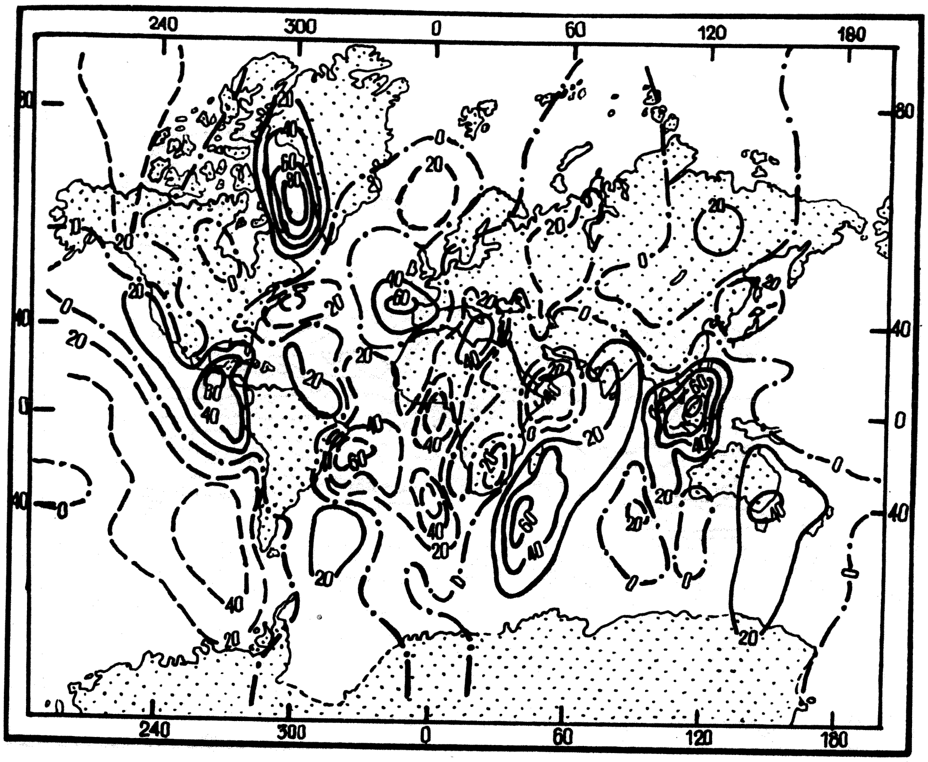

Figure 2. Imaginary part of j_θ; $C = 1.97 \times 10^{-3}\ \mathrm{A\ m^{-1}}$.

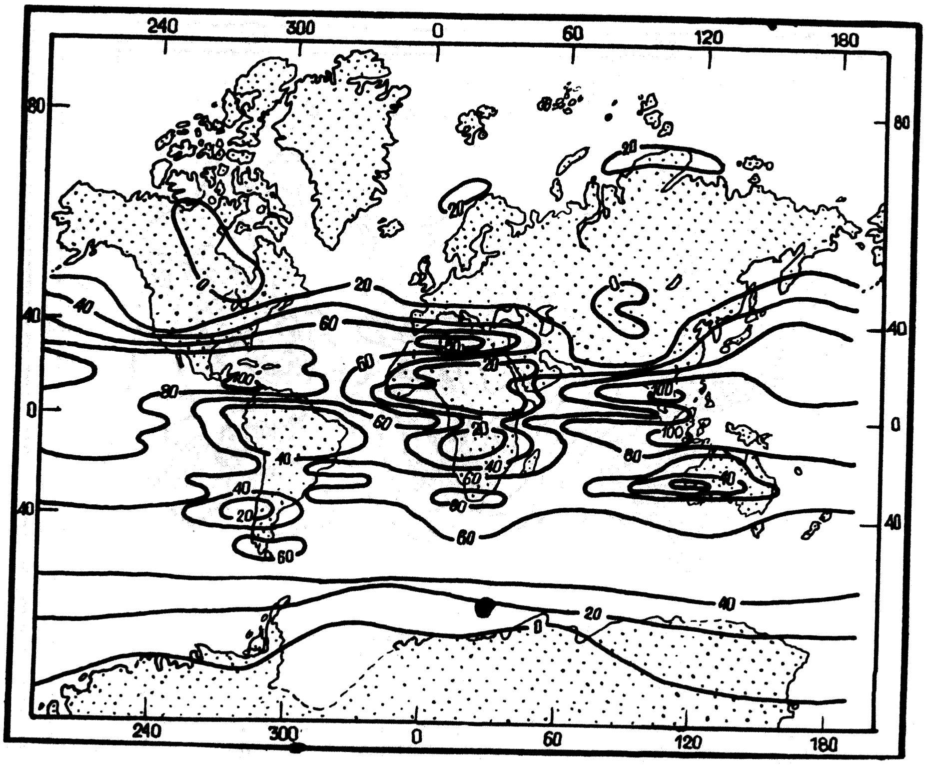

Figure 3. Real part of j_φ; $C = 1.78 \times 10^{-2}\ \mathrm{A\ m^{-1}}$.

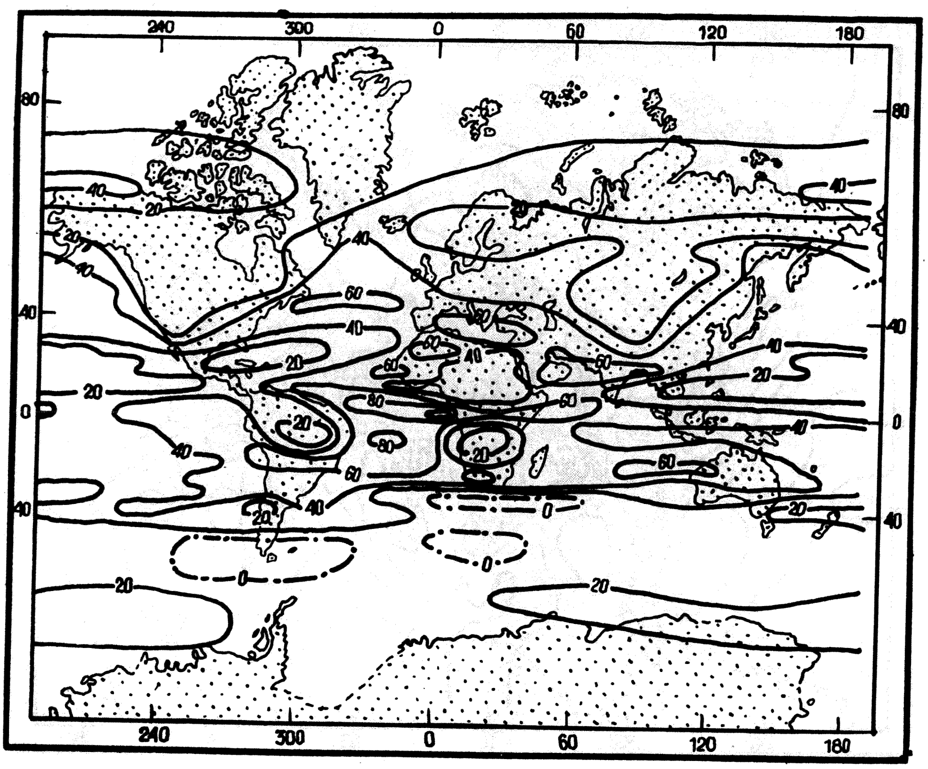

Figure 4. Imaginary part of j_φ; $C = 7.43 \times 10^{-3}$ A m^{-1}.

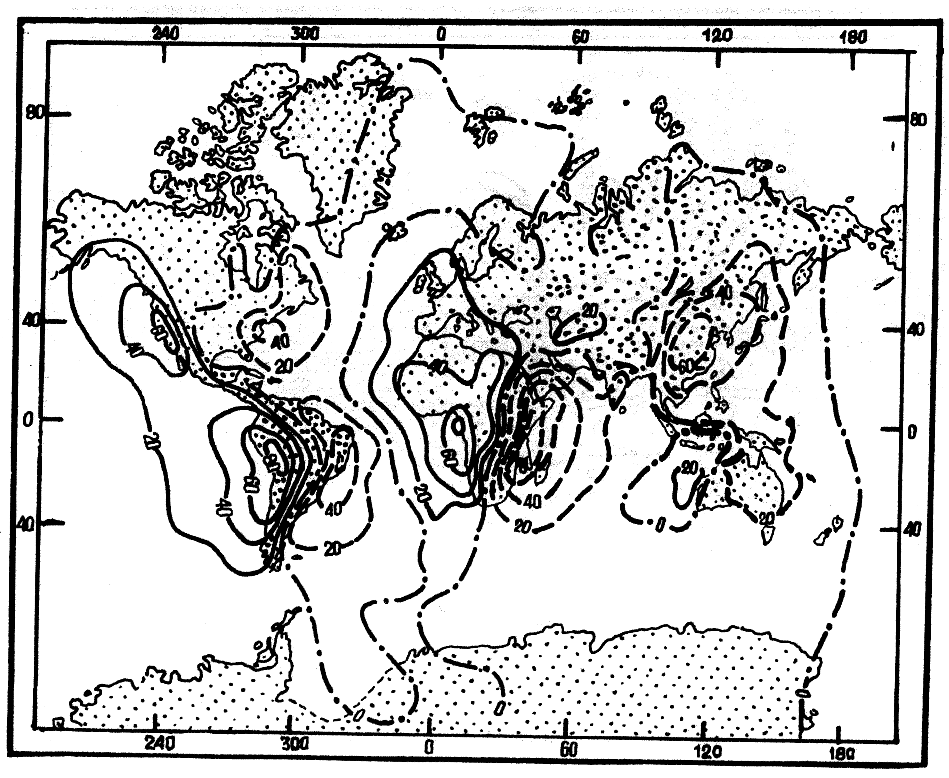

Figure 5. Real part of j_n; $C = 2.09 \times 10^{-9}$ A m^{-2}.

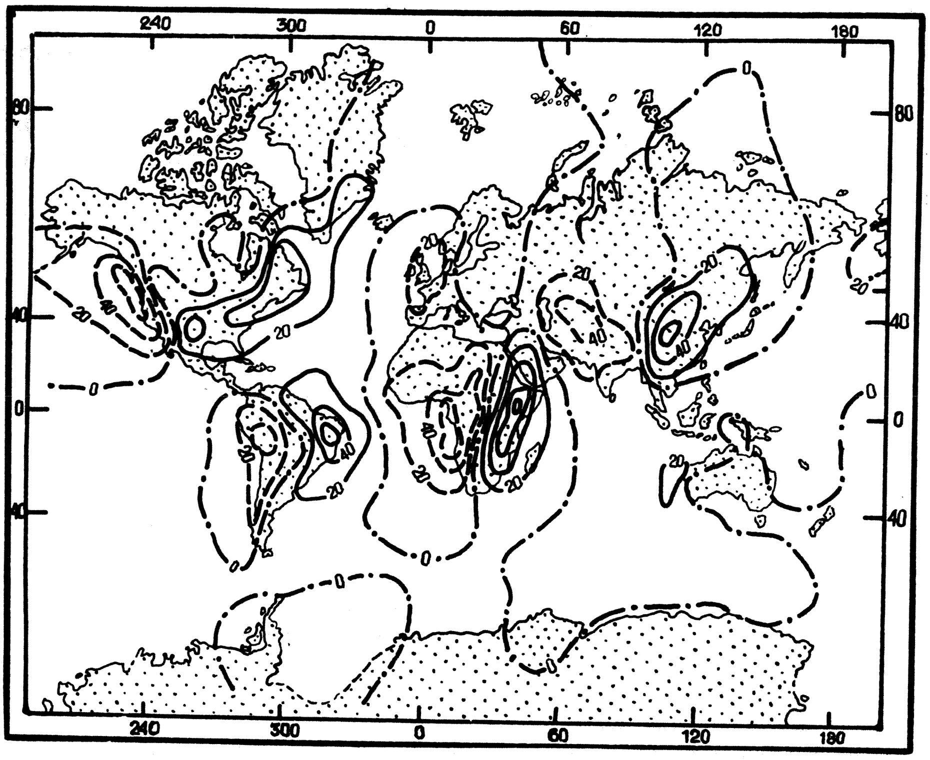

Figure 6. Imaginary part of j_n; $C = 1.91 \times 10^{-9}$ A m^{-2}.

concentration effects are clearly seen in conductive regions (Central America, the Drake Strait, northern and southern Africa and southeastern Asia). Surface inhomogeneities make the current change its direction; a meridional component reaches 30 per cent of the azimuthal one. Besides, current deviates towards the Earth's interior near one side of a continent and outwards near the other. Maximum j_n values occur in zones of sharp changes in surface conductance, i.e. at ocean–continent, mountain–geosyncline boundaries and so on. A typical width of 'flow-in' or 'flow-out' zones in oceans equals 4000–5000 km in agreement with the values $\{ST\}^{1/2}$ of the distance along which the anomalous electric field subsides. It is of interest to compare the values of horizontal and vertical currents. Estimates show that up to 50 per cent of the current flowing in the surface layer towards, for example, Africa deviate downards and flow under the continent in spite of high transverse resistance of underlying medium in the model. A similar relationship occurs near other continents. This also indicates the important role of leakage currents in the generation of surface electromagnetic fields.

REFERENCES

Cox, C. S., 1980. Electromagnetic induction in the oceans and inferences on the constitution of the Earth, *Geophys. Surv.*, **4**, 137–156.

Fainberg, E. B., 1983. Global geomagnetic sounding, in *Mathematical Modeling of Electromagnetic Fields*, pp. 79–121, IZMIRAN, Moscow (in Russian).

Fainberg, E. B. & Sidorov, V. A. (eds), 1978. *The Total Longitudinal Conductance of Sedimentary Cover and Water Shell of the Earth,* Nauka, Moscow (in Russian).

Fainberg, E. B. & Singer, B. Sh., 1980. Electromagnetic induction in a non-uniform spherical model of the Earth, *Ann. Geophys.*, **36**, 127–134.

Fainberg, E. B. & Singer, B. Sh., 1981. Electromagnetic induction in a spherical Earth model with a real distribution of the near-surface conductivity, *Phys. Earth planet. Inter.*, **25**, 52–56.

Fainberg, E. B. & Singer, B. Sh., 1987. The influence of surface inhomogeneities on deep electromagnetic sounding of the Earth, *Geophys. J. R. astr. Soc.*, **90**, 61–73.

Fainberg, E. B., Singer, B. Sh. & Kuvshinov, A. V., 1983. Electromagnetic fields induced in the world's oceans and the spatial distribution of electrical conductivity functions, *Phys. Earth planet. Inter.*, **32**, 293–300.

Jamaletdinov, A. A., 1982. Normal electrical structure of crystallic basement and its geothermal interpretation on data of MHD-sounding at the Kola peninsula, in *Deep Electromagnetic Sounding with the use of Pulse MHD-Generators,* pp. 35–36, Kola Branch USSR Academy of Science, Apatites (in Russian).

Price, A. T., 1949. The induction of electric currents in non-uniform thin sheets and shells, *Q. J. Mech. Appl. Math.*, **2**, 283–310.

Singer, B. Sh. & Fainberg, E. B., 1979. *Electromagnetic Induction in a Non-uniform Sheet,* Preprint, IZMIRAN, Moscow, (in Russian).

Singer, B. Sh. & Fainberg, E. B., 1985. *Electromagnetic Induction in Non-uniform Thin Layers,* IZMIRAN, Moscow (in Russian).

Vasseur, G. & Weidelt, P., 1977. Bimodal electromagnetic induction in non-uniform thin sheets with an application to the northern Pyrenean induction anomaly, *Geophys. J. R. astr. Soc.*, **51**, 669–690.

APPENDIX A: SPECTRAL IMPEDANCE FUNCTIONS OF THE LATERALLY UNIFORM SPHERICAL MODEL

The expressions for the kernels (18), (19) and (33) involve the impedances $Z_n^i = -i\omega\mu_0\lambda_n$ and Z_n^g with $n = 1, 2, \ldots$ up to very large numbers. The upper bound for n is dependent upon the convergence rate of the series. As shown in Appendix B, it is possible to substitute these series by faster convergent ones. Nevertheless, it is necessary to use a rather accurate and fast algorithm to calculate λ_n and Z_n^g. The functions are defined as

$$\lambda_n(r) = [\partial_n \ln V_n(r)]^{-1}, \tag{A1}$$

$$Z_n^g(r) = \sigma^{-1}(z)\partial_n \ln W_n(r), \tag{A2}$$

where $V_n(r)$ and $W_n(r)$ satisfy the equations

$$[\partial_r^2 - n(n+1)r^{-2} + i\omega\mu_0\sigma]V = 0, \tag{A3}$$

$$[\sigma\partial_r\sigma^{-1}\,\partial_r - n(n+1)r^{-2} + i\omega\mu_0\sigma]W = 0 \tag{A4}$$

and boundary conditions $V_n(0) = 0$, $W_n(0) = 0$.

In a laterally uniform flat model, the conductivity distribution is usually approximated by a piecewise-constant function. Within each homogeneous layer, solutions of equations (A3) and (A4) are a superposition of two exponential functions. This permits construction of simple formulae to express a spectral impedance at the top of a homogeneous layer via its value at the bottom. It is possible to use a piecewise-constant approximation of $\sigma(r)$ for a spherical earth model also. In this case, solutions of equations (A3) and (A4) within a homogeneous layer are a superposition of two cylindrical $(n + 1/2)$th order functions which, in turn, can be expressed via exponentials and the nth degree polynomials of r^{-1}. For large n, these expressions become rather awkward. In the case when conductivity varies within a layer as r^{-2}, the linearly independent solutions of (A3) and (A4) appear to be much more simple. We shall assume therefore that the underlying medium consists of N layers, where the kth layer $r_k < r \le r_{k-1}$ has conductivity $\sigma(r) = \sigma_{k-1}r_{k-1}^2/r^2$. In this case, the linearly independent solutions within the layer are $(r_k/r)_k^{\beta^-}$, $(r/r_k)_k^{\beta^+}$ for equation (A3) and functions $(r_k/r)_k^{\beta^+}$, $(r/r)_k^{\beta^-}$ for (A4), where $\beta_k^- = \beta_k - 1/2$, $\beta_k^+ = \beta_k + 1/2$, $\beta_k = [(n + 1/2)^2 - i\omega\mu_0\sigma_{k-1}r_{k-1}^2]^{1/2}$. Making use of the fact that $\lambda_n(r)$ and $Z_n^g(r)$ are continuous at the layer boundaries, one can obtain the recurrence relations

$$u_{k-1} = \frac{(\beta_k - \tau_k)u_k + 2\tau_k}{\beta_k + \tau_k + 2\gamma_k\tau_k u_k}, \tag{A5}$$

$$w_{k-1} = \frac{(\beta_k - \tau_k)w_k + 2\gamma_k\tau_k v_k}{\beta_k v_k + \tau_k + 2\tau_k w_k}, \tag{A6}$$

for functions $u_k = \lambda_n(r_k)/r_k$, $w_k = Z_n^g(r_k)\sigma_k r_k$, $k \in \{1, 2, \ldots, N\}$. Here

$$\gamma_k = n(n+1) - i\omega\mu_0\sigma_{k-1}r_{k-1}^2, \qquad v_k = \frac{\sigma_k r_k^2}{\sigma_{k-1}r_{k-1}^2},$$

$$\tau_k = \tfrac{1}{2}\frac{1-\xi_k}{1+\xi_k}, \qquad \xi_k = (r_k/r_{k-1})^{2\beta_k}.$$

The recursion starts with the values

$$u_{N-1} = 1/\beta_N^+, \qquad w_{N-1} = \beta_N^-. \tag{A7}$$

APPENDIX B: EVALUATION OF THE INTEGRAL EQUATION KERNELS

To calculate the tensor elements (44) it is necessary to sum poorly convergent series (18) and (19). Their convergence becomes worse as z tends to 1. To improve the convergence, the asymptotic expressions $\alpha_n(r) \simeq 1$, $\eta_n \simeq \eta(n + 1/2)$, where $\eta = [\sigma(r_0 - 0)r_0R_0]^{-1}$, for large n can be used. Setting

$$F_n = \frac{i\zeta_n}{1 - i\zeta_n} - \frac{i\zeta}{2n+1},$$

where $\zeta = \omega\mu_0 r_0 R_0^{-1}$, one can reduce equation (18) to

$$4\pi Q^i(z) = \sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}P_n(z) + i\zeta\sum_{n=1}^{\infty}\frac{1}{n(n+1)}P_n(z) + \sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}F_nP_n(z). \tag{B1}$$

The third series converges faster than the original one in (18), as $F_n \sim n^{-2}$ when $n \to \infty$. As to the first two series in (B1), they can be calculated explicitly using the Legendre polynomial definition

$$w(z, t) = \sum_{n=0}^{\infty} t^n P_n(z) \tag{B2}$$

in terms of the generating function $w(z, t) = (1 - 2zt + t^2)^{-1/2}$. Then

$$\sum_{n=0}^{\infty}\frac{P_n(z)}{n+1} = \int_0^1 w(z, t)\,dt = -\ln\mu + \ln(1+\mu),$$
$$\sum_{n=1}^{\infty}\frac{P_n(z)}{n} = \int_0^1 [w(z, t) - 1]\frac{dt}{t} = -\ln\mu - \ln(1+\mu), \tag{B3}$$

where $\mu = [(1-z)/2]^{1/2}$ (if $z = \cos\gamma$ then $\mu = \sin\gamma/2$). These relations lead to

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}P_n(z) = 1 - 2\ln(1+\mu),$$
$$\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}P_n(z) = -1 - 2\ln\mu. \tag{B4}$$

Thus,

$$4\pi Q^i(z) = -1 - 2\ln\mu + i\zeta[1 - 2\ln(1+\mu)] + \sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}F_nP_n(z). \tag{B5}$$

In the same way, one gets

$$4\pi Q^g(z) = 2\eta^{-1}[1 - 2\ln(1+\mu)] + \sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\phi_nP_n(z), \tag{B6}$$

where

$$\phi_n = \frac{1}{1+\eta_n} - \frac{2\eta^{-1}}{2n+1}.$$

Expressions for the first derivatives of kernels Q^i and Q^g can be obtained from (B5), (B6) and the well-known relation

$$(1 - z^2)P_n'(z) = \frac{n(n+1)}{2n+1}[P_{n+1}(z) - P_{n-1}(z)]. \tag{B7}$$

The result is

$$4\pi(1-z^2)\,\partial_z Q^{\mathrm{i}} = (1+F_1) + z(1+F_2) + 2i\zeta\mu(1-\mu) + \sum_{n=2}^{\infty}(F_{n+1}-F_{n-1})P_n(z), \tag{B8}$$

$$4\pi(1-z^2)\,\partial_z Q^{\mathrm{g}} = \phi_1 + z\phi_2 + 4\eta^{-1}\mu + \sum_{n=2}^{\infty}(\phi_{n+1}-\phi_{n-1})P_n(z). \tag{B9}$$