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MATHEMATICS

Convergence of Eigenfunction Expansions of a Differential Operator with Integral Boundary Conditions

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Abstract—For a second-order ordinary differential operator on an interval of the real line with integral boundary conditions, conditions for the unconditional basis property and uniform convergence of the expansion of a function in terms of the eigen- and associated functions of this operator are established. The convergence and equiconvergence rates of this expansion and the equiconvergence rate of the trigonometric Fourier expansion of this function are estimated. The uniform convergence of its expansion in the adjoint system is studied.

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Il'in's theorems on the unconditional basis property of the systems of eigen- and associated functions (briefly, root functions) of a second-order differential operator on the interval $\overline{G} = [0, 1]$ cover the cases of two-point boundary conditions, i.e., data specified at the points $x = 0$ and 1 [1] or multipoint conditions with a finite number of interior points [2]. Naturally, the problem arises of extending these results to more general boundary forms.

In the general case, boundary forms have to be treated as linear continuous functionals in the space $C(\overline{G})$ (or C^1). However, according to the Riesz theorem, each functional can then be represented in the form of a Stieltjes integral with respect to the measure generated by a function of bounded variation. Thus, in a natural way, we need to study a boundary value problem with integral conditions. This problem was partially investigated in [3, 4], but they addressed only model operators (the domain of the adjoint operator did not depend on the coefficients of the differential operation).

Since a function $v(x)$ of bounded variation can be represented in the form of the sum $v(x) = v^{\mathrm{sa}}(x) + v^{\mathrm{c}}(x) + v^{\mathrm{sc}}(x)$, where v^{sa} is a jump function, v^{c} is an absolutely continuous function, and v^{sc} is a continuous singular function, the boundary forms also split into a sum of three terms: a discrete part containing the values of the function and its derivatives at isolated points of the interval $\overline{G}$ (these points can make up a dense set on $\overline{G}$), an integral of the product by the function $(v^{\mathrm{c}})'$, which belongs to the class $L^1(G)$, and an integral with respect to a continuous singular measure.

The adjoint L^* of an operator L with integral boundary forms is described in [5] and has a rather complex structure. The system biorthogonal adjoint to the system of root functions of L consists of functions that, together with their derivatives, can have jump discontinuities at a countable number of points. The discontinuity points and the jump sizes are determined by the discrete component of the measure v^{sa} and by the singular function v^{sc}. The absolutely continuous part of the measure v^{c} influences the differential operation for L^*: it becomes "loaded," i.e., contains functionals of the solution—the values of an unknown function or its derivatives at the endpoints (and possibly interior points) of $\overline{G}$.

Let us illustrate what was said above on the structure of the adjoint operator by considering a simple example of a nonlocal boundary value problem. Suppose that the operator L acting on the space $L^2(G)$, $G = (0, 1)$, is generated by the differential operation

$$lu(x) = u''(x), \quad x \in G, \quad u \in C^2(\overline{G}) \tag{1}$$

and the boundary conditions

$$u(0) = 0, \quad u(1) + 2u\left(\frac{1}{2}\right) = \int_0^1 u(x)\,dx. \tag{2}$$

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Applying integration by parts, we see that the adjoint operator is generated by the loaded differential operation $l^*v(x) = v''(x) - v'(1)$, $x \in G$, the boundary conditions $v(0) = v(1) = 0$, and the following conditions at the discontinuity point $x = \frac{1}{2}$: $v\left[\frac{1}{2}\right] = 0$, $v'\left[\frac{1}{2}\right] = -2v'(1)$, $v(x) \in W_2^1(G) \cup C^2\left[0, \frac{1}{2}\right) \cup C^2\left[\frac{1}{2}, 1\right)$, where the jump of the function is denoted as $v\left[\frac{1}{2}\right] = v\left(\frac{1}{2}+0\right) - v\left(\frac{1}{2}-0\right)$. At the point $x = \frac{1}{2}$, the functions $v(x)$ from the domain of the operator L^* are continuous, while $v'(x)$ can have a jump discontinuity (if $v'(1) \neq 0$).

Problems with integral boundary conditions have been extensively studied. A useful overview can be found in [5]. Among more recent works, we note the following ones. For an ordinary differential operator of arbitrary order with integral boundary conditions, Shkalikov [6] introduced the concept of regular boundary conditions and proved that, under such conditions, the system of root functions of an operator forms in $L^2(G)$ a Riesz basis with brackets, a block basis (in the case of strongly regular of conditions, a usual Riesz basis). A similar result was obtained in [7] for the vector functional-differential equation $y^{(n)} + Fy = \lambda y$ (F is a subordinated operator) with integral boundary conditions. Problems with integral boundary conditions were also addressed in [8–11] and other works. The adjoint operator was not used or introduced in these and subsequent works.

Below, to prove a theorem on the Riesz basis property, we follow the Il'in approach, which is based on the following Bari theorem [12]. Let $\{u_n\}$ and $\{v_n\}$ be biorthogonal systems in the space $H = L^2(G)$. The system $\{u_n\}$ is a Riesz basis in H if and only if $\{u_n\}$ and $\{v_n\}$ are complete in H and the Bessel inequality holds for them in H. In the case under study, the system biorthogonal adjoint to the system of root functions of an operator H is the system of root functions of the adjoint operator. Therefore, we need to introduce the adjoint operator and to examine the properties of nonsmooth root functions.

Formulation of the problem and the operators L, L^+, L^*. In the space H, we consider the operator L generated by the differential operation $ly = y'' + p_1(x)y' + p_2(x)y$, $x \in G$,

$$p_1(x) \in C(\bar{G}), \qquad p_2(x) \in L^1(G) \tag{3}$$

on the set of functions $D = \{y(x) \in H$: $y \in W_2^2(G) \cap N(U)$, $ly \in H\}$, where $W_2^2(G) = \{y(x) \in H$: $y \in AC^2(\bar{G}), y^{(2)} \in H\}$, $AC^k(\bar{G})$, $k = 1, 2$, is the class of functions $f(x)$ that are absolutely continuous together with $f^{(k-1)}(x)$ on $\bar{G}$, $AC^1 \equiv A$, $N(U)$ is the kernel of the functional $U(y)$: $W_2^2(G) \to \mathbf{C}^2$,

$$U(y) = \int_0^1 y'(x)dv_1(x) + \int_0^1 y(x)dv_2(x), \qquad v_i = \begin{pmatrix} v_{i1} \\ v_{i2} \end{pmatrix}, \quad i = 1, 2, \tag{4}$$

and $v_{ij}(x)$ are functions of bounded variation on $\bar{G}$ that are right continuous at each point of G.

According to what was said above, functions of bounded variation are written as the sum $v_i = v_i^{\mathrm{c}} + v_i^{\mathrm{s}}$, $i = 1, 2$; here, v_i^{c} is a vector with absolutely continuous components and v_i^{s} is the singular part of the function $v_i(x)$: $v_i^{\mathrm{s}} = v_i^{\mathrm{sc}} + v_i^{\mathrm{sa}}$, where v_i^{sc} is a vector with continuous components and a singular function and v_i^{sa} is a jump function. It is well known that $v_i^{\mathrm{s}}(x) \in W_1^1(G)$ and, almost everywhere on G, the derivative $(v_i^{\mathrm{s}})'(x)$ is equal to θ. Let the functions $v_i^{\mathrm{sa}}(x)$ have jumps $\alpha_i^p \in \mathbf{C}^2$ ($i = 1, 2$, $p = 0, 1, \ldots$) at the points $\xi_p \in \bar{G}$. The partition $T = \{\xi_p\}$ of the interval $\bar{G}$ is such that $\xi_0 = 0$, $\xi_1 = 1$, $\xi_p \in G$, $p \geq 2$. The set of points $\{\xi_p\}$ can be finite (e.g., consisting of two points ξ_0, ξ_1) or infinite. The partition T can be dense on the entire $\bar{G}$ or on its part.

By using the notation $\beta_i(x) = \dfrac{dv_i^{\mathrm{c}}(x)}{dx}$, the boundary forms (4) can be written as

$$U(y) = \sum_{i=1}^{2}\left[\sum_{p=0}^{\infty} y^{(2-i)}(\xi_p)\alpha_i^p + \int_0^1 \beta_i(x)y^{(2-i)}(x)dx + \int_0^1 y^{(2-i)}(x)dv_i^{\mathrm{sc}}(x)\right]. \tag{5}$$

We also use the notation $v_i^{\mathrm{s}}[0,x] = \int_{0^+}^{1^-} dv_i^{\mathrm{s}}(t) = \int_0^x dv_i^{\mathrm{sc}}(t) + \sum_{p=0,x}^{\infty}\alpha_i^p\chi(\xi_p, 1]$, $i = 1, 2$, where $\chi(x, 1]$ is the characteristic function of the set $(x, 1]$ and, for each x, the sum on the right-hand side of the formula is taken over those p for which $0 \leq \xi_p < x$. As usual, v_i^* is the conjugation operation for the vector v_i: $v_i^* = \overline{v_i^{\mathrm{T}}} = (\bar{v}_{i1}, \bar{v}_{i2})$.

Consider the following operator L^+ [5]. Let $l_0^+z = z$, $l_1^+z = -z' + \bar{p}_1 z$, and $l_2^+ = z'' - (\bar{p}_1 z)' + \bar{p}_2 z$, $x \in G$, be auxiliary differential operations and D^* be the set of functions $D^* = z \in H$: $\exists \varphi \in \mathbf{C}^2$, $\varepsilon_{j+1} \equiv l_j^* z(x) + v_{j+1}^{s*}[0, x]\varphi \in A(G)$, $j = 0, 1$, $l_0^+ z(0^+) = -\alpha_1^{0*}\varphi$, $l_0^+ z(1^-) = \alpha_1^{1*}\varphi$, $l_1^+ z(0^+) = -\alpha_2^{0*}\varphi + \beta_1^*(0)\varphi$, $l_1^+ z(1^-) = \alpha_2^{1*}\varphi$, $l^*z \equiv l_2^+ z + (\beta_1^*)'(x)\varphi - \beta_2^*(x)\varphi \in H\}$, where φ is a parametric vector.

The operator L^+ acting in H is generated by the differential operation l^*z on the set of functions D^*. The operator L^+ is formally adjoint to L, and the Lagrange identity holds for L and L^+: $(ly, z) = (y, l^*z)$, $y \in D$, $z \in D^*$ [13]. If $\bar{D} = H$, i.e., the domain of L is dense in H, then $L^+ = L^*$ is the adjoint of the operator L.

According to the results of [6], if $p_1 \equiv 0$ on G and the boundary forms (4) of L are regular, then $\bar{D} = H$, i.e., in this case, $L^+ = L^*$.

An example of constructing an adjoint operator in the sense of Brown [5] and Lagrange for the case of multipoint boundary conditions ((5) contains only the first sum over p) and the derivation of an expression for the vector φ from the definition of the class D^* can be found in [14].

Basis property of the root functions of L and L^+. Define $V(x) = (\beta_1^*)'(x) - \beta_2^*(x)$, where $\beta_k = (v_k^{\mathrm{c}})'$ for $k = 1, 2$. We need to impose constraints on the vector function $v_k(x)$, i.e., to establish the relation between matrices the v_k^{sa}, v_k^{sc}, $V(x)$ acting on the vector φ. The kernels of these matrices are defined as

$$N_V = \bigcap_{x \in \bar{G}} N(V(x)),$$

$$N_{ka} = \bigcap_{x \in G} N(v_k^{\mathrm{sa}*}[x]) = \bigcap_{p=2} N(\alpha_k^{p*}),$$

$$N_{kc} = \bigcap_{x \in \bar{G}} N(v_k^{\mathrm{sc}*}[0, x]), \quad k = 1, 2.$$

Let the following embeddings hold:

$$N_{1a} \subseteq N_{1c}, \quad \exists k = 1, 2\colon N_{ka} \subseteq N_{2c}, \tag{6}$$

$$\begin{gathered} \exists k = 1, 2\colon N_{ka} \subseteq N_V, \\ \text{for} \quad k = 2\colon v_{2j}^{\mathrm{c}}(x) \in W_2^1(G), \quad j = 1, 2. \end{gathered} \tag{7}$$

Recall that the Riesz basis in H is a basis equivalent to an orthonormal one, i.e., it is obtained by applying a bounded invertible operator to an orthonormal basis in H [12]. The Riesz basis converges unconditionally, i.e., its convergence is not violated by any rearrangement of the series terms. For a Riesz basis $\{u_n\}$ in H, there exists a unique biorthogonal system $\{v_n\}$, which is also a Riesz basis in H. Both these systems are almost normalized in H, i.e., $\inf_n \|u_n\|_H > 0$ and $\sup_n \|u_n\|_H < \infty$.

Let $\{u_n(x)\}$, $\{v_n(x)\}$ be a biorthonormal (in H) pair of systems of root functions of the operators L and L^+, i.e., for every $n \in \mathbf{N}$, $u_n \in D$ and $v_n \in D^*$ and, for some number $\lambda_n \in \mathbf{C}$, the relations $lu_n + \lambda_n^2 u_n = \theta_n \mu_n u_{n-1}$ and $l^* v_n + \bar{\lambda}_n^2 v_n = \theta_{n+1} \bar{\mu}_n v_{n+1}$ hold almost everywhere in G; here, $\theta_n = 0$ or 1 (in the latter case, $\lambda_n = \lambda_{n-1}$), $\theta_1 = 0$, and $(u_n, v_k) = \delta_{nk}$ for $n, k \in \mathbf{N}$. The numbers μ_n are chosen depending on whether spectral problem 1 or 2 is considered: $\mu_n = 1$ (problem 1) or $\mu_n = \omega\lambda_n$ for $|\lambda_n| \ge 1$ and $\mu_n = \omega = \mathrm{const} \ne 0$ for $|\lambda_n| < 1$ (problem 2). The coefficient μ_n influences only the normalization of the associated functions.

Note that, if the operator L is essentially nonselfadjoint, i.e., the total number of associated functions in its system of root functions is infinite, then the Riesz basis is made up of only the root functions solving the spectral problem 2.

For an arbitrary fixed number $\gamma \ge 0$, we introduce the spectral set $\Pi_\gamma = \{\lambda \in \mathbf{C}\colon \lambda = \rho + i\mu$, $\rho, \mu \in \mathbf{R}$, $\rho \ge 0$, $|\mu| \le \gamma\}$. Consider numbers $\{\lambda_n\}$ such that

$$\lambda_n \in \Pi_\gamma, \quad \sum_{\lambda \le |\lambda_n| \le \lambda + 1} 1 \le c_0 \quad \forall \lambda \ge 0, \tag{8}$$

where $c_0 > 0$ and $\gamma \ge 0$ are constants; i.e., the numbers λ_n lie in a strip near the real line, there are no finite accumulation points, and the number of associated functions corresponding to a single eigenvalue is uniformly bounded.

Theorem 1. *Let the coefficients $p_k(x)$ satisfy conditions* (3), *the system of numbers $\{\lambda_n\}$ obey conditions* (8), *conditions* (6) *and* (7) *hold for the functions $v_k(x)$, $v_{1j}^{\mathrm{c}}(x) \in W_2^2(G)$, $j = 1, 2$, and let the systems $\{u_n\}$ and $\{v_n\}$ be both complete in H. Then each of the systems $\{u_n\}$ and $\{v_n\}$ is a Riesz basis in H if and only if they are almost normalized in H. Each of the systems $\{u_n\}$ and $\{v_n\}$ is an unconditional basis in H if and only if*

$$\|u_n\|_2 \cdot \|v_n\|_2 \le c, \quad n = 1, 2, \ldots, \tag{9}$$

where $c > 0$ is a constant and $\|\cdot\|_2 = \|\cdot\|_{L^2(G)}$. Each of the systems $\{u_n \cdot \|u_n\|_2^{-1}\}$ and $\{v_n \cdot \|v_n\|_2^{-1}\}$ is a Riesz basis in H if and only if condition (9) *holds.*

Remark 1. For an operator L with regular boundary conditions and the coefficient $p_1 \equiv 0$ in G, for some numbers $\gamma \ge 0$ and $c_0 > 0$, conditions (8) are satisfied and the systems $\{u_n\}$ and $\{v_n\}$ are both complete in H, which follows from [6]. Thus, in this case, only conditions (6) and (7) on the kernels of the vectors and the almost normalization condition for the systems or condition (9) have to be checked in Theorem 1 (for this purpose, it is sufficient to know the leading terms of the asymptotics of the functions u_n, v_n).

Examples. In example (1), (2), all the conditions of Theorem 1 are satisfied, except for the almost normalization of $\{v_n\}$ and condition (9) [15]. Neither $\{u_n\}$ nor $\{v_n\}$ is a basis in H. Let us drop the integral term from (2). Then, with a proper choice of associated functions (so that condition (9) holds), all the conditions of Theorem 1 are satisfied. The systems $\{u_n\}$ and $\{v_n\}$ form a Riesz basis in H. In each of the examples, the operator L is essentially nonself-adjoint.

Uniform convergence of biorthogonal expansions. We continue to study the above-introduced systems $\{u_n\}$, $\{v_n\}$. For an arbitrary function $f(x) \in H$, we consider two partial sums of its biorthogonal expansions

$$\sigma_\lambda(x,f) = \sum_{|\lambda_n|\le\lambda} (f, v_n) u_n(x),$$

$$\tilde{\sigma}_\lambda(x,f) = \sum_{|\lambda_n|\le\lambda} (f, u_n) v_n(x), \qquad \forall\lambda > 0, \qquad x \in \bar{G}.$$

Theorem 2. *Let conditions* (3), (6), (7) (*condition* (7) *for* $k = 1$), (8), *and* (9) *be satisfied*, *and let* $v_{1j}^{c}(x) \in W_2^2(G)$, $j = 1, 2$. *Suppose that* $f(x) \in W_2^1(G)$ *and*

$$\int_0^1 f(x)\, dv_2^{s}(x) + f(1)(v_1^{c})'(1) - f(0) v_1^{c}(0) = 0. \quad (10)$$

Then the expansion $\sigma_\lambda(x,f)$ *converges absolutely and uniformly on* $\bar{G}$ *as* $\lambda \to +\infty$. *If*, *additionally*, *the systems* $\{u_n\}$ *and* $\{v_n\}$ *are assumed to be complete in* H, *then the expansion* $\sigma_\lambda(x,f)$ *converges to* $f(x)$ *uniformly on the interval* $\bar{G}$ *and*

$$f(x) - \sigma_\lambda(x,f) = o((\sqrt{\lambda})^{-1}), \qquad \lambda \to +\infty, \qquad (11)$$

holds uniformly with respect to $x \in \bar{G}$. *The remainder of the series of the moduli* $|f_n u_n(x)|$ *has the same estimate* (11) *uniformly with respect to* $x \in \bar{G}$.

Let us compare the partial sum $\sigma_\lambda(x,f)$ of the expansion of $f(x)$ in the root functions of the operator L and the partial sum $S_\lambda(x,f)$ of the trigonometric Fourier series viewed as an orthogonal expansion of $f(x)$ in terms of the eigenfunctions of the operator L_0: $l_0 u(x) = u''(x)$, $x \in G$, $u^{(j)}(0) = u^{(j)}(1)$, $j = 0, 1$, $u(x) \in C^2(\bar{G})$. Let $p \in [1,\infty)$ be an arbitrary fixed number.

Theorem 3. *Let conditions* (3) *and* (7)–(9) *be satisfied*; $v_{1j}^{c}(x) \in W_2^2(G)$, $j = 1, 2$; *the system* $\{u_n\}$ *of root functions of the operator L be complete and minimal in* H; $f(x) \in W_2^1(G)$; *and condition* (10) *be fulfilled. Then, for all sufficiently large numbers* λ *and any interval* $K \subset G$, *the rate of equiconvergence of the expansions* $\sigma_\lambda(x,f)$ *and* $S_\lambda(x,f)$ *satisfies the estimate*

$$\|\sigma_\lambda(x,f) - S_\lambda(x,f)\|_{L^p(K)} \le \frac{c}{\eta}\left[\frac{\ln\lambda}{\lambda} + \|p_1\|\frac{\ln^2\lambda}{\lambda}\right],$$

where $\eta = \rho(K, \partial G) > 0$ *is the distance to the boundary of the interval G. A similar estimate holds on the entire interval* $\bar{G}$:

$$\|\sigma_\lambda(x,f) - S_\lambda(x,f)\|_p \le \frac{c}{\lambda^{1-\frac{1}{\delta}}},$$

$$\delta = \min(2,q), \qquad q = \frac{p}{p-1}.$$

Corollary to Theorem 3. *If the conditions of Theorem 3 hold*, $p \ge 2$, *and* $f(x)$ *is a function of bounded variation on* $\bar{G}$, *then* $\|f(x) - \sigma_\lambda(x,f)\|_p \le c\lambda^{-\frac{1}{p}}$ *as* $\lambda \to +\infty$, *which coincides with a sharp convergence rate estimate for the trigonometric Fourier series of* $f(x)$.

Consider the "adjoint" expansion $\tilde{\sigma}_\lambda(x,f)$. The convergence of such expansions is required, for example, in the Il'in method for proving equiconvergence theorems.

Theorem 4. *Let conditions* (3) *and* (7)–(9) *be satisfied*; $v_{1j}^{c}(x) \in W_2^2(G)$, $j = 1, 2$; *and* $f(x) \in W_2^1(G)$ *with* $f(0) = f(1) = 0$. *Then the expansion* $\tilde{\sigma}_\lambda(x,f)$ *converges absolutely and uniformly on* $\bar{G}$ *as* $\lambda \to +\infty$.

If $\{u_n\}$ *and* $\{v_n\}$ *are complete systems in* H, *then the expansion* $\tilde{\sigma}_\lambda(x,f)$ *converges to* $f(x)$ *almost everywhere in* G; *moreover*, *almost everywhere in* G,

$$f(x) - \tilde{\sigma}_\lambda(x,f) = o((\sqrt{\lambda})^{-1}), \qquad \lambda \to +\infty$$

(*if the functions* v_n *are continuous on* $\bar{G}$, *then*, *in both cases*, *the convergence is uniform on the interval* $\bar{G}$).

Examples. In example (1), (2), since condition (9) is violated, the expansion $\sigma_\lambda(x,f)$ of $f(x) \in W_2^1(G)$ with $f(0) = 0$ and $f(1) + 2f\left(\frac{1}{2}\right) = 0$ (condition (10)) and the expansion $\tilde{\sigma}_\lambda(x,f)$ for $f(0) = f(1) = 0$ cannot be guaranteed to converge uniformly on $\bar{G}$. If the integral term is dropped from (2), then, with a proper choice of associated functions for any function $f(x) \in W_2^1(G)$ with $f(0) = 0$ and $f(1) + 2f\left(\frac{1}{2}\right) = 0$, the expansion

$\sigma_\lambda(x, f)$ converges to $f(x)$ uniformly on $\bar{G}$ and $f(x) - \sigma_\lambda(x, f) = o((\sqrt{\lambda})^{-1})$, $\lambda \to +\infty$, uniformly on $\bar{G}$. For $f(x) \in W_2^1(G)$ with $f(0) = f(1) = 0$, the expansion $\tilde{\sigma}_\lambda(x, f)$ converges to $f(x)$ uniformly on $\bar{G}$ (since the functions $v_n(x)$ are continuous) and the convergence rate satisfies the estimate $f(x) - \tilde{\sigma}_\lambda(x, f) = o((\sqrt{\lambda})^{-1})$, $\lambda \to +\infty$, uniformly on $\bar{G}$.

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