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The streamline flow around a permeable plate in a plane-parallel channel☆

P.R. Andronov, S.V. Guvernyuk

Lomonosov Moscow State University, Institute of Mechanics, Moscow, Russia

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ABSTRACT

An analytical solution of the problem of a symmetric streamline flow of an ideal incompressible fluid around a permeable plate in a plane-parallel channel is constructed. The boundary conditions on the plate correspond to the linear Darcy law and to the condition of the controlling action of the pore structure. The effect of the production of a distributed vorticity when a continuous medium flows through a permeable boundary is taken into account. An exact solution is obtained in a form containing a Schwarz integral. The relation between the resistance of the plate and its relative size and porosity is investigated. The result is used to construct a theory of combined permeability. A relation between the hydraulic loss coefficient and the physical parameters of the combined permeability containing porous and perforated elements is obtained in an explicit form.

The difficulty in studying flows around bodies containing components made of different gas-permeable materials (perforated or porous plates, fabrics, meshes, etc.) is due to the fact that a certain large-scale gas flow has to be described taking account of the effect of a large number of small-scale rigid bodies, that is, the parts of the structure of the permeable wall. A simplified approach is known[1–3] in which the “main” large-scale flow on both sides of a permeable wall is assumed to be ideal and the permeable wall together with its surrounding boundary layer of local dissipative flow is replaced by a surface of discontinuity. Here, although we are dealing with the boundaries of a discrete structure, in the mathematical model it is assumed that there is a certain seepage rate at each point of the permeable surface and that the boundary values of the main flow parameters are related to one another by special compatibility conditions on the discontinuity.[3] We distinguish between permeability of the first kind, to which perforated plates and shells correspond, where their local hydraulic resistance is mainly associated with vortex losses during the disruption and subsequent re-mixing of local flows formed when the medium passes through the aperture of a perforation[4] and of the second kind when the local hydraulic resistance is determined by internal friction in the seepage of the medium through a porous layer or a fine mesh fabric.[3,4]

In addition to the types of permeability of the first and second kinds mentioned above, a third type is possible, that we shall call “combined permeability”, when a porous boundary of the second kind has an additional perforation of the first kind. The canopy of a parachute made of an air-permeable fabric when it additionally has a distributed structural perforation with openings or slits that are large relative to the thickness of the fabric serves as a practical example of combined permeability. No validated system of boundary conditions is known for this type of permeability at the present time.

Obviously, the problem of streamline flow in a plane-parallel channel around a permeable plate with permeability of the second kind, since the gap between the edge of the plate and the channel wall can be considered as structural permeability of the first kind, is a useful local model of combined permeability. Data on the relation between the overall resistance of the plate and its relative width and coefficient of permeability make it possible to obtain a law for the flow of the medium in terms of the combined permeability. These facts were the motivation for studying the problem of a streamline flow around a permeable plate in a channel.

A classical solution of the problem of the streamline flow of an ideal fluid around an impermeable plate located between the parallel impermeable channel walls is known.[5] A special case is the unbounded streamline flow around a plate with the flow separation, according to the Kirchhoff scheme. However, the corresponding classical methods for obtaining an exact analytical solution of this problem do not

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E-mail address: guv@imec.msu.ru (S.V. Guvernyuk).

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have extensions to the case of a permeable plate. The difficulty lies in the fact that a permeable boundary, unlike a continuous boundary, is not a streamline.

An analytical method of constructing steady flows in a channel with a permeable screen that does not completely block the cross-section of the channel has been proposed.[6] However, the corresponding flow [6] with regions of non-zero vorticity is not an exact solution of the problem of the flow around a permeable screen on account of the fact that not all the boundary conditions on the surface of discontinuity are satisfied, although, in several special cases, it is in satisfactory agreement with the results of physical experiments. The issue has been discussed in Ref. 3.

An exact analytical solution of the problem of the unbounded streamline flow of an ideal incompressible fluid around a permeable plate with the flow separation was obtained for the first time by Buchin.[7] The statement of the problem assumed a permeability of the second kind that made it possible to justify a scheme with an isobaric vortex jet in the bottom region behind the permeable plate and to formulate a closed boundary value problem for determining the irrotational part of the flow. A similar boundary value problem for finding the irrotational component of the flow had been studied earlier [8,9] but linearization of the degree of permeability of the screen with respect to a small parameter was used. The method proposed by Buchin [7] for constructing an exact solution without linearization can be called the "method of a derivative of analytical function" and it enables the exact solution to be found for any coefficient of permeability of a plate. The result is obtained in a parametric form containing a Schwarz integral. This method has been used [10,11] to solve complex problems of the interaction of a plane jet of an ideal fluid with an unbounded permeable screen. In some simpler cases, the idea of introducing an isobaric whirling jet with straight streamlines into the scheme for the interaction of the fluid with the permeable boundaries enables a complete solution of the problem to be constructed in terms of elementary functions[12].

1. Statement of the problem

We shall consider a plane steady flow of an ideal incompressible fluid around a permeable plate arranged symmetrically across an infinite plane-parallel channel (Fig. 1). The flow of the fluid, of constant density ρ, is described by a system of continuity and Euler equations. The permeable plate is simulated by a surface of hydrodynamic discontinuity with boundary conditions expressing the law of conservation of mass, the linear Darcy law and the condition for the complete loss of tangential momentum of the seeping medium:[3]

$$u_+ = u_-, \quad p_+ - p_- = -ku_-, \quad \upsilon_+ = 0 \tag{1.1}$$

Here u and υ are the normal and tangential components of the velocity vector $\mathbf{V}$ at the discontinuity, p is the pressure, the minus and plus subscripts indicate the windward and the leeward sides of the discontinuity and k is a generalized physical parameter, proportional to the dynamic viscosity of the medium μ and inversely proportional to some effective linear dimension b of the porosity structure of the plate: $k = \mu/b$ (the relation between b and the flowrate characteristic of the air-permeability of real fabrics and other porous materials is considered below in Section 4).

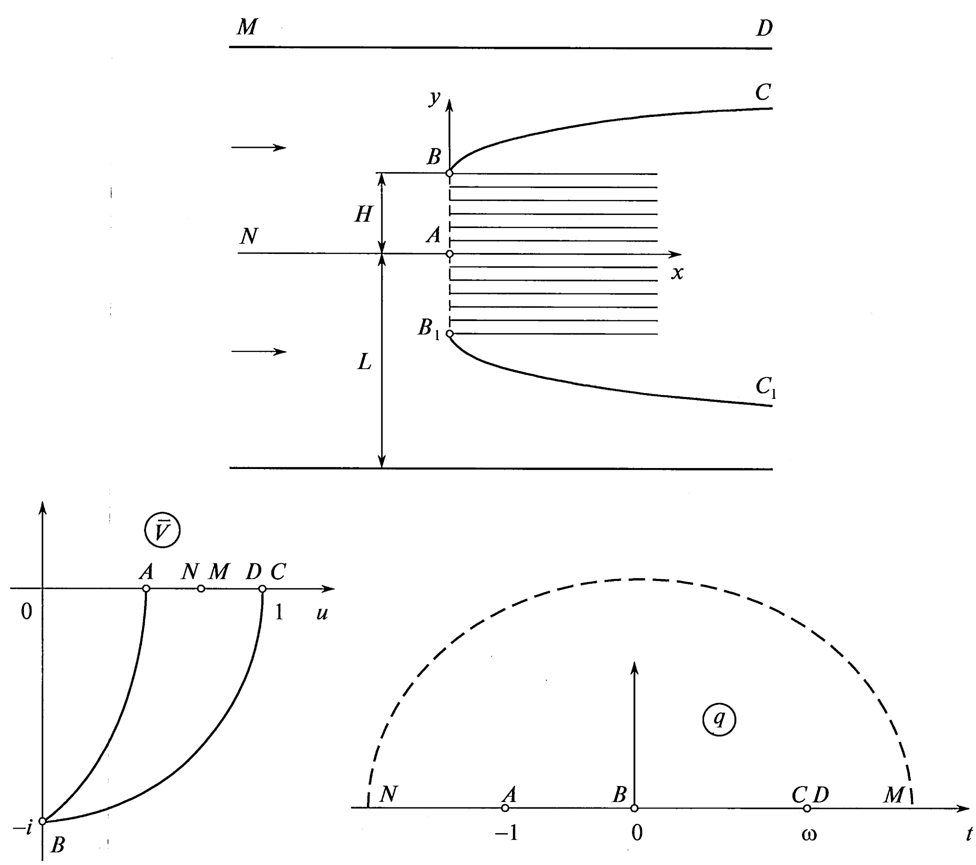

Fig. 1.

The unperturbed flow velocity in the channel at infinity in front of the plate is constant and equal to V_∞. The bottom region of the flow behind the plate extends up to infinity in the channel, the pressure p_0 in it is constant (without loss of generality, we shall assume that $p_0 = 0$), and the structure of the velocity field is the whirling stream of fluid that has seeped through with straight streamlines that spreads along the stagnation zone. The bottom flow is separated from the main flow in the channel by the discontinuities BC and B_1C_1 (Fig. 1). Outside the bottom region, all the streamlines come from infinity without crossing the permeable plate and the flow is therefore irrotational there and a first integral of the equations of motion (a Bernoulli integral)

$$\rho V^2 + 2p = \rho V_0^2 \tag{1.2}$$

exists, where V_0 is the constant value of the flow velocity in the streams BC and B_1C_1.

The relative width of the plate $h = H/L$ and the local Reynolds number for the seeping fluid

$$R = \rho V_\infty/k = \rho b V_\infty/\mu \tag{1.3}$$

are the dimensionless governing parameters.

We introduce a Cartesian system of coordinates x and y with origin at the centre of the plate (at point A in the upper part of Fig. 1) and direct the x axis in the same direction as the fluid flow. Taking account of the symmetry about the x axis, we shall subsequently only consider the flow in the upper part of the channel $0 < y < L$. The boundary conditions on the lines $y = 0$ and $y = L$ for $\mathbf{V} = \{u, \upsilon\}$ are the impermeability conditions $\upsilon = 0$. The kinematic boundary condition for the windward side of the plate

$$x = -0, \quad 0 < y < H\colon \rho(u^2 + \upsilon^2) + 2ku = \rho V_0^2 \tag{1.4}$$

is obtained from conditions (1.1) and (1.2) taking account of the equality $p_+ = p_0 = 0$.

Hence, the problem of the fluid flow in a channel with regions of non-zero vorticity is broken up: the irrotational part of the velocity field can be constructed regardless of the flow around the plate as a whole. Using the results of the solution of this basic problem and boundary conditions (1.1), the two parameters

$$\delta(R, h) = \frac{L - y_C}{L - H}, \quad c_x(R, h) = \frac{2}{\rho H V_\infty^2}\int_0^H p_- dy \tag{1.5}$$

that are important in practical applications, are determined as well as the shear profile of the velocity $u(x, y) = u_+(y)$ in the vortex part of the bottom region $x > 0$, $0 < y < H$ behind the plate. Here, δ is the coefficient of contraction of the fluid streams for the free outflow through a slit with permeable edges into a flooded space and c_x is the aerodynamic drag coefficient of the permeable plate in the channel.

We take the half-width of the channel L and the velocity V_0 on the free streamline as the characteristic scales of length and velocity and change to the dimensionless variables

$$x_* = \frac{x}{L}, \quad y_* = \frac{y}{L}, \quad u_* = \frac{u}{V_0}, \quad \upsilon_* = \frac{\upsilon}{V_0}, \quad w = \frac{V_\infty}{V_0}, \quad \varepsilon = \frac{k}{\rho V_0}$$

(henceforth we shall omit the asterisk in the notation for dimensionless variables). We will use the following notation: $z = x + iy$ is the complex coordinate of a point in the physical plane and $\bar{V} = u - i\upsilon$ is the value of the complex-conjugate fluid velocity at a point z. The following boundary conditions:

$$\text{AB: } u^2 + \upsilon^2 + 2\varepsilon u = 1; \quad \text{BC: } u^2 + \upsilon^2 = 1; \quad \text{DM, NA: } \upsilon = 0 \tag{1.6}$$

are used to find the analytical function $\bar{V}(z)$ in the region bounded by the contour $ABCDMNA$ (the left-hand lower part of Fig. 1) when $0 < h < 1$, $0 < \varepsilon < \infty$.

Here, the domain of the values of $\bar{V}$ is known and it is one half of the lune formed by the arcs of the two circles (1.6) with the following correspondence of the points:

$$\bar{V}_\text{A} = u_\text{A} = \frac{1}{\varepsilon + \sqrt{1 + \varepsilon^2}}, \quad \bar{V}_\text{B} = -i, \quad \bar{V}_\text{C, D} = 1, \quad \bar{V}_\text{N, M} = w \tag{1.7}$$

The value of the real constant w is bounded by the interval $u_A < w < 1$, but it is not known in advance.

2. Construction of the solution

We shall seek the mapping $z \leftrightarrow \bar{V}$ in the parametric form

$$z = z(q), \quad \bar{V} = \bar{V}(q) \tag{2.1}$$

where $q = t + i\theta$ is a parametric variable which is such that the upper half-plane $\operatorname{Im} q > 0$ with the following correspondence of the points (the left-hand lower part of Fig. 1):

$$q_\text{A} = -1, \quad q_\text{B} = 0, \quad q_\text{C} = q_\text{D} = \omega, \quad q_\text{M} = +\infty, \quad q_\text{N} = -\infty \tag{2.2}$$

corresponds to the flow domain bounded by the contour $ABCDMNA$ in the q plane.

Conformal mapping of the upper half-plane of q onto one half of the circular lune with the proper correspondence of the points (1.7), (2.2) gives the formula

$$\bar{V}(q) = [\xi^{-}(q) - i\xi^{+}(q)]/[\xi^{+}(q) - i\xi^{-}(q)] \tag{2.3}$$

where

$$\xi^{\pm}(q) = (\sqrt{\omega(1+q)} \pm \sqrt{\omega - q})^{\beta}, \quad 0 < \omega < \infty, \quad \beta = \pi^{-1}\operatorname{arctg}\varepsilon, \quad 0 < \beta < 1/2$$

From equality (2.3), when $q = t \to \infty$, we obtain the relation between the parameters w, ω and β in the form

$$w = [1 - \operatorname{tg}(\beta \operatorname{arctg}\omega^{-1/2})]/[1 + \operatorname{tg}(\beta \operatorname{arctg}\omega^{-1/2})] \tag{2.4}$$

With the aim of finding the conformal mapping $z = z(q)$, we consider the interval BC on the real axis of the parametric plane q. It corresponds to the free boundary of the stream passing around the upper edge of the plate (the upper part of Fig. 1). At each point $0 < t < \omega$ of this interval, formula (2.3) gives the value of the conjugate velocity $\bar{V}(t)$ and, in particular, enables us to find

$$-\arg\bar{V}(t) = \operatorname{arctg}(\upsilon/u)$$

which, in turn, determines the direction of the velocity vector $\mathbf{V}$ in the stream BC in the physical plane z. The angle of inclination of the tangent to the free streamline BC at any point $z(t)$ of the physical plane is thereby known. At the same time, the change in the angle of inclination of the boundary BC for the conformal mapping $z(q)$ has the mathematical meaning of the argument of the derivative dz/dq. Hence, for the part of the boundary BC, we have

$$\arg(dz/dq) = -\arg\bar{V}(t), \quad 0 < t < \omega$$

The function $\arg\left(dz/dq\right)$ is also known on the remaining parts of the real axis t since these parts correspond to known rectilinear segments of the boundary of the irrotational flow region in the physical plane z. Consequently, the imaginary part of the function $F(q) = \ln\left(dz/dq\right)$:

$$\operatorname{Im}F(q) = \begin{cases} 0, & -\infty < t < -1 \\ \pi/2, & -1 < t < 0 \\ \lambda(t), & 0 < t < \omega; \quad \lambda(t) = -\arg\bar{V}(t) = \operatorname{arctg}\dfrac{\xi^{+}(t)^2 - \xi^{-}(t)^2}{2t^{\beta}(1+w)^{\beta}} \\ \pi, & \omega < t < +\infty \end{cases}$$

is known everywhere on the real axis t of the parametric plane q:

It is convenient to seek the function F in the form of the sum $F(q) = F_1(q) + F_2(q)$ with the boundary conditions

$$\operatorname{Im}F_1 = 0, \quad \operatorname{Im}F_2 = 0, \quad -\infty < t < -1$$

$$\operatorname{Im}F_1 = \pi/2, \quad \operatorname{Im}F_2 = 0, \quad -1 < t < 0$$

$$\operatorname{Im}F_1 = 0, \quad \operatorname{Im}F_2 = \lambda(t), \quad 0 < t < \omega$$

$$\operatorname{Im}F_1 = \pi, \quad \operatorname{Im}F_2 = 0, \quad \omega < t < +\infty$$

The function

$$F_1(q) = i\pi + \frac{1}{2}\ln q - \ln(q - \omega) - \frac{1}{2}\ln(1+q)$$

can be determined at once and the function $F_2(q)$ can be recovered using the Schwarz integral[13] apart from an arbitrary real constant:

$$F_2(q) = \frac{1}{\pi}\int_0^{\omega}\frac{\lambda(t')}{t - q}dt + \text{const}$$

Carrying out involution and integration with respect to q, we find the required mapping

$$z(q) = \frac{1}{\pi}\int_{-1}^{q}\frac{Q^{1/2}}{(\omega - Q)(1+Q)^{1/2}}\exp\left(\frac{1}{\pi}\int_0^{\omega}\frac{\lambda(t)}{t - Q}dt\right)dQ \tag{2.5}$$

The coefficient in front of the integral with respect to Q is determined from the condition that the assumed normalization of the channel width $z(\omega + 0) = 1$ is satisfied, and, for this, it is necessary to integrate in the parametric plane q along the contour $ANMD$ that includes the semicircle NM of infinitely large radius (the right-hand lower part of Fig. 1). From equality (2.5), we obtain the expression for the width of the plate

$$h = \frac{1}{\pi}\int_{-1}^{0}\frac{(-\tau)^{1/2}}{(\omega-\tau)(\tau+1)^{1/2}}\exp\left(\frac{1}{\pi}\int_{0}^{\omega}\frac{\lambda(t)}{t-\tau}dt\right)d\tau \tag{2.6}$$

According to equality (1.1), the shear velocity profile $u(x, y) = u_+(y)$ on the leeward side of the plate AB in the domain $x > 0$, $|y| < h$ is given by the expression $u_-(y)$ for the longitudinal velocity on the windward side of the plate, Fig. 1. Separating the real part in equality (2.3) when $q = t$, $-1 \le t \le 0$, we obtain

$$u_- = \frac{2(-t)^{\beta}(1+\omega)^{\beta}\cos\pi\beta}{\eta^+(t)^2+\eta^-(t)^2+2(-t)^{\beta}(1+\omega)^{\beta}\sin\pi\beta}$$
$$\eta^{\pm}(t) = (\sqrt{\omega-t}\pm\sqrt{\omega(1+t)})^{\beta} \tag{2.7}$$

3. Analysis of the results

Relations (2.5) - (2.7) enable us to represent the overall flow rate through the permeable plate in the form

$$g = \int_0^h u_- dy = \frac{1}{\pi}\int_{-1}^{0}\frac{u_-(\tau)(-\tau)^{1/2}}{(\omega-\tau)(\tau+1)^{1/2}}\exp\left(\frac{1}{\pi}\int_{0}^{\omega}\frac{\lambda(t)}{t-\tau}dt\right)d\tau \tag{3.1}$$

By virtue of the fact that, when $\beta < 1/2$, the pressure drop on the plate is linearly related to the seepage rate u_-, the overall hydrodynamic load on the plate is equal to εg. Correspondingly, the drag coefficient (1.5) is calculated as

$$c_x = \frac{2g\,\mathrm{tg}\,\pi\beta}{hw^2}$$

The relative width $1 - y_C$ of the potential stream at infinity (the upper part of Fig. 1) determines the dimensionless flow rate in this stream. The balance relation $g + 1 - y_C = w$, together with expressions (2.4) and (3.1), provide a convenient method for calculating the contraction coefficient of the potential stream flowing out from the slit between the permeable plate and the channel wall

$$\delta = (1-y_C)/(1-h) = (w-g)/(1-h)$$

When $\beta \to 1/2$, we obtain

$$w \to 0,\quad h \to 1,\quad \delta \to \pi/(2+\pi)\ \text{when}\ \omega \to 0$$
$$w \to 1,\quad h \to 0,\quad \delta \to 1,\quad c_x \to 2\pi/(4+\pi)\ \text{when}\ \omega \to +\infty$$

which agrees with the classical solutions of problems of the contraction coefficient of a stream in the case of the outflow through a slit in an infinitely permeable plane and on the drag coefficient of an impermeable plane in an unbounded flow around which a flow occurs according to the Kirchhoff scheme with flow separation.

The special problem of the jet flow around an impermeable plate in a channel ($\beta = 1/2$, $0 < h < 1$) has an exact solution in terms of elementary functions.[5,14] Introducing the notation

$$f(w) = 1 - w - \frac{1-w^2}{\pi}\mathrm{arctg}\frac{2w}{1-w^2},\quad f_0(w) = \left(\frac{1}{2}+\frac{1+w}{\pi(1-w)}\mathrm{arctg}\frac{1-w^2}{2w}\right)^{-1} \tag{3.2}$$

this solution can be represented in the form

$$w = f^{-1}(h),\quad c_x = f_0(w)w^{-2},\quad \delta = w/(1-h)$$

It is convenient to use the functions (3.2) in constructing rough approximations of the solutions of the general problem for $\beta \in (0, 1/2)$, $h \in (0, 1)$. As a result, the analytical approximations

$$\delta(h,\beta) \approx 1 - 2\beta(1.4 - 0.8\beta)\left(1 - \frac{f_1(h)}{1-h}\right), \quad w(h,\beta) \approx \frac{1 + (2f_1(h) - 1)\operatorname{tg}\frac{\pi\beta}{2}}{1 + \operatorname{tg}\frac{\pi\beta}{2}}$$

$$c_{x0}(h,\beta) \approx \frac{2\sin\pi\beta}{1 + \sin\pi\beta} - (1 - f_2(h))(1 - \cos\pi\beta - 0.266\beta(1 - 2\beta)) \tag{3.3}$$

are obtained where, when notation (3.2) is taken into account,

$$f_1(h) = f^{-1}(h) \approx 1 - \frac{2}{\pi}\arccos(1-h) - \frac{4 + 2\pi - \pi^2}{2\pi + \pi^2}(1-h)h^{2/5}$$

$$f_2(h) = f_0(f_1(h))$$

The exact relations $w(h,\beta)$, $\delta(h,\beta)$, $c_{x0} = w^2 c_x(h,\beta)$, determined by relations (2.4) and (3.1) when $\beta \in (0, 1/2)$, $h \in (0, 1)$, are constructed in Fig. 2. Approximate relations (3.3) are not shown as they are practically indistinguishable from the graphs of the corresponding exact solutions. Here, in the limit cases when approaching the domains of definition with respect to h and β, approximations (3.3) are asymptotically exact.

The relation between the drag coefficient c_{x0} and the degree of porosity of the plate β is non-monotonic for all h, $0 < h < 1$, and a maximum value is attained within the interval $0 < \beta < 1/2$ (Fig. 2, bottom right). This property was noted for the first time by Buchin [7] in the special case of the unbounded flow around a porous plate (in this case $c_{x0} = c_x$).

Explicit approximate expressions for the drag coefficient of a porous plate in an unbounded flow and the contraction coefficient of a potential stream flowing out through a slit in an infinite porous plane can be obtained by taking to the limit as $h \to 0$ and $h \to 1$ in approximations (3.3) in the form

$$c_x = \frac{c_{x0}(0,\beta)}{w(0,\beta)^2} \approx \frac{2\sin\pi\beta}{1 + \sin\pi\beta} - \frac{(4-\pi)(1-\cos\pi\beta)}{4+\pi} + 0.032\beta(1-2\beta)$$

$$\delta = \delta(1,\beta) \approx 1 - \frac{4\beta(1.4 - 0.8\beta)}{2+\pi} \tag{3.4}$$

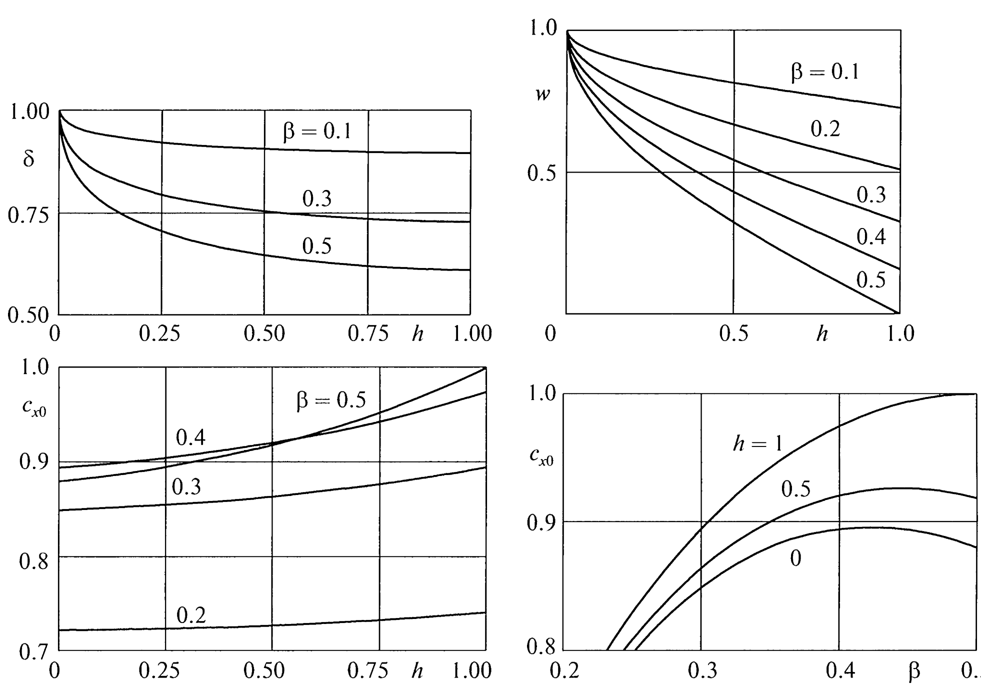

Fig. 2.

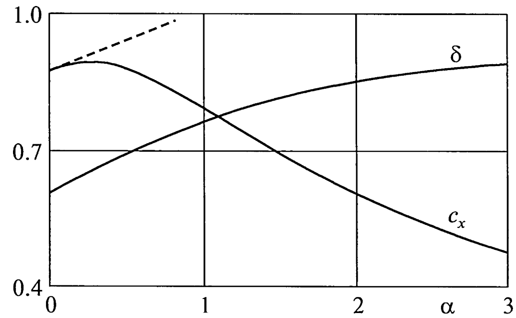

Fig. 3.

Relations (3.4), with scaling by the porosity parameter $\alpha = \varepsilon^{-1} = \text{ctg}\ \pi\beta$, are shown in Fig. 3. In the case of small α, we have

$$c_x = \frac{2\pi}{4+\pi} + 0.130\alpha + O(\alpha^2)$$

$$\delta = \frac{\pi}{2+\pi} + 0.10\alpha + O(\alpha^2) \tag{3.5}$$

Linear expression (3.5) for the drag coefficient of the porous plate corresponds to the solution [8] obtained in the linearized formulation of the problem for small α. The paradoxical conclusion [8,9] concerning the increase in the drag coefficient of the plate when the degree of permeability increases (the dashes in Fig. 3) followed from it. The exact solution of the corresponding problem in the non-linear formulation has been found [7] (it is described to a high degree of accuracy by approximation (3.4)). The increase in the drag coefficient over a range of small α is explained by the joint action of opposing factors: the pressure drop at the centre of the plate decreases as the permeability increases but, simultaneously, the mean value of this drop over the span of the plate increases due to smoothing of the velocity profile $u_-(y)$.

The velocity profile in the isobaric vortex stream behind the plate is determined using boundary conditions (1.1) and (2.7). For all $\beta \in \left(0, 1/2\right)$, $h \in (0, 1)$, it is convex, a maximum value $u_+(y)$ is attained on the axis of symmetry and a minimum value on the boundary of the stream:

$$\max(u_+) = u_+(0) = u_A(\beta), \quad \min(u_+) = u_+(h) = 0$$

Here, according to the first relation of (1.7) and taking account of the equality $\varepsilon = \text{tg}\pi\beta$, we have

$$u_A = \frac{\cos\pi\beta}{1+\sin\pi\beta}$$

Examples of a calculation using formulae (2.5) and (2.7) for different values of h and β are shown in Fig. 4; curves 1, 2, and 3 correspond to values of h equal to 0.07, 0.59 and 0.86.

The expression

$$u_+ \approx u_A(\beta)\left(1-(y/h)^{1/((1-h)\beta)}\right)^{\beta/2}$$

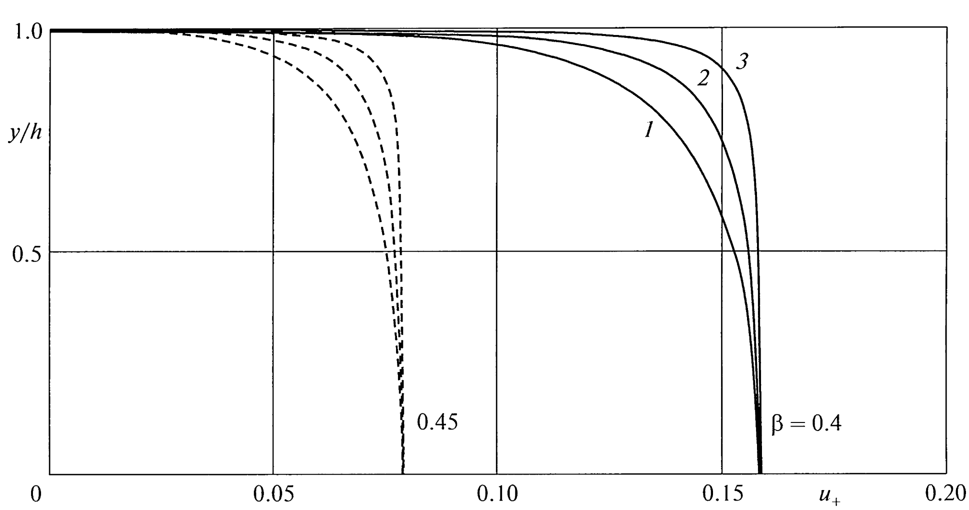

Fig. 4.

approximately describes the velocity profile over the whole range of variation in the parameters h and β. The vorticity level du/dy in the main body of the stream is close to zero and tends to infinity on the periphery.

All the calculations were carried out using the program application for symbolic calculations Wolfram Mathematica 8.0.

4. The theory of combined permeability

Consider the fluid motion through an area of a uniform porous material when there are additional structural perforations in it. This combined permeability is characterized by two parameters: the specific air-permeability B_0 of the porous material proper and the relative area σ of the open apertures of the structural perforation per unit area of this material. In technology, the parameter B_0 is customarily defined as the number of litres of air under normal conditions flowing per second through a square metre of a fabric or other porous material for a pressure drop $\Delta_0 = 49$ Pa. In the practice of parachute construction, the air-permeability B_0 can vary over a range from 0 to 2000 l/(m^{2}s) [15]. The corresponding nominal rate of permeation is obviously $u_0 = 10^{-3}B_0$ m/s. Substitution of these values of the parameters into equality (1.1) gives the relation $\Delta_0 = ku_0 = \mu_0 b^{-1} u_0$, in which μ_0 is the dynamic coefficient of viscosity of air under normal conditions and b is the effective linear size of the porous structure of the material. Hence,

$$b = u_0 \mu_0 \Delta_0^{-1} \tag{4.1}$$

We will assume that the local jet stream in the neighbourhood of the apertures of the perforation is similar to the flow considered above past a porous plate in a channel with the parameters $h = 1 - \sigma$, $u_1 = V_\infty$, Fig. 5. We make the assumption that is usual in permeability theory [1–4] that, downstream beyond the permeable boundary, the velocity profile is completely smoothed out as a result of the development of instabilities in the layers of mixing and the dissipation of local inhomogeneities. The arbitrary sections 1 and 2 shown in Fig. 5 bound a layer of locally non-uniform boundary layer flow in the neighbourhood of unit area of the material permeable boundary. If a layer 1 – 2 is interpreted as a surface of hydrodynamic discontinuity, the laws of conservation of mass $\rho u_2 = \rho u_1$ and of the change in the momentum $\rho u_1^2 = p_1 - X = \rho u_2^2 + p_2$ of a moving incompressible fluid give the compatibility conditions on the discontinuity in the form

$$u_2 = u_1, \quad p_1 - p_2 = X = \rho u_1^2 \zeta / 2$$

where X is the specific hydrodynamic force acting per unit area of the combined permeable boundary and ζ is the hydraulic loss coefficient that depends on the degree of perforation $\sigma = 1 - h$ and the local Reynolds number $R = \rho_1 b u_1 / \mu_1$ which, taking account of relations (1.3) and (4.1), can be represented in the form

$$R = UR_0; \quad U = \frac{u_1}{u_\Delta}, \quad u_\Delta = \sqrt{\frac{\Delta_0}{\rho_1}}, \quad R_0 = \frac{u_0 \mu_0}{u_\Delta \mu_1} = \frac{\mu_0 10^{-3} B_0}{\mu_1 u_\Delta} \tag{4.2}$$

For air, when $\mu_1 = \mu_0$ and $\rho_1 = 1.23$ kg/m^3, we have $u_\Delta = 6.3$ m/s and the correspondence (4.2) between the air-permeability of the material B_0 and the parameter R_0 is shown below in Table 1.

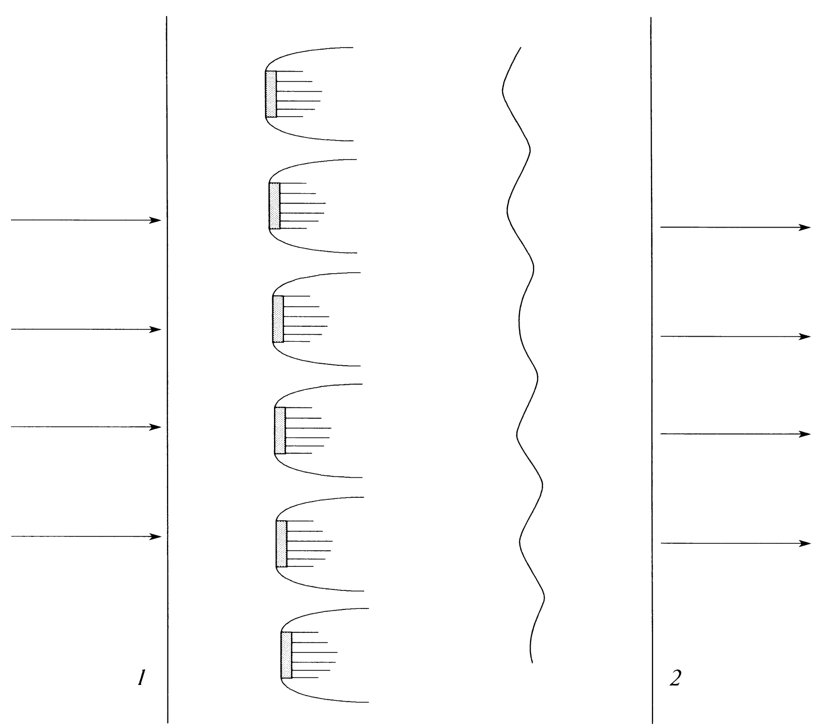

Fig. 5.

Table 1

$B_0, l/\left(m^2 \cdot s\right)$	0	100	200	400	800	1600
R_0	0	0.0158	0.0317	0.0634	0.1267	0.2535

Using the solution of the problem of the flow around a porous plate in a channel obtained in this paper, we have

$$\zeta = hc_{x0}w^{-2}, \quad R = w\operatorname{ctg}\pi\beta, \quad h = 1-\sigma, \quad 0<\beta<1/2 \tag{4.3}$$

where $w(h, \beta)$ and $c_{x0}\left(x, \beta\right)$ are known functions (3.3).

In the special case of an impermeable perforated material

$$\beta \to 1/2, \quad 0<\sigma<1$$

and, from relations (4.3), we obtain that $R = 0$ and

$$\zeta = \zeta_0(\sigma) = \left(\frac{1}{\sigma\delta_0}-1\right)^2, \quad \delta_0 = \frac{1-2\pi^{-1}\arccos\sigma}{\sigma} - \frac{4+2\pi-\pi^2}{2\pi+\pi^2}(1-\sigma)^{2/5} \tag{4.4}$$

Comparison of this result with Idel'chik's well-known empirical relation

$$\zeta = \left(\frac{0.707(1-\sigma)^{0.375}+1-\sigma}{\sigma}\right)^2$$

recommended for designing of diaphragms and perforated membranes with sharp edge of the apertures of arbitrarily shaped perforations, shows good agreement with the experimental data.[4]

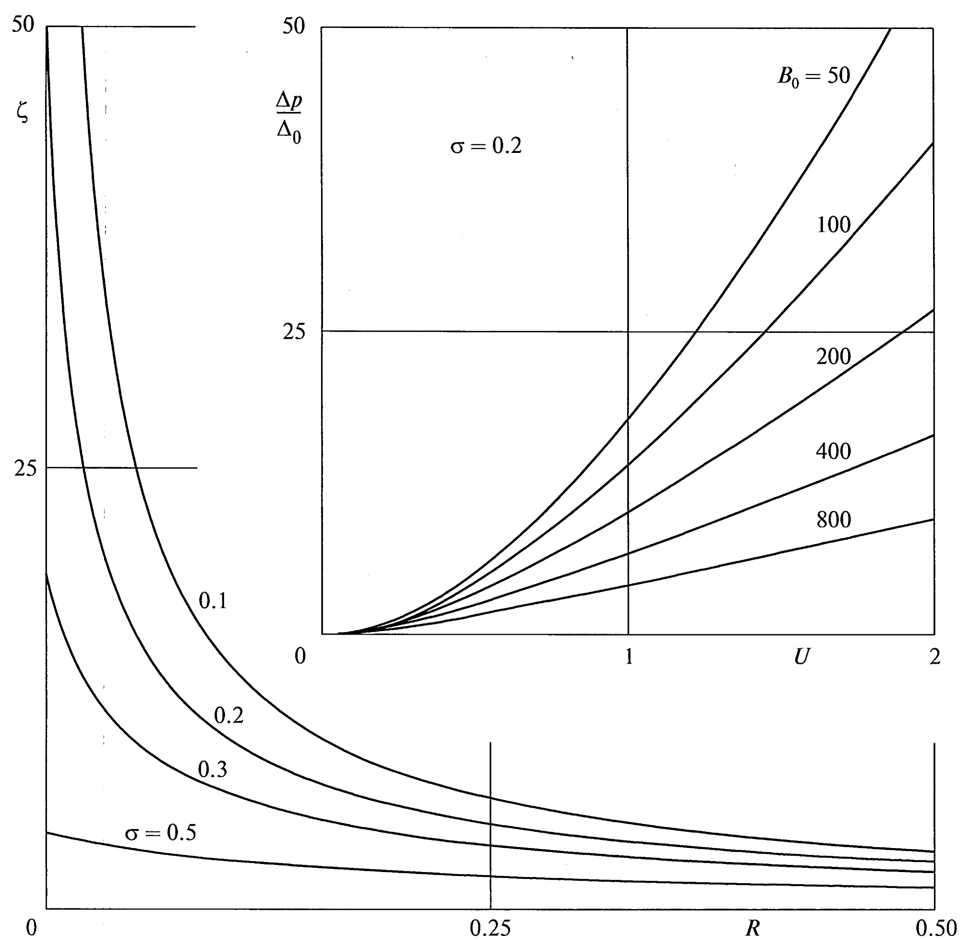

Fig. 6.

In the general case when $\beta \in (0, 1/2)$, $\sigma \in (0, 1)$, the parameter β can be eliminated using expressions (3.3) and (4.3) and an explicit expression can be obtained for the required hydraulic drag coefficient when a medium flows through a boundary with the combined type of permeability in the form

$$\zeta = \zeta(\sigma, R) = \frac{h(\Omega^2 - 1)^2 c_{x0}(h, 2\pi^{-1}\operatorname{arctg}\Omega)}{4R^2\Omega^2} \tag{4.5}$$

where

$$\Omega = \Omega(\sigma, R) = \frac{T - R - 1 + \sqrt{2R + (T-R)^2}}{2T - 1}, \quad T = f_1(h), \quad h = 1 - \sigma$$

The relations between ζ and R are shown in the lower left-hand part of Fig. 6 for different values of σ. For practical use, it is best to represent equality (4.5) in the form $p_1 - p_2 = \Delta p(U, B_0, \sigma)$, that gives the direct connection between the pressure drop Δp and the relative velocity of the passage of the medium U as a function of the two physical constants of the combined permeability: B_0 and σ. By relations (4.2) – (4.5), we have

$$\frac{\Delta p}{\Delta_0} = \frac{(1-\sigma)(\Omega^2 - 1)^2 c_{x0}(1-\sigma, 2\pi^{-1}\operatorname{arctg}\Omega)}{8R_0^2\Omega^2}; \quad \Omega = \Omega(\sigma, UR_0) \tag{4.6}$$

An example of relation (4.6) for different values of β for a fixed perforation parameter $\sigma = 0.2$ under the conditions for which the relation between B_0 and R_0 shown above was obtained is constructed in the upper right-hand part of Fig. 6. The considerable effect of the specific air-permeability of the material B_0 on the rate of permeation of the medium through this material for a specified pressure drop is observed (it increases as the air permeability increases). For relatively small pressure drops, the relation $\Delta p(U)$ is close to quadratic, changing into a linear relation when U increases.

5. Conclusion

The exact analytical solution of the general problem of steady symmetric flow past a porous plate in a plane-parallel channel with flow separation obtained agrees with the known special solutions for the limit cases of streamline flow around an impermeable plate in a channel and a porous plate in an unbounded space. For small degrees of blockage of the channel, its drag coefficient depends non-monotonically on the degree of porosity, so that the drag of a porous plate can be greater than that of a continuous plate. This is explained by an increase in the mean pressure drop on a porous plate due to the smoothing out of the pressure drop profile over its span. The explicit expression obtained for the contraction coefficient of the potential flow through a slit with permeable edges is an extension of the known formula for the outflow through a slit in an impermeable wall. The velocity profile in the vortex stream, formed in the bottom region immediately behind the porous plate, is convex in all cases.

The vortex level in the central part of this stream is close to zero and tends to infinity on approaching its boundaries.

The results obtained are of practical importance from the point of view of supplements to the theory of combined permeability. A relation between the hydraulic loss coefficient and the physical parameters of the combined permeability has been found for the first time. In the limit case when the material has no porosity, formula (4.5) agrees to a high degree of accuracy with the known empirical relation.[4] The representation of formula (4.5) in the form (4.6) is convenient for use as boundary conditions on surfaces with the combined type of permeability, for example, in formulating problems of the flow past the canopy of a parachute made of a material with an air-permeability B_0 l/(m^{2}s) when there is an additional structural perforation in it with a relative area of the apertures σ.

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