Asymptotical approximations of substantially nonlinear oscillations of the pendulum with vibrating pivot

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Pendulum with vertically vibrating pivot (PVVP) is one of the most studied classical systems with parametric excitation. Despite the popularity of the PVVP, in the literature [1, 2, 3, 4] etc., the author could not found analytical solutions with higher than the first order approximation of nonlinear oscillations.

In dimensionless parameters the equation of PVVP can be expressed as

$$\ddot{\theta} + \beta\omega\dot{\theta} + \big(\omega^2 + \varepsilon\varphi(\tau)\big)\sin(\theta) = 0, \tag{1}$$

were θ is the angle of PVVP, ω is the natural frequency, the upper dot denotes differentiation with respect to time τ and function $\varphi(\tau)$ is zero mean, and 2π periodic.

We assume that relative amplitude of excitation ε and damping β are small of the same order $\varepsilon \sim \beta \ll 1$. Then system (1) can be referred to as close to *hamiltonian system*. To solve system (1) we will use averaging method [4, 5]. To do so we need to write (1) in the *standard form* of first order differential equations with small right-hand sides. It is shown in [5] that for such transformation one needs only first integrals of the *unperturbed system* $\ddot{\theta} + \omega^2\sin(\theta) = 0$. First integral is $\dot{\theta}^2 = 2\,\omega^2\left(\cos(\theta) - \cos(a)\right)$, where a is the amplitude of oscillations. We are to express oscillating angle θ via monotonicity increasing *fast phase* ψ along with new slow variable $k = \sin\left(\frac{a}{2}\right)$ as it is usually done in the process of solving the unperturbed equation, e. g. [6]: $\sin\left(\frac{\theta}{2}\right) = k\sin(\psi)$. We differentiate the expressions for new variables $k^2 = \sin^2\left(\frac{\theta}{2}\right) + \frac{\dot{\theta}^2}{4\,\omega^2}$, $\psi = \arctan\left(\frac{2\omega}{\dot{\theta}}\,\sin\left(\frac{\theta}{2}\right)\right)$ with respect to τ and substitute in there expressions for $\ddot{\theta}$, $\dot{\theta}$, and θ in terms of k and ψ

$$\dot{k} = -k\left(\frac{\varepsilon}{\omega}\,\varphi(\tau)\sin(\psi)\cos(\psi)\sqrt{1 - k^2\sin^2(\psi)} + \beta\omega\cos^2(\psi)\right) \tag{2}$$

$$\dot{\psi} = \left(\omega + \frac{\varepsilon}{\omega}\,\varphi(\tau)\sin^2(\psi)\right)\sqrt{1 - k^2\sin^2(\psi)} + \beta\omega\sin(\psi)\cos(\psi). \tag{3}$$

The unperturbed system $\dot{k} = 0$, $\dot{\psi} = \omega\sqrt{1 - k^2\sin^2(\psi)}$ has the two first integrals

$$\tau - \tau_0 = \frac{1}{\omega}\int_0^{\psi}\frac{d\eta}{\sqrt{1 - k^2\sin^2(\eta)}}, \quad k = const. \tag{4}$$

To study resonant dynamics we introduce resonance ratio $p:q$, where p and q are natural numbers, so when time τ increments by $2\pi q$ fast phase ψ increments by $2\pi p$. This resonance condition binds time τ and phase ψ by the relation $F(\psi,k) = K(k)\frac{2\omega p}{\pi q}(\tau-\tau_0)$, where $F(\psi,k)$ denotes the first kind elliptic integral in the right hand-side of (4), and $K(k) = F(\pi/2,k)$ the complete first kind elliptic integral. From this resonant condition we have $\tau = \tau_0 + \frac{\pi q F(\psi,k)}{2\omega p K(k)}$. In the perturbed case system (2)–(3) yields

$$\frac{dk}{d\psi} = -\frac{k\left(\frac{\varepsilon}{2\omega}\,\varphi(\tau(\psi,k))\sin(2\psi)\sqrt{1-k^2\sin^2(\psi)} + \beta\omega\cos^2(\psi)\right)}{\left(\omega + \frac{\varepsilon}{\omega}\,\varphi(\tau(\psi,k))\sin^2(\psi)\right)\sqrt{1-k^2\sin^2(\psi)} + \frac{\beta\omega}{2}\sin(2\psi)} \tag{5}$$

that we present in the standard form augmented with expansion for τ

$$\frac{dk}{d\psi} = \varepsilon X_1(\tau,\psi,k) + \varepsilon^2 X_2(\tau,\psi,k) + \ldots, \tag{6}$$

$$\tau = T_0(\psi,k) + \varepsilon T_1(\psi,k) + \ldots, \tag{7}$$

where $T_0(\psi,k) = \tau_0 + \frac{\pi q F(\psi,k)}{2\omega p K(k)}$; $T_1(\psi,k)$ is chosen to eliminate secular terms in the solution $k = \bar{k} + \varepsilon u_1(\psi,\bar{k}) + \varepsilon^2 u_2(\psi,\bar{k}) + \ldots$ that transforms (6) to $\frac{d\bar{k}}{d\psi} = \varepsilon\Theta_1(\bar{k}) + \varepsilon^2\Theta_2(\bar{k}) + \ldots$.

We study resonant oscillations p:q = 1:2, 1:4 with excitation function $\varphi(\tau) = \cos(\tau)$.

References

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