Propagation of few-cycle pulses in a nonlinear medium and an integrable generalization of the sine-Gordon equation

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The generalized sine-Gordon equation is obtained under the theoretical investigation of interaction of few-cycle pulses in a nonlinear medium modeled by a set of four-level atoms. This equation is derived without the use of the slowly varying envelope approximation and is shown to be integrable in the frameworks of the inverse scattering transformation method. Its solutions describing the propagation of the solitons and breathers and their interaction are investigated. In the case of different signs of the parameters of the equation considered, it is revealed, in particular, that the collision of solitons with opposite polarities can lead to an appearance of the short-living pulse having extraordinarily large amplitude, whose dynamics is similar to that of rogue waves. Also, the solitons of "rectangular" form and the breathers with rectangular oscillations exist in the case of the same signs of the parameters.

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I. INTRODUCTION

One of the basic tendencies of development of the nonlinear optics from its origin is the generation of the pulses of ever shorter duration. Using such pulses provided additional possibilities in measurements of the relaxation times, processing of materials, testing of high-speed devices, etc.

At the beginning, since the 1960s to 1980s of the last century, the light pulses of duration τ_p from nanoseconds to picoseconds were generated. The characteristic period T_p of the electromagnetic oscillations corresponding to visible and near infrared ranges is 10^{-15} s. Then, the number N of the electromagnetic oscillations contained in such pulses is $N \sim \tau_p/T_p \sim 10^6$–$10^3$. It is possible in this case to introduce a small parameter

$$\varepsilon_1 \sim 1/N \ll 1. \qquad (1)$$

The existence of this small parameter gave us an opportunity to apply the approximation of the slowly varying envelopes (SVE). This approximation simplifies significantly the theoretical considerations of the nonlinear interaction of the nano- and picosecond pulses with matter. Many resonant and nonresonant nonlinear optical phenomena were investigated in such a manner [1,2]. The spectrum of these pulses is rather narrow: $\varepsilon_1 \sim \delta\omega/\omega \ll 1$, where ω is the carrier frequency and $\delta\omega$ is the width of the pulse spectrum. For this reason, such pulses are called quasimonochromatic.

At the beginning of the 1990s, the femtosecond pulses were involved in consideration [3]. Taking $\tau_p \sim 1$ fs $= 10^{-15}$ s, one has $N \sim 1$ and $\varepsilon_1 \sim 1$. Here the pulse contains about one period of oscillations, and the SVE approximation is inapplicable since the small parameter allowing us to exploit it disappears. It is necessary in this case to search for other approximations and to derive the simplified wave equations for the electric field E of the pulse rather than for the envelopes [4–7].

Usually the pulses containing only several oscillations of the electromagnetic field are referred to as the few-cycle pulses (FCPs) [4,7–9]. The spectrum of the FCPs is wide so that it is almost impossible to allocate the carrier frequency. Therefore, the FCPs are called the broadband pulses sometimes.

The absolute duration of the FCPs lies in the range from pico- to femtoseconds [5–8,10]. Sometimes, the picosecond FCPs are called terahertz FCPs. The intensity of such pulses is so high that the development of "nonlinear terahertz optics" gained a push. It should be noted that the terahertz range is still the least investigated with respect to an interaction of electromagnetic radiation with matter [11].

It is necessary to mention briefly the pulses of the attosecond duration [12], for which $\tau_p \sim 100$ as $= 10^{-16}$ s. The theory of interaction of such pulses with matter is very difficult and far from completion. An electric field of powerful attosecond pulses is comparable in the order of magnitude with intratomic electric field and even exceeds it sometimes. Under these conditions, the ionization processes should already be taken into account which complicates the physical model considerably. An investigation of such processes lies beyond the scope of our discussion.

A creation of the model of the medium, on which the FCPs influences, is a difficult problem. Due to the large spectral width of FCPs, many of the quantum transitions can be involved in the interaction with them. At the same time, the model of the medium has to be simple enough and adequate

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to the situation considered in each case. The simplest model of the medium is that of the two-level atoms with frequency ω_0 of quantum transition. Being rather rough, it, nevertheless, is used often to describe the interaction of FCPs with matter [13]. In certain cases, phenomenological nonlinear-oscillator models are used [14–20].

One of the most effective tools of studying the nonlinear partial differential equations is the inverse scattering transformation method (ISTM) [21–23]. The theoretical nonlinear optics is a very good "supplier" of the nonlinear wave equations and systems integrable by the ISTM [5,24]. The integrable nonlinear equations possess the solitonic solutions (solitons) that are the solitary waves capable of interacting elastically with localized structures, including other solitons. It concerns solitons of the envelope (quasimonochromatic solitons) [5] and the FCPs [5,24]. Therefore, it is no wonder that the nonlinear optics gives a powerful push to the development of mathematical physics. In turn, the progress in the mathematical physics stimulates the searches of the new integrable models in nonlinear optics.

The solitons of the integrable system of the self-induced transparency (SIT) equations are most known among the resonant quasimonochromatic solitons [23]. In the case of exact resonance of the pulse with the two-level medium ($\omega = \omega_0$), the SIT system passes into the famous sine-Gordon (SG) equation integrable by the ISTM, whose solitons ("2π pulses") are well studied [23,25].

The nonlinear Schrödinger (NLS) equation describes propagation of the quasimonochromatic nonresonant solitons in isotropic media [2,3]. An essential development of the ISTM was due to revealing the integrability of this equation [26]. With shortening of the pulse duration (with an increase of parameter ε_1), the propagation of pulses is described well by integrable [27,28] and nonintegrable [2–4,29–31] modifications of the NLS equation (higher-order NLS equations). It is assumed for the higher-order NLS equations also that the spectral width of the pulse is much smaller than its carrier frequency, so that the SVE approximation can be still applied.

The refusal from the SVE approximation in nonlinear optics was made, perhaps, in Ref. [32], where an alternative approach to describe the SIT phenomenon was offered. Instead of the SVE approximation, the so-called unidirectional propagation (UP) approximation was used. This approximation is based on the condition of a small concentration n of the two-level atoms

$$\varepsilon_2 = \frac{8\pi d^2 n}{\hbar\omega_0} \ll 1, \tag{2}$$

where d is the matrix element of the dipole moment operator of the considered transition and $\hbar$ is Planck's constant.

The first-order wave equations are obtained by applying the UP approximation. As a result, the reduced Maxwell-Bloch (RMB) system was derived in [32]. This system occurred to be integrable in the frameworks of the ISTM also. The so-called breather solutions have a particular interest here. These solutions are the solitons, whose profile oscillates periodically under propagation with constant group velocity in the medium. The breather solution possesses two free parameters: duration τ_p and central frequency $\omega = 2\pi/T_p$ of the spectrum. If $\omega\tau_p \sim N \sim 1$, then this solution describes propagation of the FCPs. In the quasimonochromatic limit $\omega\tau_p \sim N \gg 1$, the breather of the RMB system passes into the soliton of the envelope. If $|\omega - \omega_0|/\omega_0 \ll 1$, it is the soliton of the envelope of the SIT equations. In the opposite case $|\omega - \omega_0|/\omega_0 \gg 1$, the breather of the RMB system passes into the soliton of the envelope of the NLS equation.

The refusal from the SVE and UP approximations was made in Refs. [13,33–35], where the approximations of optical transparency (OT) and sudden excitations (SEs) were suggested. These approximations are exploited if the following conditions:

$$\varepsilon_3 = (\omega_0\tau_*)^{-1} \ll 1 \tag{3}$$

and

$$\varepsilon_4 = \omega_0\tau_* \ll 1 \tag{4}$$

are fulfilled, respectively. Here $\tau_* = \min\{\tau_p, \omega^{-1}\}$ is the minimum timescale of the pulse. Note that the SEs approximation was, apparently, used first in the physics of nuclear reactions [36]. It was found that the pulse electric field E obeys the SG equation in the case of the SEs approximation [13,33,34], while it obeys the so-called modified Korteweg-de Vries (MKdV) equation integrable by the ISTM also in the OT approximation case.

If the duration of laser FCPs is about 10 fs then the tunnel quantum transitions, whose characteristic frequencies $\omega_0 \sim 10^{13}\ \mathrm{s}^{-1}$, satisfy the SEs approximation condition (4), for example. At the same time, electron-optical transitions ($\omega_0 \sim 10^{13}\ \mathrm{s}^{-1}$) are in a good agreement with the OT approximation condition (3).

Several versions of the model of two-component two-level media with different frequencies of quantum transitions were proposed [25,27,28,32]. The OT approximation was assumed to be valid for one of the components, and the SEs approximation was applied to another component.

If the pulse spectrum is located below the so-called zero-dispersion frequency, where the second-order dispersion vanishes, then its dynamics is described by the short-pulse equation (SPE) [37,38]. This equation was derived from Maxwell's equations using the renormalization group method and was revealed to appear in differential geometry in an attempt to construct integrable differential equations associated with pseudospherical surfaces [39,40]. The ISTM integrability of the SPE has been studied from various points of view [41,42]. Several improvements in the model leading to the SPE were suggested later [43,44]. Under certain conditions, the integrable Konno-Kameyama-Sanuki equations were obtained for the electric field of the pulse [45–48].

At the present time, there is a need for creation of a more complicated and realistic model of the medium, in which different quantum transitions are connected among themselves. Obviously it demands a refusal from the two-level model.

This paper is devoted to a derivation of the nonlinear wave equation for the pulse propagating in the medium with the interconnected quantum transitions, which meet conditions of the SEs and OT approximations, a consideration of its integrability, and to a construction of the soliton and breather solutions.

The paper is organized as follows. In the next section the model of the four-level medium, which is formed by the

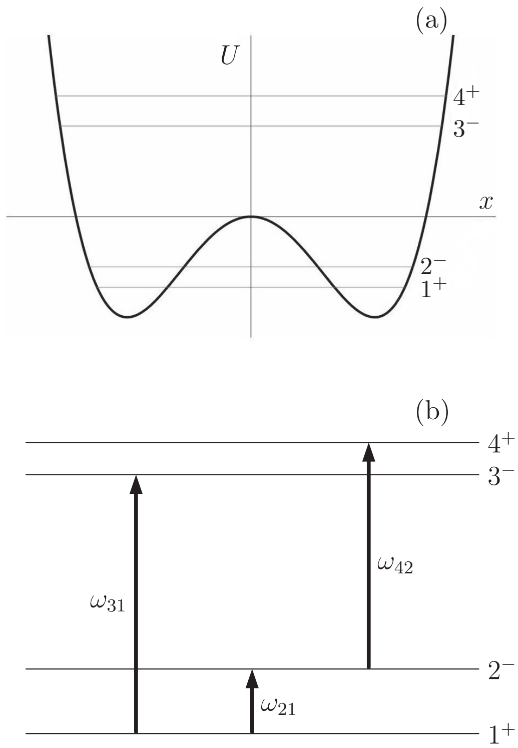

FIG. 1. (a) Quantum levels of symmetric two-pit potential. Levels 1^+ and 2^- correspond to the tunnel splitting; 3^- and 4^+ correspond to the remote quantum levels; the superscripts "+" and "−" designate positive and negative parity, respectively. (b) The scheme of the allowed quantum transitions.

tunnel splitting of quantum states in the two-pit potential and by two other remote quantum levels differing by the energy and parity, is offered. The procedure of an exclusion of the material variables is carried out in Sec. III by using the conditions (3) and (4), and the generalized SG equation is derived. It is shown in Sec. IV that this equation is integrable by the ISTM, and families of its soliton solutions and breather ones in the form of the FCPs are constructed. The processes of collisions of various solitons and breathers are investigated. Also, the NLS equation for the envelope of the breather is obtained in limit (1) from the generalized SG equation in this section. Finally, in Sec. V we summarize the results of our consideration.

II. BASIC MODEL

Consider the quantum medium formed by the tunnel splitting of the levels of quantum states in the two-pit potential [Fig. 1(a)]. It can be the proton states of the order-disorder type in a ferroelectric material [49], the electron states in the quantum dots, wells [50], etc. Besides the two quantum states 1 and 2 given by this splitting, there are remote quantum states placed above on the energy scale. We approximate them by two quantum levels 3 and 4 (see Fig. 1) differing by energy and parity.

For the sake of definiteness, we assume that states 1 and 4 possess positive parity, while the parity of states 2 and 3 is negative (Fig. 1). Then, transitions $1 \leftrightarrow 2$, $1 \leftrightarrow 3$, and $2 \leftrightarrow 4$ are allowed in the electrodipole approximation. Transitions $1 \leftrightarrow 4$ and $2 \leftrightarrow 3$ are forbidden by the selection rule on the parity.

Suppose that the frequency ω_{12} [see Fig. 1(b)] of the tunnel splitting satisfies condition (4) taking into account replacement $\omega_0 \to \omega_{12}$, and the frequencies ω_{31}, ω_{42} satisfy condition (3) replacing $\omega_0 \to \omega_{31}$, $\omega_0 \to \omega_{42}$. Owing to condition (3), the quantum transitions $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$ are excited much more weakly than transition $1 \leftrightarrow 2$. We assume also that levels 3 and 4 are not occupied before the pulse action. Therefore, we neglect a contribution from the allowed transition $3 \leftrightarrow 4$.

As a result, we come to the scheme of the quantum transitions presented in Fig. 1(b). According to it, we have the following system of the evolution equations on the elements of corresponding density matrix:

$$
\begin{aligned}
\frac{\partial \rho_{21}}{\partial t} &= -i\omega_{21}\rho_{21} + i\Omega_{21}(\rho_{11} - \rho_{22}) + i\Omega_{42}\rho_{41} - i\Omega_{31}\rho_{32}^*, \\
\frac{\partial \rho_{31}}{\partial t} &= -i\omega_{31}\rho_{31} + i\Omega_{31}(\rho_{11} - \rho_{33}) - i\Omega_{21}\rho_{32}, \\
\frac{\partial \rho_{42}}{\partial t} &= -i\omega_{42}\rho_{42} + i\Omega_{42}(\rho_{22} - \rho_{44}) - i\Omega_{21}\rho_{41},
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
\frac{\partial \rho_{32}}{\partial t} &= -i\omega_{32}\rho_{32} + i(\Omega_{31}\rho_{21}^* - \Omega_{21}\rho_{31} - \Omega_{42}\rho_{43}^*), \\
\frac{\partial \rho_{41}}{\partial t} &= -i\omega_{41}\rho_{41} + i(\Omega_{42}\rho_{21} - \Omega_{31}\rho_{43} - \Omega_{21}\rho_{42}), \\
\frac{\partial \rho_{43}}{\partial t} &= -i\omega_{43}\rho_{43} + i(\Omega_{42}\rho_{32}^* - \Omega_{31}\rho_{41}),
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
\frac{\partial \rho_{11}}{\partial t} &= i\Omega_{21}(\rho_{21} - \rho_{21}^*) + i\Omega_{31}(\rho_{31} - \rho_{31}^*), \\
\frac{\partial \rho_{22}}{\partial t} &= -i\Omega_{21}(\rho_{21} - \rho_{21}^*) + i\Omega_{42}(\rho_{42} - \rho_{42}^*), \\
\frac{\partial \rho_{33}}{\partial t} &= -i\Omega_{31}(\rho_{31} - \rho_{31}^*), \\
\frac{\partial \rho_{44}}{\partial t} &= -i\Omega_{42}(\rho_{42} - \rho_{42}^*).
\end{aligned} \tag{7}
$$

Here $\Omega_{lk} = d_{lk}E/\hbar$ and d_{lk} is the dipole moment of the allowed quantum transitions, which are assumed to be real without loss of generality ($l,\ k = 1,\ 2,\ 3,\ 4$).

Equations (5) and (6) describe the dynamics of the nondiagonal elements of the density matrix operator $\hat{\rho}$ in the cases of allowed and forbidden transitions, respectively. System (7) is responsible for the dynamics of the populations of quantum levels.

Let us supplement the system of material equations (5)–(7) by the wave equation on an electric field of the pulse, presupposing that it propagates along the z axis and its radial dependence is negligible. Then, this equation is written as

$$
\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{4\pi n}{c^2}\frac{\partial^2}{\partial t^2}[d_{21}(\rho_{21} + \rho_{21}^*) + d_{31}(\rho_{31} + \rho_{31}^*) + d_{42}(\rho_{42} + \rho_{42}^*)], \tag{8}
$$

where c is the speed of light in vacuum.

Equations (5)–(8) represent the self-consistent system describing the nonlinear interaction of an electromagnetic pulse with medium, whose quantum transitions are presented in Fig. 1(b).

III. THE GENERALIZED SINE-GORDON EQUATION

Now we express the elements ρ_{21}, ρ_{31}, and ρ_{42} from Eqs. (5)–(7) through the pulse electric field by using the OT and SEs approximations and substitute these expressions into the right-hand side of Eq. (8). Conditions of the type (3) and (4) are assumed to be fulfilled in the appropriate cases discussed in the previous section. We restrict ourselves by the first-order approximations with respect to the small parameters ε_3 and ε_4.

In accordance with condition (3), the derivatives on the left-hand sides of the second and third equations in system (5) and in Eqs. (6) should be neglected. We then obtain

$$\rho_{31} = \frac{\Omega_{31}(\rho_{11}-\rho_{33}) - \Omega_{21}\rho_{32}}{\omega_{31}},$$

$$\rho_{42} = \frac{\Omega_{42}(\rho_{22}-\rho_{44}) - \Omega_{21}\rho_{41}}{\omega_{42}}, \tag{9}$$

$$\rho_{32} = \frac{\Omega_{31}\rho_{21}^* - \Omega_{21}\rho_{31} - \Omega_{42}\rho_{43}^*}{\omega_{32}},$$

$$\rho_{41} = \frac{\Omega_{42}\rho_{21} - \Omega_{21}\rho_{42} - \Omega_{31}\rho_{43}}{\omega_{41}}, \tag{10}$$

$$\rho_{43} = \frac{\Omega_{42}\rho_{32}^* - \Omega_{31}\rho_{41}}{\omega_{43}}. \tag{11}$$

Owing to the type (3) inequalities, we have $\omega_{31} \approx \omega_{32}$, $\omega_{41} \approx \omega_{42}$. Therefore, in the first-order approximation with respect to the small parameter ε_3 taking into account replacements $\omega_0 \to \omega_{31}$ and $\omega_0 \to \omega_{42}$, one has to discard the last terms in the numerators of expressions (9) and to neglect element ρ_{43} in what follows [see relations (11) and (10)]. Thus,

$$\rho_{31} = \frac{\Omega_{31}(\rho_{11}-\rho_{33})}{\omega_{31}}, \quad \rho_{42} = \frac{\Omega_{42}(\rho_{22}-\rho_{44})}{\omega_{42}}. \tag{12}$$

It is easy to see from Eqs. (12), (10), and (7) that the change of the populations of the third and fourth levels has to be neglected also in the approximation accepted. We put $\rho_{33} = \rho_{44} = 0$ as they were unpopulated before the pulse impact. Then we have

$$\rho_{31} = \frac{\Omega_{31}}{\omega_{31}}\rho_{11}, \quad \rho_{42} = \frac{\Omega_{42}}{\omega_{42}}\rho_{22}. \tag{13}$$

Substitution of the expressions for ρ_{31} and ρ_{42} into Eqs. (10) leads to an account for the processes of the second order with respect to ε_3. For this reason, the first term in the numerator of expressions (10) has to be retained only. As a result, we obtain

$$\rho_{32} = \frac{\Omega_{31}}{\omega_{32}}\rho_{21}^*, \quad \rho_{41} = \frac{\Omega_{42}}{\omega_{41}}\rho_{21}. \tag{14}$$

Inserting these expressions into the first equation of system (5) and using approximate equalities $\omega_{31} \approx \omega_{32}$ and $\omega_{41} \approx \omega_{42}$, we get

$$\frac{\partial\rho_{21}}{\partial t} = -i\left(\omega_{21} + \frac{\Omega_{31}^2}{\omega_{31}} - \frac{\Omega_{42}^2}{\omega_{42}}\right)\rho_{21} + i\Omega_{21}(\rho_{11}-\rho_{22}). \tag{15}$$

After the substitution of expressions (12) into the right-hand side of the first and second equations in (7), we find

$$\frac{\partial\rho_{11}}{\partial t} = i\Omega_{21}(\rho_{21}-\rho_{21}^*), \quad \frac{\partial\rho_{22}}{\partial t} = -i\Omega_{21}(\rho_{21}-\rho_{21}^*). \tag{16}$$

Thus, an influence of the quantum transitions $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$ on transition $1 \leftrightarrow 2$ is reduced in the first order in small parameter ε_3 to dynamic Stark shift of frequency ω_{21} [see Eq. (15), the first term on the right-hand side]. In turn, transition $1 \leftrightarrow 2$ has dynamic impact on transitions $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$ by changing the susceptibility formed by them due to the change of populations of levels 1 and 2 [see Eqs. (13)].

Let us introduce the Bloch's variables

$$U = \frac{\rho_{21}+\rho_{21}^*}{2}, \quad V = \frac{\rho_{21}^*-\rho_{21}}{2i}, \quad W = \frac{\rho_{22}-\rho_{11}}{2}. \tag{17}$$

Then, Eqs. (15) and (16) are rewritten as

$$\frac{\partial U}{\partial t} = -\tilde{\omega}_{21}V, \quad \frac{\partial V}{\partial t} = \tilde{\omega}_{21}U + 2\Omega_{21}W,$$
$$\frac{\partial W}{\partial t} = -2\Omega_{21}V, \tag{18}$$

where

$$\tilde{\omega}_{21} = \omega_{21} + \frac{\Omega_{31}^2}{\omega_{31}} - \frac{\Omega_{42}^2}{\omega_{42}}. \tag{19}$$

It follows from Eqs. (18) that $|\Omega_{21}| \sim 1/\tau_*$. Taking into account the expressions for Ω_{lk} given after system (7), we conclude that $|\Omega_{31}| \sim |\Omega_{42}| \sim 1/\tau_*$. Now it is seen that the relation of the first term on the right-hand side of Eq. (19) to the second and third ones is of the order of $\varepsilon_4/\varepsilon_3$. This means that all terms on the right-hand side of (19) are the quantities, generally speaking, of the same order.

In the zeroth approximation in small parameter ε_4, we put formally $\tilde{\omega}_{21} = 0$ in Eqs. (18). Then $U = 0$,

$$V = W_0 \sin\theta, \quad W = W_0\cos\theta, \tag{20}$$

where $W = (w_1 - w_2)/2$ and w_1 and w_2 are initial populations of the first and second levels, respectively,

$$\theta = 2\int_{-\infty}^{t}\Omega_{21}dt' = 2\frac{d_{21}}{\hbar}\int_{-\infty}^{t}E dt'. \tag{21}$$

Next, we find from Eqs. (17)–(20) in the first order approximation

$$\frac{\partial U}{\partial t} = -W_0\left(\omega_{21} + \frac{\Omega_{31}^2}{\omega_{31}} - \frac{\Omega_{42}^2}{\omega_{42}}\right)\sin\theta. \tag{22}$$

From the normalization condition $\rho_{11} + \rho_{22} = 1$ and definition of W in (17), we have $\rho_{11} = W - 1/2$, $\rho_{22} = W + 1/2$. Using these expressions and relations (20) and (13), we obtain

$$\rho_{31} = \frac{\Omega_{31}}{\omega_{31}}\left(\frac{1}{2} - W_0\cos\theta\right), \quad \rho_{42} = \frac{\Omega_{42}}{\omega_{42}}\left(\frac{1}{2} + W_0\cos\theta\right). \tag{23}$$

Now, substituting Eqs. (22) and (23) into the right-hand side of Eq. (8), integrating the equation resulting on time, and taking into account expression (21), we obtain the nonlinear wave equation

$$\frac{\partial^2\theta}{\partial z^2}-\frac{n_0^2}{c^2}\frac{\partial^2\theta}{\partial t^2}=\frac{2n_0}{c}\left\{\left[\alpha-\beta\left(\frac{\partial\theta}{\partial t}\right)^2\right]\sin\theta -4\beta\frac{\partial^2\theta}{\partial t^2}\sin^2\frac{\theta}{2}\right\}, \tag{24}$$

where

$$\alpha=-\frac{8\pi d_{21}^2 n\omega_{21}}{\hbar c n_0}W_0,\quad \beta=\frac{\alpha}{4d_{21}^2\omega_{21}}\left(\frac{d_{31}^2}{\omega_{31}}-\frac{d_{42}^2}{\omega_{42}}\right),$$

the inertialess part n_0 of the refraction index is defined by expression

$$n_0=\sqrt{1+\frac{8\pi n}{\hbar}\left[\frac{d_{31}^2}{\omega_{31}}\left(\frac{1}{2}-W_0\right)+\frac{d_{42}^2}{\omega_{42}}\left(\frac{1}{2}+W_0\right)\right]}.$$

The right-hand side of Eq. (24) contains small parameters ε_3 and ε_4. This allows us to apply the UP approximation [32] to Eq. (24). In accordance with this approximation, we introduce "local" time $\tau=t-n_0z/c$ and "slow" coordinate $\zeta=\varepsilon z$, where $\varepsilon\sim\varepsilon_3,\ \varepsilon_4$. We have

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial\tau},\quad \frac{\partial}{\partial z}=-\frac{n_0}{c}\frac{\partial}{\partial\tau}+\varepsilon\frac{\partial}{\partial\zeta},$$
$$\frac{\partial^2}{\partial z^2}\approx\frac{n_0^2}{c^2}\frac{\partial^2}{\partial\tau^2}-2\varepsilon\frac{n_0}{c}\frac{\partial^2}{\partial\tau\partial\zeta}.$$

The term quadratic in ε is neglected in the last relation. After returning to the initial variable z, we finally obtain from Eq. (24) the following equation:

$$\frac{\partial^2\theta}{\partial z\partial\tau}=-\left[\alpha-\beta\left(\frac{\partial\theta}{\partial\tau}\right)^2\right]\sin\theta+4\beta\frac{\partial^2\theta}{\partial\tau^2}\sin^2\frac{\theta}{2}. \tag{25}$$

If transitions $1\leftrightarrow 3$ and $2\leftrightarrow 2$ are absent, then $\beta=0$. In this case, Eq. (25) passes into the sine-Gordon equation derived by applying the SEs approximation to the two-level system in Ref. [13].

It is important to emphasize once again that Eq. (25) is written not for the envelope but for the full electric field as a whole.

It is easy to see that Eq. (25) can be written in the following form:

$$\frac{\partial^2\theta}{\partial z\partial\tau'}=-\left[\alpha+\beta\left(\frac{\partial\theta}{\partial\tau'}\right)^2\right]\sin\theta-2\beta\frac{\partial^2}{\partial\tau'^2}\sin\theta, \tag{26}$$

where $\tau'=\tau+2\beta z=t-(n_0/c-2\beta)z$.

Note also that condition (2) of a small concentration of atoms was not exploited under the derivation of Eq. (25). Indeed, parameter α is proportional to $\varepsilon_2\varepsilon_4\ll 1$, while parameter β is proportional to $\varepsilon_2\varepsilon_3\ll 1$. Since we have $\varepsilon_3\ll 1$ and $\varepsilon_4\ll 1$ here [see conditions (3) and (4)], parameter ε_2 can be rather large. As a consequence, the inertialess refraction index can differ from 1 considerably.

Equation (25) was obtained in Ref. [51] within the physical model accepted here. In the present consideration, we derived it in a shorter way. Besides, the integrability of Eq. (25) with arbitrary values of parameters α and β in the frameworks of the ISTM will be shown in the next section. This allows us to construct and investigate comprehensively its two-soliton and breather solutions.

We will refer to Eq. (25) as the generalized sine-Gordon (GSG) equation below. It has to be stressed that this equation cannot be considered as a weakly perturbed SG equation. The terms containing parameter β in (25) can be comparable with the term $\alpha\sin\theta$ or can even exceed it [see the discussion after Eq. (19)]. Thus, the GSG equation (25) [or its equivalent form (26)] represents independent mathematical interest, and its solutions of various types deserve separate attention and physical analysis.

IV. INTEGRABILITY OF THE GSG EQUATION: SOLITONS AND BREATHER SOLUTIONS

The GSG equation (25) can be obtained in the appropriate limit from the integrable version of the nonlinear wave equation considered in [52]. Hence, Eq. (25) is integrable by the ISTM also. It admits the representation as the zero-curvature condition

$$\frac{\partial\hat{L}}{\partial z}-\frac{\partial\hat{A}}{\partial\tau}+[\hat{L},\hat{A}]=0, \tag{27}$$

where

$$\hat{L}=\frac{1}{2}\begin{pmatrix} i\frac{\partial\theta}{\partial\tau} & \lambda\left[\alpha+\beta\left(\frac{\partial\theta}{\partial\tau}\right)^2-i\kappa\frac{\partial\theta}{\partial\tau}\right]\\ \lambda\left[\alpha+\beta\left(\frac{\partial\theta}{\partial\tau}\right)^2+i\kappa\frac{\partial\theta}{\partial\tau}\right] & -i\frac{\partial\theta}{\partial\tau}\end{pmatrix},\quad \hat{A}=-\frac{1}{2}\begin{pmatrix}-i\kappa\sin\theta & \frac{e^{i\theta}}{\lambda}\\ \frac{e^{-i\theta}}{\lambda} & i\kappa\sin\theta\end{pmatrix}+4\beta\sin^2\frac{\theta}{2}\hat{L},$$

λ is the spectral parameter,

$$\kappa=2\sqrt{-\alpha\beta},$$

and is connected by the change of variables $(\tilde{x},\ \tilde{t},\ \xi)\to(\tau,\ z,\ \theta)$ defined in the following manner:

$$d\tau=\frac{1}{2\alpha}(1+\sqrt{1-\kappa^2u^2})\,d\tilde{x}-4\beta\sin^2\frac{\xi}{2}\,d\tilde{t},\quad dz=d\tilde{t},\quad \theta(\tau,z)=\xi(\tilde{x},\tilde{t}), \tag{28}$$

where

$$u=\frac{\partial\xi}{\partial\tilde{x}},$$

with the so-called modified SG (MSG) equation

$$\frac{\partial^2 \xi}{\partial \tilde{x} \partial \tilde{t}} = -\sqrt{1-\kappa^2 u^2}\ \sin\xi. \tag{29}$$

It follows from these relations that

$$\frac{\partial\theta}{\partial\tau} = \frac{2\alpha u}{1+\sqrt{1-\kappa^2 u^2}}. \tag{30}$$

The MSG equation is known to be integrable in the frameworks of the ISTM [53–56]. It admits the representation as the zero-curvature condition

$$\frac{\partial \hat{L}_{\rm MSG}}{\partial \tilde{t}} - \frac{\partial \hat{A}_{\rm MSG}}{\partial \tilde{x}} + [\hat{L}_{\rm MSG}, \hat{A}_{\rm MSG}] = 0, \tag{31}$$

where matrices $\hat{L}_{\rm MSG}$ and $\hat{A}_{\rm MSG}$ are defined as

$$\hat{L}_{\rm MSG} = \frac{1}{2}\begin{pmatrix} iu & \lambda[\sqrt{1-\kappa^2u^2} - i\kappa u] \\ \lambda[\sqrt{1-\kappa^2u^2} + i\kappa u] & -iu \end{pmatrix},$$

$$\hat{A}_{\rm MSG} = -\frac{1}{2}\begin{pmatrix} -i\kappa\sin\xi & \dfrac{e^{i\xi}}{\lambda} \\ \dfrac{e^{-i\xi}}{\lambda} & i\kappa\sin\xi \end{pmatrix}.$$

The change of variables (28) transforms the zero-curvature condition (31) into the zero-curvature condition (27).

The MSG equation (29) was derived under an investigation of the propagation of a two-component electromagnetic and acoustic FCPs in the anisotropic media [57,58]. Its multisoliton solutions were studied in detail in [57]. These solutions and the change of variables (28) will be used below to construct the multisoliton solutions of the GSG equation (25). Note that the previous considerations of the MSG equations were performed only if parameter κ is real or, respectively, $\alpha\beta < 0$. In the physical problem studied here, the sign of the product $\alpha\beta$ is determined by the sign of difference

$$\frac{d_{31}^2}{\omega_{31}} - \frac{d_{42}^2}{\omega_{42}}$$

and can take any value.

Let us start from the case, when parameter κ is real ($\alpha\beta < 0$). Then, the one-soliton solution of the MSG equation (29) is written in the following manner:

$$\xi = (-1)^k 2\arccos\frac{\mu\kappa - \tanh\chi}{\sqrt{1-2\mu\kappa\tanh\chi + \mu^2\kappa^2}}. \tag{32}$$

Here

$$\chi = \mu\tilde{x} - \frac{\tilde{t}}{\mu} + \chi^{(0)},$$

μ and $\chi^{(0)}$ are real constants and $k = 0, 1$. Its topological charge $S = (\xi|_{\tilde{x}\to\infty} - \xi|_{\tilde{x}\to-\infty})/\pi$ takes the following values:

$$S = \begin{cases} (-1)^k\,\mathrm{sgn}(\mu), & |\mu\kappa| < 1, \\ 0, & |\mu\kappa| > 1. \end{cases}$$

Thus, one-soliton pulses are divided into three families characterized by the topological charges 1, −1, and 0, respectively. The first two families are analogous to the kinks and antikinks (2π pulses) of the SG equation. The solitons of the last family were referred to as neutral kinks in [52]. Unlike the zero-area breather solution to the SG equation (0π pulse), these solitons are steady in the co-moving frame of reference.

From Eq. (32) we have

$$u = \frac{\partial\xi}{\partial\tilde{x}} = 2\mu\ \mathrm{sech}\,\chi\ \frac{1-\mu\kappa\tanh\chi}{1-2\mu\kappa\tanh\chi+\mu^2\kappa^2}. \tag{33}$$

Then, the amplitude of u is equal to

$$\max|u| = \begin{cases} 2|\mu|\sqrt{1-\mu^2\kappa^2}, & |\mu\kappa| < \frac{1}{\sqrt{2}}, \\ \frac{1}{|\kappa|}, & |\mu\kappa| \geqslant \frac{1}{\sqrt{2}}. \end{cases} \tag{34}$$

We see that the amplitude is independent of parameter μ if $|\mu| > 1/\sqrt{2}|\kappa|$. In this case, the profile of u consists of two peaks with amplitude $1/|\kappa|$, which are separated by the interval depending on μ [52]. The polarities of the peaks coincide if $1/\sqrt{2} < |\mu\kappa| < 1$, while they are opposite in the neutral kink case $|\mu\kappa| > 1$.

The one-soliton solution of the GSG equation (25) is obtained in the case $\alpha\beta < 0$ by the substitution of the expressions (32) and (33) into Eqs. (28). Then, variable τ is expressed as

$$\tau = \frac{\tilde{x}}{\alpha} + \frac{\kappa}{2\alpha}\ln[1-2\mu\kappa\tanh\chi+\mu^2\kappa^2]. \tag{35}$$

It follows from this relation that the one-soliton solution of Eq. (25) is steady.

It should be noted that the square roots in Eqs. (28) and (30) change the branch in the points, where $|u|$ takes its maximum value equal to $1/|\kappa|$. Taking into account also that variable u changes the sign in the case of the neutral kinks ($|\mu| > 1/|\kappa|$), we see that the corresponding one-soliton solutions of Eq. (25) are singular.

Let us define the decreasing of $\partial\theta/\partial\tau$ of the one-soliton solution of the GSG equation (25) in the case considered as $\partial\theta/\partial\tau \sim \exp(-|t - z/v|/\tau_p)$ on the tails, where parameters τ_p and v are a characteristic duration of the one-soliton pulse and its velocity in the laboratory frame of reference (z, t), respectively. From Eqs. (30), (33), (35), and expressions for κ and χ, we have

$$\tau_p = \frac{1}{|\mu\alpha|}, \quad v = \frac{c}{1+c(\alpha\tau_p^2+2\beta)}. \tag{36}$$

These formulas coincide with ones for the SG and MSG equations if $\beta = 0$. It follows from these relations that the duration of the nonsingular one-soliton pulse satisfies condition $\tau_p > 2\sqrt{-\beta/\alpha}$. Its velocity v is smaller than c if $\alpha > 0$. In this case $W_0 < 0$, i.e., the medium is in the equilibrium state.

The profiles of the variable $\partial\theta/\partial\tau$ of the one-soliton solution of Eq. (25) for different values of the parameter μ are presented in Fig. 2. Corresponding profiles of the variable u of the one-soliton solution of the MSG equation (29) are depicted here by thin lines. In the last case, the designations of the axes are given in the parentheses.

As we can see from Fig. 2(c), the amplitude of $|\partial\theta/\partial\tau|$ tends to infinity and the form of the soliton becomes sharp in the limit $|\mu| \to 1/|\kappa|$. This is a sequence of the change of the branch by the square root in Eq. (30) in the points, where $|u|$

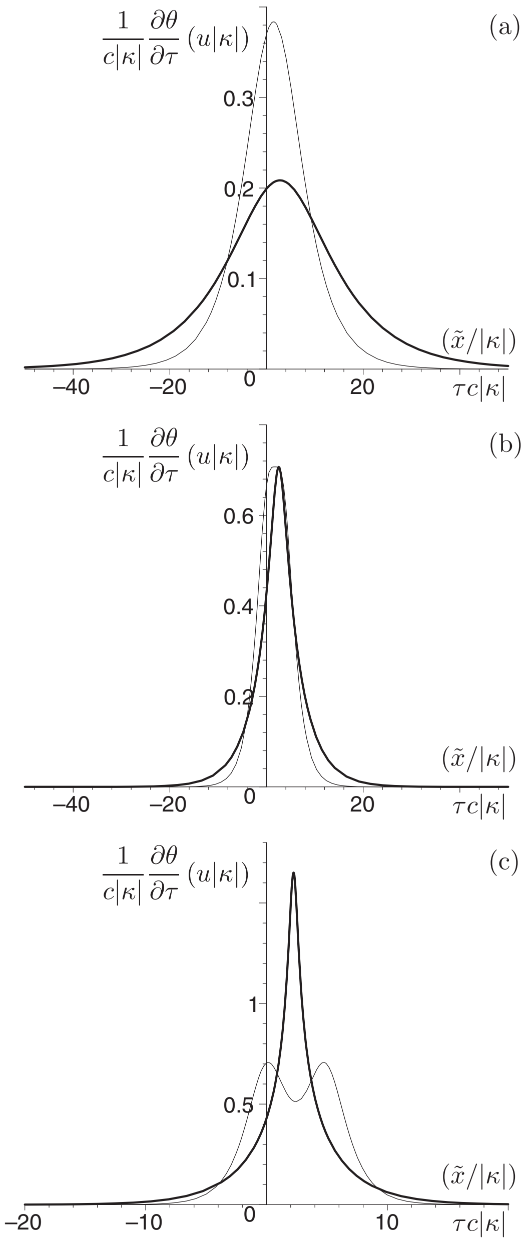

FIG. 2. Profiles of $\partial\theta/\partial\tau$ and u (thin line, the designations of the axes are in the parentheses) of one-soliton solutions with parameters $\alpha = 0.5\kappa^2 c$, $\beta = -0.5/c$, $\chi^{(0)} = 0$, $k = 0$, and $\mu = 0.2/|\kappa|$ (a), $\mu = 0.5/|\kappa|$ (b), and $\mu = 0.65/|\kappa|$ (c).

is equal to $1/|\kappa|$. We can say that the sign "−" has to be at the square root in expression (30) between the peaks of variable u. In this case, the absolute value of $\partial\theta/\partial\tau$ increases as variable u decreases.

The two-soliton solution of the MSG equation (29) in the case of real parameter κ is defined as

$$\xi = -2\arctan\frac{\mu_+ \sinh\chi_-}{\mu_- \cosh\chi_+} - 2\arctan\frac{\mu_+[\varepsilon_- \sinh\chi_- - 2\mu_-\kappa\cosh\chi_-]}{\mu_-[\varepsilon_+ \cosh\chi_+ - 2\mu_+\kappa\sinh\chi_+]}, \quad (37)$$

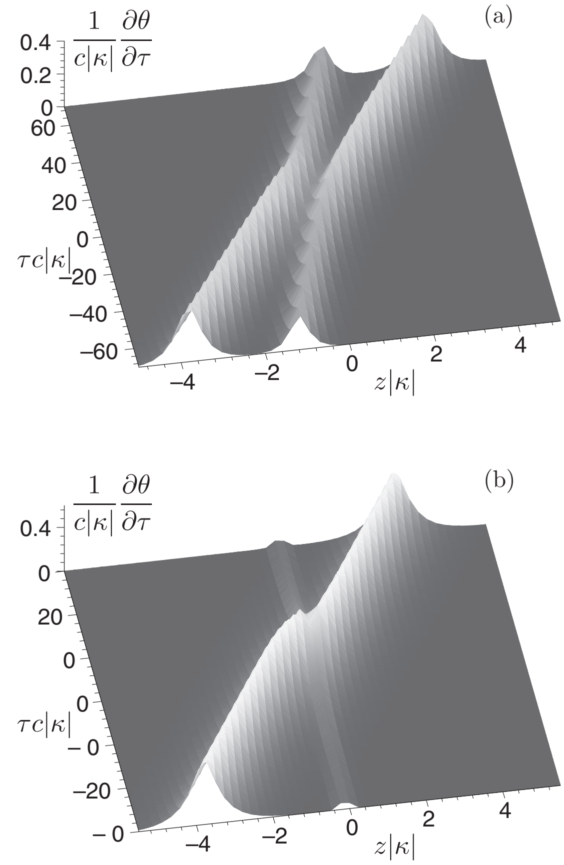

FIG. 3. Profiles of $\partial\theta/\partial\tau$ of two-soliton solutions with parameters $\alpha = 0.5\kappa^2 c$, $\beta = -0.5/c$, $\chi_1^{(0)} = \chi_2^{(0)} = 0$, $k_1 = k_2 = 0$ and $\mu_1 = 0.2/|\kappa|$, $\mu_2 = -0.3/|\kappa|$ (a) and $\mu_1 = 0.1/|\kappa|$, $\mu_2 = -0.45/|\kappa|$ (b).

where

$$\mu_\pm = \frac{\mu_1 \pm \mu_2}{2}, \quad \chi_\pm = \frac{\chi_1 \pm \chi_2}{2}, \quad \varepsilon_\pm = 1 \pm \mu_1\mu_2\kappa^2,$$
$$\chi_{1,2} = \mu_{1,2}\tilde{x} - \frac{\tilde{t}}{\mu_{1,2}} + \chi_{1,2}^{(0)} + ik_{1,2}\pi,$$

$\mu_{1,2}$ and $\chi_{1,2}^{(0)}$ are real constants and $k_{1,2} = 0, 1$. The substitution of the expression (37) into Eqs. (28) gives us the two-soliton solution of the GSG equation (25) in the case $\alpha\beta < 0$. Note that this solution can be well defined only if $|\mu_1| < 1/|\kappa|$ and $|\mu_2| < 1/|\kappa|$. In that case, the two-soliton solution describes the collision of the nonsingular solitons, whose durations and velocities are defined by the relations (36) with $\mu = \mu_1$ and $\mu = \mu_2$.

Let us discuss the collision of the solitons of Eq. (25) in detail in the terms of the variable $\partial\theta/\partial\tau$. In the case $(-1)^{k_1+k_2}\mu_1\mu_2 < 0$, the solitons of the same polarities collide. Here the character of the soliton interaction is similar to that for the Korteweg–de Vries and MKdV equations (see, e.g., Refs. [59,60]). If the absolute values of the parameters μ_1 and μ_2 of the solitons are close, the process of exchange of the energy between them is observed [Fig. 3(a)]. An interval separating the solitons exists always in such a collision. If the amplitudes of the solitons are very different, then the fast one passes through the other soliton. The monopolar pulse consisting of a single peak with the amplitude smaller than the amplitude of the larger soliton is formed in this case [Fig. 3(b)].

If $(-1)^{k_1+k_2}\mu_1\mu_2 > 0$, then the two-soliton solution of Eq. (25) describes an interaction of solitons with opposite polarities. In the case when the absolute values of μ_1 and μ_2 are much smaller than $1/\sqrt{2}|\kappa|$, this interaction is similar to that for the MKdV equation and leads to an appearance of the pulse with an amplitude equal approximately to a sum of the amplitudes of the colliding solitons [59,60]. If the absolute values of the parameters μ_1 and μ_2 are close enough to $1/\sqrt{2}|\kappa|$, then the collision of the well-defined solitons leads to an appearance of the short-living pulse with extraordinarily large amplitude or to the blow-up of the two-soliton solution.

Figure 4 illustrates the main stages of the collision of the solitons of the GSG equation (25) with opposite polarities in the last case. As we see from Fig. 4(b), the amplitude of the short-living pulse appearing under the interaction of solitons is larger than the ones of the colliding solitons on the order. The dynamics of these short-living pulses is similar to that of rogue waves [61]. Note that the rogue waves are solutions evolving on constant (or periodic) background.

The breather solution of the MSG equation (29) is written in the case of real κ as follows:

$$\xi = 2\arctan\frac{\mu_R\sin\chi_I}{\mu_I\cosh\chi_R} + 2\arctan\frac{\mu_R[(1-|\mu|^2\kappa^2)\sin\chi_I - 2\mu_I\kappa\cos\chi_I]}{\mu_I[(1+|\mu|^2\kappa^2)\cosh\chi_R - 2\mu_R\kappa\sinh\chi_R]}, \quad (38)$$

where

$$\chi_R = \mu_R\left(\tilde{x} - \frac{\tilde{t}}{|\mu|^2}\right) + \chi_R^{(0)}, \quad \chi_I = \mu_I\left(\tilde{x} + \frac{\tilde{t}}{|\mu|^2}\right) + \chi_I^{(0)},$$

μ_R, μ_I, $\chi_R^{(0)}$, and $\chi_I^{(0)}$ are real constants and $\mu = \mu_R + i\mu_I$. To obtain the breather solution of the GSG equation (25), we substitute expression (38) into Eqs. (28).

Let us introduce the characteristic parameters of the breather solution of Eq. (25). We assume that the variable $\partial\theta/\partial\tau$ is represented at the breather tails in the following manner: $\exp(-|t - z/v_g|/\tau_b)\cos[\omega_b(t - z/v_{\rm ph})]$, where parameters τ_b, ω_b, v_g, and $v_{\rm ph}$ are the duration of the breather, its carrier frequency, group, and phase velocities in the laboratory frame of references (z, t), respectively. From the first relation in (28), Eqs. (30) and (38) and expressions for χ_R and χ_I, we find

$$\tau_b = \frac{1}{|\mu_R\alpha|}, \quad \omega_b = |\mu_I\alpha|, \quad (39)$$

$$v_g = c\left[1 + c\left(\frac{\alpha\tau_b^2}{1+\omega_b^2\tau_b^2} + 2\beta\right)\right]^{-1}, \quad (40)$$

$$v_{\rm ph} = c\left[1 - c\left(\frac{\alpha\tau_b^2}{1+\omega_b^2\tau_b^2} - 2\beta\right)\right]^{-1}. \quad (41)$$

These formulas coincide with ones of the SG equation if $\beta = 0$.

Now, let us consider the case when parameter κ is pure imaginary (or $\alpha\beta > 0$). The formulas of the Darboux transformation derived in [57] allows us to construct the real

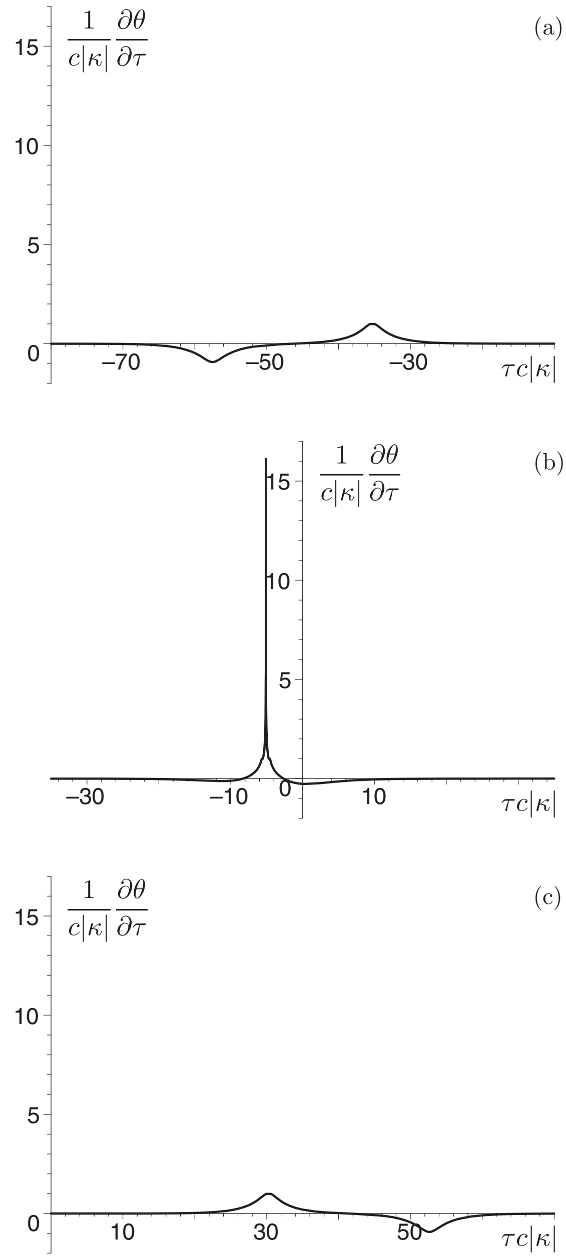

FIG. 4. Profiles of $\partial\theta/\partial\tau$ of two-soliton solution with parameters $\alpha = 0.5\kappa^2 c$, $\beta = -0.5/c$, $\chi_1^{(0)} = \chi_2^{(0)} = 0$, $k_1 = k_2 = 0$, $\mu_1 = -0.69/|\kappa|$, $\mu_2 = -0.7/|\kappa|$, and $z = -15|\kappa|$ (a), $z = 0$ (b), and $z = 15|\kappa|$ (c).

multisoliton solutions of the MSG equation (29) in this case. The corresponding one-soliton solution reads as

$$\xi = 2\arctan\frac{2\exp\chi}{1 + \mu^2K^2 - \exp(2\chi)}, \quad (42)$$

where

$$K = -i\kappa = 2\sqrt{\alpha\beta}$$

and μ are real parameters. Note that variable ξ is here nothing but the real part of the complex solution of the usual SG equation, which satisfies additional reduction. The maximum of the absolute value of $u = \partial\xi/\partial\tilde{x}$ is equal here to

$$2|\mu|\sqrt{1+\mu^2K^2}.$$

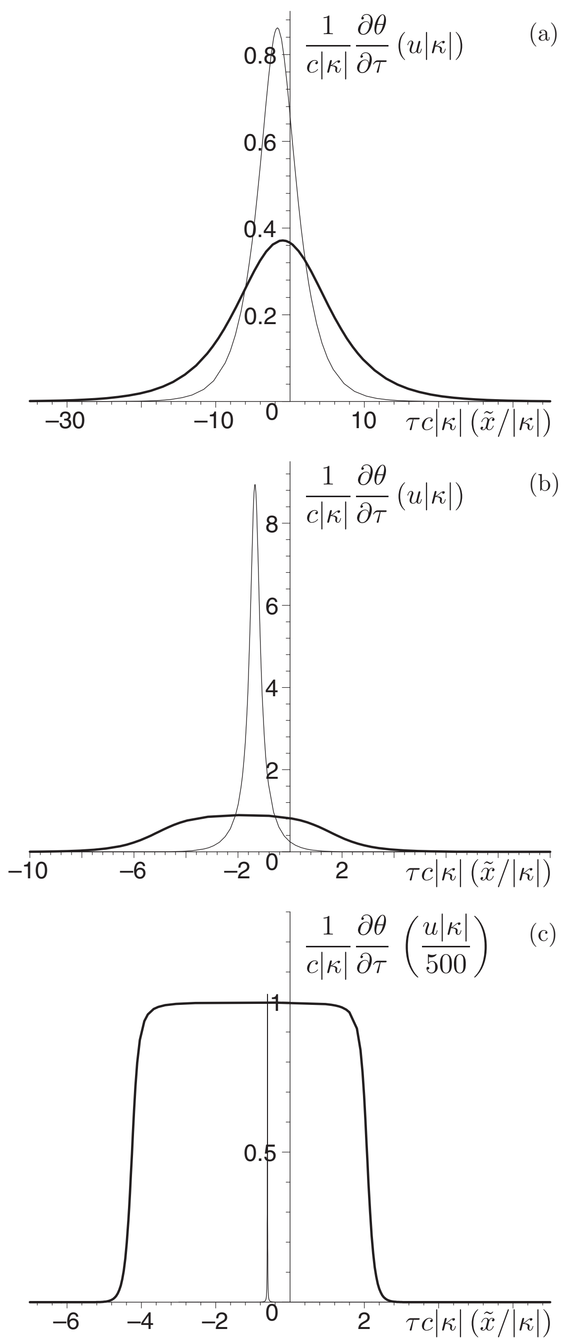

FIG. 5. Profiles of $\partial\theta/\partial\tau$ and u (thin line, the designations of the axes are in the parentheses) of one-soliton solutions with parameters $\alpha = 0.5|\kappa|^2 c$, $\beta = 0.5/c$, $\chi^{(0)} = 0$, and $\mu = 0.2/|\kappa|$ (a), $\mu = 1/|\kappa|$ (b), and $\mu = 16/|\kappa|$ (c).

Substituting expressions (42) into Eqs. (28), we obtain the one-soliton solution of the GSG equation (25) in the case $\alpha\beta > 0$. Here we have the following expression for variable τ:

$$\tau = \frac{\tilde{x}}{\alpha} + \frac{4\beta}{K} \arctan \frac{1 + \exp(2\chi) - \mu^2 K^2}{2\mu K}. \tag{43}$$

It follows from this relation that the one-soliton solution of Eq. (25) is steady also in the case considered. Figure 5 shows the profiles of variable $\partial\theta/\partial\tau$ of the one-soliton solution and corresponding ones of u for different values of parameter μ.

It is seen from Eq. (43) also that the duration of the one-soliton pulse tends to

$$\tau_{\min} = \frac{4|\beta|\pi}{|K|}$$

in the limit $|\mu| \to \infty$. The amplitude of $\partial\theta/\partial\tau$ tends in this limit to its maximum value $2|\alpha/K|$ [see Eq. (30)]. It is interesting that the form of the one-soliton pulse becomes "rectangular" under increasing $|\mu|$ [see Fig. 5(c); the scale of variable u is reduced here to 500 times]. These properties of the one-soliton solution were found in [51].

The two-soliton solution of the MSG equation (29) is written in the case of pure imaginary κ as

$$\xi = 2 \arctan \frac{(\mu_1 + \mu_2)s_+}{r_-} + 2 \arctan \frac{(\mu_1 + \mu_2)s_-}{r_+}, \tag{44}$$

where

$$s_\pm = \exp(-\chi_1) - \exp(-\chi_2) \pm K(\mu_1 - \mu_2)\exp(-\chi_1 - \chi_2),$$
$$r_\pm = (\mu_1 - \mu_2)[1 + (1 - K^2\mu_1\mu_2)\exp(-\chi_1 - \chi_2)]$$
$$\pm K(\mu_1 + \mu_2)[\mu_1 \exp(-\chi_1) - \mu_2 \exp(-\chi_2)].$$

Substitution of expression (44) into Eqs. (28) gives us the two-soliton solution of Eq. (25) describing the collision of the solitons in the case $\alpha\beta > 0$. As an example, the profiles of the variable $\partial\theta/\partial\tau$ corresponding to the collision of the rectangular solitons with opposite polarities are presented in Fig. 6. The slow soliton conserves almost its form during the interaction, which becomes more rectangular with the duration equal to $\tau_{\min}$ approximately [see Fig. 6(b)]. Such kind of soliton interaction can be called "seepage."

The breather solution of the MSG equation (29) is written in the case of pure imaginary κ in the following manner:

$$\xi = -2 \arctan \frac{\mu_R q_+}{\mu_I p_+} + 2 \arctan \frac{\mu_R q_-}{\mu_I p_-}, \tag{45}$$

where

$$p_\pm = 1 + K\mu_I \pm 2K\mu_R \sin(\chi_I)\exp(-\chi_R)$$
$$+ (1 - K\mu_I)\exp(-2\chi_R),$$

$$q_\pm = p_\pm - 1 \pm 2\cos(\chi_I)\exp(-\chi_R) - \exp(-2\chi_R).$$

Substituting expression (45) into Eqs. (28), we obtain the breather solution of the GSG equation (25) in the case $\alpha\beta > 0$. Note that the maximum of the absolute value of $u = \partial\xi/\partial\tilde{x}$ tends to infinity in the limit $\mu_R \to 0$ if

$$|\mu_I| > \frac{1}{|K|}. \tag{46}$$

In this case, the form of oscillation of $\partial\theta/\partial\tau$ becomes rectangular in the center of the breather. The corresponding plot of variable $\partial\theta/\partial\tau$ of the breather solution of Eq. (25) is presented in Fig. 7. As it can be seen, the period of the rectangular oscillations is equal approximately to $\tau_{\min}$. Their amplitude reaches the maximum value $2|\alpha/K|$.

The multisoliton solutions of the GSG equation (25) obtained on the zero background describe elastic collisions of

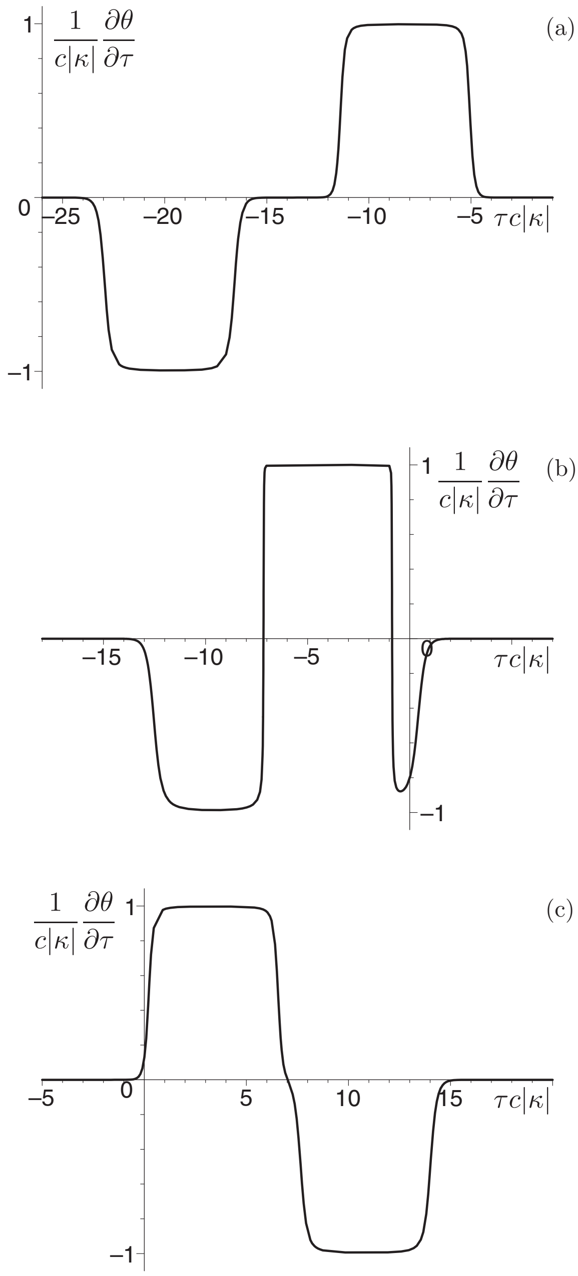

FIG. 6. Profiles of $\partial\theta/\partial\tau$ of two-soliton solution with parameters $\alpha = 0.5\kappa^2 c$, $\beta = 0.5/c$, $\chi_1^{(0)} = \chi_2^{(0)} = 0$, $\mu_1 = -10/|\kappa|$, $\mu_2 = -14/|\kappa|$, and $z = -500|\kappa|$ (a), $z = 7|\kappa|$ (b), and $z = 100|\kappa|$ (c).

the steady one-soliton pulses and breathers considered in this section.

The breather solution of the GSG equation (25) passes into the envelope soliton under condition (1), i.e., if $\omega\tau_p \gg 1$. It is interesting in this regard to derive from Eq. (25) an equation for slowly varying complex envelope ψ defined as

$$2\Omega_{21} = \frac{\partial\theta}{\partial\tau} = \psi\, e^{i(\omega\tau - qz)} + \text{c.c.}, \tag{47}$$

where parameter q is connected with the wave number k by expression $k = n_0\omega/c + q$.

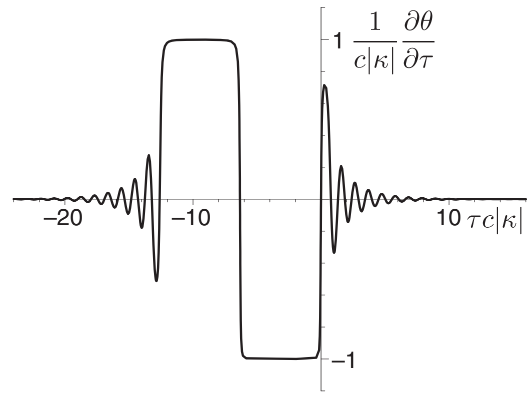

FIG. 7. Profile of $\partial\theta/\partial\tau$ of breather solution with parameters $\alpha = 0.5\kappa^2 c$, $\beta = 0.5/c$, $\chi_R^{(0)} = \chi_I^{(0)} = 0$, $\mu_R = 1/|\kappa|$, $\mu_I = 12/|\kappa|$, $z = 0$.

Taking the integral by parts successively, we have

$$\theta = \int_{-\infty}^{\tau} \psi\, e^{i(\omega\tau' - qz)}\, d\tau' + \text{c.c.}$$
$$= \left[\frac{\psi}{i\omega} - \left(\frac{1}{i\omega}\right)^2 \frac{\partial\psi}{\partial\tau} + \left(\frac{1}{i\omega}\right)^3 \frac{\partial^2\psi}{\partial\tau^2} - \left(\frac{1}{i\omega}\right)^4 \frac{\partial^3\psi}{\partial\tau^3} + t\cdots\right] \times e^{i(\omega\tau - qz)} + \text{c.c.} \tag{48}$$

Since the breather amplitude decreases with an increase of the carrier frequency, we will restrict in the subsequent consideration by the cubic nonlinearity and by the terms having an order up to $(\omega\tau_p)^{-3}$. Retaining the terms oscillating at the main frequency ω only, we write

$$\sin\theta \approx \theta - \frac{\theta^3}{6} \approx \left[-i\frac{\psi}{\omega} + \frac{1}{\omega^2}\frac{\partial\psi}{\partial\tau} + \frac{i}{\omega^3}\frac{\partial^2\psi}{\partial\tau^2} - \frac{1}{\omega^4}\frac{\partial^3\psi}{\partial\tau^3} + \frac{i}{2\omega^3}|\psi|^2\psi + \frac{1}{2\omega^4}\frac{\partial}{\partial\tau}(|\psi|^2\psi) - \frac{2}{\omega^4}|\psi|^2\frac{\partial\psi}{\partial\tau} + \cdots\right] \times e^{i(\omega\tau - qz)} + \text{c.c.} \tag{49}$$

Substitution of relations (47)–(49) into Eq. (25) gives us the following equation:

$$i\frac{\partial\psi}{\partial z} = \frac{\alpha}{\omega^3}\frac{\partial^2\psi}{\partial T^2} + i\frac{\alpha}{\omega^4}\frac{\partial^3\psi}{\partial T^3} + \frac{b}{2\omega}|\psi|^2\psi - \frac{ib}{2\omega^2}\frac{\partial}{\partial T}(|\psi|^2\psi) + \frac{ig}{2\omega^2}|\psi|^2\frac{\partial\psi}{\partial T}, \tag{50}$$

where

$$b = \frac{\alpha}{\omega^2} - 4\beta, \quad g = \frac{4\alpha}{\omega^2} - \beta, \quad T = \tau - \frac{\alpha z}{\omega^2} = t - \frac{z}{v_g},$$

linear group velocity v_g is defined by the relation

$$\frac{1}{v_g} = \frac{n_0}{c} + \frac{\alpha}{\omega^2}.$$

Expression for corresponding wave number

$$k = \frac{n_0}{c} - \frac{\alpha}{\omega}$$

is found by equalizing the coefficient at the free term of ψ to zero.

If the dispersion of group velocity and nonlinearity in the higher-order NLS equation (50) are kept in the minimal order in parameter ε_1, then it is reduced to the usual NLS equation

$$i\frac{\partial\psi}{\partial z} = \frac{\alpha}{\omega^3}\frac{\partial^2\psi}{\partial T^2} + \frac{b}{2\omega}|\psi|^2\psi. \tag{51}$$

Its soliton solution reads as

$$\psi = \frac{1}{\tau_p\sqrt{1-4\beta\omega^2/\alpha}}\exp\left(-i\frac{\alpha z}{\omega^3\tau_p^2}\right)\mathrm{sech}\left(\frac{T}{\tau_p}\right). \tag{52}$$

Here the expression for b given after Eq. (50) is used.

It follows from (52) that there is a restriction from above on the carrier frequency

$$\omega < \omega_{\max}, \tag{53}$$

where

$$\omega_{\max} = \frac{1}{2}\sqrt{\frac{\alpha}{\beta}},$$

in the case $\alpha\beta > 0$. Assuming $\omega = |\mu_I\alpha|$ in accordance with the second relation in (39) and taking into account definition of K, we see that the inequality (53) is opposite to (46). If the inequality (46) takes place and $\mu_R \to 0$ and, consequently, $\tau_p \to \infty$ [see the first relation in (39)], then the oscillations of Ω_{21} are rectangular in the center of the breather. Owing to this, the assumptions used under the derivation of the NLS equation (51) are not valid if $\alpha\beta > 0$ and $\omega > \omega_{\max}$.

In the case $\alpha\beta < 0$, the formal restrictions on the frequency of the breather and the envelope soliton are absent.

V. CONCLUSIONS

Thus, taking into account the transitions from the two allocated quantum levels to the ones lying above on the energy scale led us to integrable generalization (25) of the SG equation. It is very important that the additional terms in this equation containing the coefficient β cannot be considered as a slight adjustments of the SG equation. Due to these terms, the properties of the soliton and breather solutions of the equation differ significantly from that of the SG equation. So, depending on the sign of $\alpha\beta$, the solitons and breathers of various types can exist. If $\alpha\beta > 0$, then there exist the solitons of rectangular form and the breathers with rectangular oscillations. The interaction of the solitons displays new features also. Unlike the SG equation, the interaction of solitons of Eq. (25) having opposite polarities can give rise in the case $\alpha\beta < 0$ to an appearance of the short-living pulse with extraordinarily large amplitude, whose dynamics is similar to that of rogue waves. Also, the blow-up of the two-soliton solution can take place in some region of its parameters.

We emphasize that a simple search of the steady solutions of Eq. (25) was carried out in [51]. The solutions in the form of the rectangular and pointed pulses were obtained this way. In the present paper, these solutions appeared as special cases that correspond to the one-soliton solutions. This circumstance is important argument in the favor of the approach used here, which allowed us to obtain and analyze more complicated two-soliton and breather solutions.

Note the integrable GSG equation (25) is obtained here by considering the concrete physical model of nonlinear interaction of the FCPs with matter. This can evoke an interest to the investigation of another model in a similar manner. We hope that such investigations will stimulate the further development of nonlinear optics of short laser pulses.

ACKNOWLEDGMENT

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