ON NECESSARY OPTIMALITY CONDITIONS FOR RAMSEY-TYPE PROBLEMS

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We study optimal control problem in infinite time, where integrand does not depend explicitly on state variable. Special case of such problem is Ramsey optimal capital accumulation in centralized economy. To complete optimality conditions of Pontryagin maximum principle so called transversality conditions of different types are used in the literature. Here instead of transversality condition additional maximum condition is considered.

1. Statement of the problem

Let X be a nonempty open convex subset of R, U be an arbitrary nonempty set in R. Let us consider the following optimal control problem:

$$\int_{t_0}^{\infty} e^{-\rho t} g(u(t))\,\mathrm{d}t \to \max_u, \tag{1}$$

$$\dot{x}(t) = f(x(t), u(t)), \quad x(t_0) = x_0, \tag{2}$$

where $u(t) \in U$ and exists state variable $x(t) \in X$ for all $t \in (t_0, +\infty)$. We call such control $u(\cdot)$ and state variable $x(\cdot)$ trajectories *admissible*. Functions f and g are differentiable w.r.t. all their arguments, and together with the partial derivatives f is continuous in (x, u). Moreover function g is strictly concave and $\rho \geq 0$.[1]

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[1]Improper integral in (1) might not converge for any candidate for optimal control $\hat{u}(\cdot)$, i.e. the limit

$$\lim_{T\to\infty} J(\hat{u}(\cdot), x_0, t_0, T), \tag{3}$$

might fail to exist, or might be infinite, where we introduce the finite time horizon functional:

$$J(u(\cdot), x_0, t_0, T) = \int_{t_0}^{T} e^{-\rho t} g(u(t))\,\mathrm{d}t,$$

subject to state equation (2). Thus functional J may be unbounded as $T \to \infty$. So we can involve the following more general definitions of optimality.

An admissible control $\hat{u}(\cdot)$ is *overtaking optimal* (OO) if for every admissible control $u(\cdot)$ holds

$$\limsup_{T\to\infty} \left(J(u(\cdot), x_0, t_0, T) - J(\hat{u}(\cdot), x_0, t_0, T)\right) \leq 0.$$

2. Optimality conditions

2.1. **Pontryagin's maximum principle.** With the use of the adjoint variable ψ we introduce current value *Hamiltonian*

$$\mathcal{H}(x,u,\psi,\lambda) = \lambda\, g(u) + \psi\, f(x,u). \tag{4}$$

Theorem 1 ([1, 2, 3])**.** *There exist $\lambda \geq 0$ and ψ_0, such that $(\lambda, \psi_0) \neq 0$ and the* maximum condition *holds:*

$$\mathcal{H}(\hat{x}(t), \hat{u}(t), \psi(t), \lambda) = \max_{u\in U} \mathcal{H}(\hat{x}(t), u, \psi(t), \lambda), \tag{5}$$

along with the adjoint equation*:*

$$-\dot{\psi}(t) = -\rho\psi(t) + \psi(t)\frac{\partial f}{\partial x}(\hat{x}(t), \hat{u}(t)), \quad \psi(t_0) = \psi_0. \tag{6}$$

In this theorem ψ_0 remains undetermined. Notice that for $\psi \equiv \psi_0 = 0$ maximum condition (5) might have no solution with $\lambda > 0$, while $\lambda = \psi_0 = 0$ contradicts the theorem. Additional arguments are used to refine solutions of (5)–(6) and single out nonzero value of ψ_0.

It turns out that maximum condition (5) with $\psi \equiv 0$ and $\lambda = 1$ yields additional nesessary optimality condition if we substitute set U by set $\hat{U}(\hat{x}(\tau))$, defined as follows.

Definition 1.

$$\hat{U}(x) = \{u : (u,x) \in G\},$$

where $G \subset U \times X$ is the set of all admissible trajectories $(u(\cdot), x(\cdot))$ satisfying maximum principle (5)–(6) and state equation (2).

2.2. **Additional maximum condition.** In order to use the following condition we need first to make synthesis of control and calculate sets $\hat{U}(x)$.

Proposition 1 (Necessary optimality condition)**.** *Let there exists an admissible pair $(\hat{u}(\cdot), \hat{x}(\cdot))$.*
If control $\hat{u}$ is optimal, then for almost all $\tau \in [t_0, \infty)$ and all $u \in \hat{U}(\hat{x}(\tau))$

$$g(u) \leq g(\hat{u}(\tau)). \tag{7}$$

Example 1 (Ramsey problem with $\rho = 0$)*.* We maximize aggregated constant relative risk aversion utility

$$\int_0^\infty \frac{c(t)^{1-\theta}}{1-\theta}\,\mathrm{d}t \to \max_{c>0},$$

An admissible control $\hat{u}(\cdot)$ is weakly overtaking optimal (WOO) if for every admissible control $u(\cdot)$ holds

$$\liminf_{T\to\infty} \left(J(u(\cdot), x_0, t_0, T) - J(\hat{u}(\cdot), x_0, t_0, T)\right) \leq 0.$$

It is clear that if $\hat{u}(\cdot)$ is OO, then it is also WOO. When ordinal optimality holds, i.e. finite limit exists in (3) and for all admissible controls $u(\cdot)$

$$\limsup_{T\to\infty} J(u(\cdot), x_0, t_0, T) \leq \lim_{T\to\infty} J(\hat{u}(\cdot), x_0, t_0, T),$$

then $\hat{u}(\cdot)$ is also both OO and WOO.

subject to dynamics of capital

$$\dot{k}(t) = k(t)^{\alpha} - \delta k(t) - c(t), \quad k(t) > 0,$$

where $k(0) = k_0 > 0$, $\theta \neq 1$, $\theta > 0$, and $\alpha \in (0,1)$.

Hamiltonian:

$$\mathcal{H}(k, c, \psi, \lambda) = \lambda \frac{c^{1-\theta}}{1-\theta} + \psi \left(k^{\alpha} - \delta k - c\right), \quad \lambda \geq 0, \quad (\lambda, \psi) \neq 0.$$

Abnormal case ($\lambda = 0$) would lead to $\psi = 0$ and thus impossible. Stationarity condition $c(t)^{-\theta} = \psi(t)$ and the adjoint equation $-\dot{\psi}(t) = \left(\alpha k(t)^{\alpha-1} - \delta\right) \psi(t)$ result in the Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{\alpha k(t)^{\alpha-1} - \delta}{\theta}.$$

Due to Euler and state equations any feasible pair (k, c), not violating constraints $c(t) \geq 0$ and $k(t) > 0$, converges either to steady state (k_*, c_*), where $k_* = (\delta/\alpha)^{\frac{1}{\alpha-1}}$ and $c_* = (1-\alpha)\, k_* > 0$, or to $(\delta^{\frac{1}{\alpha-1}}, 0)$, where $k_* < \delta^{\frac{1}{\alpha-1}}$. Solid lines in Fig. 1 constitute set G, which is all space below saddle path

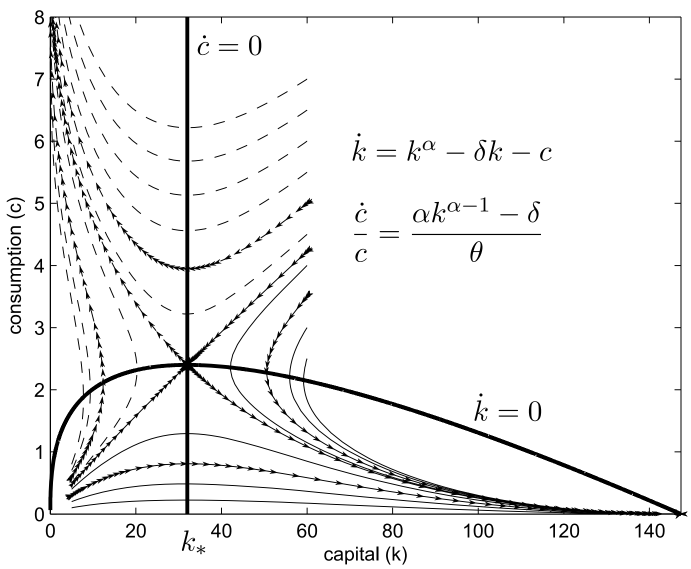

FIG. 1. Bold lines are the stationary curves, $\dot{k} = 0$ and $\dot{c} = 0$. Solid lines are the trajectories governed by the state and Euler equations, for which $k(t) > 0$ for all $t > t_0$.

and horizontal line $c = 0$. Thus condition (7)

$$\frac{\hat{c}(t)^{1-\theta}}{1-\theta} \geq \frac{c^{1-\theta}}{1-\theta}, \quad \text{for all } c \in \{c \,:\, (c, \hat{k}(t)) \in G\}, \tag{8}$$

selects saddle path as the only possible optimal.

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