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COHOMOLOGY OF THE VARIATIONAL COMPLEX IN FIELD–ANTIFIELD BRST THEORY

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We show that cohomology of the variational complex in the field-antifield BRST theory on an arbitrary manifold X equals the de Rham cohomology of X. It follows that there is no topological obstruction to constructing global descent equations in BRST theory.

1. Introduction

In field–antifield BRST theory, the antibracket is defined by means of the variational operator.[1] This operator can be introduced in a rigorous algebraic way as the coboundary operator of the variational complex of exterior forms on the infinite jet space of physical fields, ghosts and antifields.[2–4] Herewith, the antibracket and the BRST operator are expressed in terms of jets of ghosts and physical fields. Furthermore, the variational complex in BRST theory on a contractible manifold $X = \mathbb{R}^n$ is exact.[3,5] It follows that the kernel of the variational operator δ equals the image of the horizontal (total) differential d_H. Therefore, several objects in field–antifield BRST theory on $\mathbb{R}^n$ are determined modulo d_H-exact forms. In particular, let us mention the iterated cohomology $H^{k,p}(\mathrm{s}|d_H)$ of the BRST bicomplex, defined with respect to the BRST operator s and the horizontal differential d_H, and graded by the ghost number k and the form degree p. The iterated cohomology of form degree $p = n = \dim X$ coincides with the local BRST cohomology (i.e. the s-cohomology modulo d_H). If $X = \mathbb{R}^n$, an isomorphism of the local BRST cohomology $H^{k,n}(\mathrm{s}|d_H)$, $k \neq -n$, to the cohomology H^{k+n}_{tot} of the total BRST operator $\mathrm{s} + d_H$ has been proved by constructing the descent equations.[5]

In the recent article,[6] we have generalized this result to an arbitrary connected manifold X for the even field sector of BRST theory. Here, it is extended to the whole classical basis of field–antifield BRST theory including physical fields, ghosts and antifields. For this purpose, we provide a (global) differential geometric definition of jets of odd ghosts and antifields, and extend the variational complex to

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the space of these jets. Similarly to the case of even classical fields,[6,7] we show that cohomology of this complex equals the de Rham cohomology of a manifold X. In other words, the obstruction to the exactness of the variational complex in BRST theory lies only in closed exterior forms on X. These forms contribute to the right-hand side of the descent equations, but the BRST operator vanishes on them. Therefore, global descent equations can be constructed and a desired isomorphism takes place.

For the sake of simplicity, we consider BRST theory of even physical fields and finitely reducible gauge transformations. Its finite classical basis consists of even physical fields of zero ghost number, even and odd ghosts (including ghosts-for-ghosts) of strictly positive ghost number, and even and odd antifields of strictly negative ghost number. For instance, this is the case of Yang–Mills theory.

2. Global Differential Calculus in Odd Ghosts

There exist different geometric models of ghosts. For instance, ghosts in Yang–Mills theory are often represented by the Maurer–Cartan form on the gauge group.[8,9] This representation, however, is not extended to other gauge models. We describe all odd fields as elements of simple graded manifolds.[10–12]

Let $Y \to X$ be a vector bundle with a typical fibre V and $Y^* \to X$ its dual. Let us consider the exterior bundle

$$\wedge Y^* = \mathbb{R} \bigoplus_X \left(\bigoplus_{k=1}^{m} \overset{k}{\wedge} Y^* \right),$$

whose typical fibre is the finitely generated Grassmann algebra $\wedge V^*$. Sections of this exterior bundle are called graded functions on X. Let $\mathcal{A}_Y$ denote the sheaf of germs of these graded functions. The pair $(X, \mathcal{A}_Y)$ is a simple graded manifold, determined by the characteristic vector bundle $Y \to X$. Graded functions constitute the graded commutative $C^\infty(X)$-ring $\mathcal{A}_Y(X)$ of sections of the sheaf $\mathcal{A}_Y$. Given a coordinate chart $(U; x^\lambda, y^a)$ of Y with transition functions $y'^a = \rho^a_b(x) y^b$, let $\{c^a\}$ be the corresponding fibre bases for $Y^* \to X$ possessing the transition functions $c'^a = \rho^a_b(x) c^b$. We will call (x^λ, c^a) the local basis for the simple graded manifold $(X, \mathcal{A}_Y)$. With respect to this basis, graded functions read

$$f = \sum_{k=0} \frac{1}{k!} f_{a_1 \cdots a_k} c^{a_1} \cdots c^{a_k},$$

where $f_{a_1 \cdots a_k}$ are local smooth real functions on X, and we omit the symbol of the exterior product of coframes c.

In BRST theory, the basis elements c^i for a simple graded manifold describe odd ghosts and antifields (see Sec. 5). For instance, in Yang–Mills theory on a principal bundle $P \to X$ with the structure group G, the characteristic vector bundle Y of a graded manifold is the quotient $V_G P = VP/G$ of the vertical tangent bundle VP of P. Its typical fibre is the Lie algebra $\mathfrak{g}$ of group G. The typical fibre of the dual $V^*_G P$ of $V_G P$ is the Lie coalgebra $\mathfrak{g}^*$. Let $\{\varepsilon_r\}$ be a basis for $\mathfrak{g}$, $\{e_r\}$ the corresponding fibre bases for V_GP, and $\{C^r\}$ the dual coframes in V_G^*P. Covectors C^r play the role of ghosts in the BRST extension of Yang–Mills theory. Indeed, the canonical section $C = C^r \otimes e_r$ of the tensor product $V_G^*P \otimes V_GP$ is the above-mentioned Maurer–Cartan form on the gauge group. In heuristic formulation of BRST theory, it plays the role of a generator of gauge transformations with odd parameters, i.e. is the BRST operator.

Let $\mathfrak{d}\mathcal{A}_Y(X)$ be the algebra of graded derivations of the ring $\mathcal{A}_Y(X)$ of graded functions on X, i.e.

$$u(ff') = u(f)f' + (-1)^{[u][f]} fu(f'), \qquad u \in \mathfrak{d}\mathcal{A}_Y(X), \qquad f, f' \in \mathcal{A}_Y(X),$$

where [.] denotes the Grassmann parity. It is a Lie superalgebra with respect to the bracket

$$[u, u'] = uu' + (-1)^{[u][u']+1} u'u.$$

Its elements are called graded vector fields on the graded manifold $(X, \mathcal{A}_Y)$.

Graded vector fields on a simple graded manifold can be seen as sections of a vector bundle as follows. Due to the canonical splitting $VY \cong Y \times Y$, the vertical tangent bundle $VY \to Y$ of $Y \to X$ can be provided with the fibre bases $\{\partial/\partial c^a\}$, dual of $\{c^a\}$. Then graded vector fields take the coordinate form

$$u = u^\lambda \partial_\lambda + u^a \frac{\partial}{\partial c^a}, \tag{1}$$

where u^λ, u^a are local graded functions. They act on graded functions by the rule

$$u(f_{a\cdots b}c^a \cdots c^b) = u^\lambda \partial_\lambda (f_{a\cdots b})c^a \cdots c^b + u^d f_{a\cdots b} \frac{\partial}{\partial c^d} \rfloor (c^a \cdots c^b). \tag{2}$$

This rule implies the corresponding coordinate transformation law

$$u'^\lambda = u^\lambda, \qquad u'^a = \rho^a_j u^j + u^\lambda \partial_\lambda(\rho^a_j)c^j$$

of graded vector fields. It follows that graded vector fields (1) can be represented by sections of the vector bundle $\mathcal{V}_Y \to X$, which is locally isomorphic to the vector bundle

$$\mathcal{V}_Y|_U \approx \wedge Y^* \bigotimes_X \left(Y \bigoplus_X TX \right)\Big|_U,$$

and is equipped with bundle coordinates $(x^\lambda_{a_1\cdots a_k}, v^i_{b_1\cdots b_k})$, possessing the transition functions

$$x'^\lambda_{i_1\cdots i_k} = \rho^{-1}{}^{a_1}_{i_1} \cdots \rho^{-1}{}^{a_k}_{i_k} x^\lambda_{a_1\cdots a_k},$$

$$v'^i_{j_1\cdots j_k} = \rho^{-1}{}^{b_1}_{j_1} \cdots \rho^{-1}{}^{b_k}_{j_k} \left[\rho^i_j v^j_{b_1\cdots b_k} + \frac{k!}{(k-1)!} x^\lambda_{b_1\cdots b_{k-1}} \partial_\lambda \rho^i_{b_k} \right].$$

Furthermore, there exists the exact sequence

$$0 \to \wedge Y^* \bigotimes_X Y \to \mathcal{V}_Y \to \wedge Y^* \bigotimes_X TX \to 0$$

of vector bundles over X. Its splitting

$$\tilde{\gamma}: \dot{x}^\lambda\partial_\lambda \mapsto \dot{x}^\lambda\left(\partial_\lambda + \tilde{\gamma}^a_\lambda \frac{\partial}{\partial c^a}\right) \tag{3}$$

brings every vector field τ on X into the graded vector field

$$\tau = \tau^\lambda\partial_\alpha \mapsto \nabla_\tau = \tau^\lambda\left(\partial_\lambda + \tilde{\gamma}^a_\lambda \frac{\partial}{\partial c^a}\right),$$

which is a graded derivation of the ring $\mathcal{A}_Y(X)$ satisfying the Leibniz rule

$$\nabla_\tau(sf) = (\tau\rfloor ds)f + s\nabla_\tau(f), \qquad f \in \mathcal{A}_Y(X), \qquad s \in C^\infty(X).$$

Therefore, one can think of the splitting (3) as being a graded connection on the simple graded manifold $(X, \mathcal{A}_Y)$. Note that every linear connection

$$\gamma = dx^\lambda \otimes (\partial_\lambda + \gamma_\lambda{}^a{}_b v^b \partial_a)$$

on the characteristic vector bundle $Y \to X$ yields the graded connection

$$\tilde{\gamma} = dx^\lambda \otimes \left(\partial_\lambda + \gamma_\lambda{}^a{}_b c^b \frac{\partial}{\partial c^a}\right). \tag{4}$$

For instance, let Y be the Lie algebra bundle $V_G P$ in Yang–Mills theory on the G-principal bundle P. Every principal connection A on $P \to X$ yields a linear connection

$$A = dx^\lambda \otimes (\partial_\lambda - c^r_{pq} A^p_\lambda \xi^q e_r)$$

on[12] $V_G P \to X$ and, consequently, the graded connection on ghosts

$$\tilde{A} = dx^\lambda \otimes \left(\partial_\lambda - c^r_{pq} A^p_\lambda C^q \frac{\partial}{C^r}\right),$$

where c^r_{pq} are the structure constants of the Lie algebra $\mathfrak{g}$.

Let now $\mathcal{V}^*_Y \to X$ be a vector bundle which is the pointwise $\wedge Y^*$-dual of $\mathcal{V}_Y$. It is locally isomorphic to the vector bundle

$$\mathcal{V}^*_Y|_U \approx \wedge Y^* \bigotimes_X \left(Y^* \bigoplus_X T^*X\right)\Big|_U.$$

With respect to the dual bases $\{dx^\lambda\}$ for T^*X and $\{dc^b\}$ for $\mathrm{pr}_2 V^*Y = Y^*$, sections of the vector bundle $\mathcal{V}^*_Y$ take the coordinate form

$$\phi = \phi_\lambda\, dx^\lambda + \phi_a\, dc^a,$$

together with transition functions

$$\phi'_a = \rho^{-1}{}^b{}_a \phi_b, \qquad \phi'_\lambda = \phi_\lambda + \rho^{-1}{}^b{}_a \partial_\lambda(\rho^a_j)\phi_b c^j.$$

They are called graded one-forms on the graded manifold $(X, \mathcal{A}_Y)$. Graded exterior forms of higher degree are defined as sections of the graded exterior bundle $\wedge\mathcal{V}^*_Y$

such that

$$\phi \wedge \sigma = (-1)^{|\phi||\sigma| + [\phi][\sigma]} \sigma \wedge \phi \,,$$

where $|.|$ denotes the form degree.

Graded exterior forms constitute a graded differential algebra $\mathcal{O}^*\mathcal{A}_Y$, provided with the graded exterior differential d. In particular, $\mathcal{O}^0\mathcal{A}_Y = \mathcal{A}_Y(X)$ and $\mathcal{O}^1\mathcal{A}_Y$ is the dual of the module $\mathfrak{d}\mathcal{A}_Y(X)$ of graded vector fields on X, where the duality morphism is given by the interior product

$$u\rfloor\phi = u^\lambda \phi_\lambda + (-1)^{[\phi_a]} u^a \phi_a \,.$$

The graded differential is introduced on graded functions by the equality $u\rfloor df = u(f)$ for any graded vector field u, and is uniquely extended to graded exterior forms by the rules

$$d(\phi \wedge \sigma) = (d\phi) \wedge \sigma + (-1)^{|\phi|} \phi \wedge (d\sigma) \,, \qquad d \circ d = 0 \,.$$

It takes the coordinate form

$$d\phi = dx^\lambda \wedge \partial_\lambda(\phi) + dc^a \wedge \frac{\partial}{\partial c^a}(\phi) \,,$$

where the left derivatives ∂_λ, $\partial/\partial c^a$ act on coefficients of graded exterior forms by the prescription (2), and they are graded commutative with the graded forms dx^λ, dc^a.

If elements c^a of the graded manifold $(X, \mathcal{A}_Y)$ model ghosts, graded exterior forms $\phi \in \mathcal{O}^*\mathcal{A}_Y$ are provided with a ghost number by the rule

$$\mathrm{gh}(dc^a) = \mathrm{gh}(c^a) \,, \qquad \mathrm{gh}(dx^\lambda) = 0 \,.$$

3. Jets of Odd Ghosts

As was mentioned above, the antibracket and the BRST operator in field–antifield BRST theory of Refs. 2–4 are expressed in terms of jets of ghosts. For example, the BRST transformation of gauge potentials a^r_λ in Yang–Mills theory reads

$$sa^r_\lambda = C^r_\lambda + c^r_{pq} a^p_\lambda C^q \,,$$

where C^r_λ are jets of ghosts C^r introduced in a heuristic way. We will describe jets of odd fields as elements of a particular graded manifold.

Let $Y \to X$ be the characteristic vector bundle of a simple graded manifold $(X, \mathcal{A}_Y)$. The r-order jet manifold J^rY of Y is also a vector bundle over X (see, e.g., Refs. 12 and 13 for an exposition of jet formalism). Let us consider the simple graded manifold $(X, \mathcal{A}_{J^rY})$, determined by the characteristic vector bundle $J^rY \to X$. Its local basis is $\{x^\lambda, c^a_\Lambda\}$, $0 \le |\Lambda| \le r$, where $\Lambda = (\lambda_k, \ldots, \lambda_1)$ are multi-indices. It possesses the transition functions

$$c'^a_{\lambda+\Lambda} = d_\lambda(\rho^a_j c^j_\Lambda) \,, \qquad d_\lambda = \partial_\lambda + \sum_{|\Lambda|<r} c^a_{\lambda+\Lambda} \frac{\partial}{\partial c^a_\Lambda} \,, \tag{5}$$

where d_λ is the graded total derivative. In view of the transition functions (5), one can think of $(X, \mathcal{A}_{J^rY})$ as being a graded r-order jet manifold of the simple graded manifold $(X, \mathcal{A}_Y)$.

Let $\mathcal{O}^*\mathcal{A}_{J^rY}$ be the differential algebra of graded exterior forms on the graded jet manifold $(X, \mathcal{A}_{J^rY})$. Since $Y \to X$ is a vector bundle, the canonical fibration $\pi^r_{r-1} : J^rY \to J^{r-1}Y$ is a linear morphism of vector bundles over X and, thereby, yields the corresponding morphism of graded jet manifolds $(X, \mathcal{A}_{J^rY}) \to (X, \mathcal{A}_{J^{r-1}Y})$ accompanied by the pull-back monomorphism of differential algebras $\mathcal{O}^*\mathcal{A}_{J^{r-1}Y} \to \mathcal{O}^*\mathcal{A}_{J^rY}$. Then we have the direct system of differential algebras

$$\mathcal{O}^*\mathcal{A}_Y \longrightarrow \mathcal{O}^*\mathcal{A}_{J^1Y} \longrightarrow \cdots \mathcal{O}^*\mathcal{A}_{J^rY} \xrightarrow{\pi^{r+1*}_r} \cdots .$$

Its direct limit $\mathcal{O}^*_\infty\mathcal{A}_Y$ consists of graded exterior forms on all graded jet manifolds $(X, \mathcal{A}_{J^rY})$ modulo the pull-back identification. It is a locally free graded $C^\infty(X)$-algebra generated by the elements

$$(1, c^a_\Lambda, dx^\lambda, \theta^a_\Lambda = dc^a_\Lambda - c^a_{\lambda+\Lambda}\, dx^\lambda) , \qquad 0 \leq |\Lambda| ,$$

where dx^λ and θ^a_Λ are called horizontal and contact forms, respectively. In particular, $\mathcal{O}^0_\infty\mathcal{A}_Y$ is the graded commutative ring of graded functions on all graded jet manifolds $(X, \mathcal{A}_{J^rY})$ modulo the pull-back identification.

Let us consider the sheaf $\mathfrak{P}^0_\infty\mathcal{A}_Y$ of germs of graded functions $\phi \in \mathcal{O}^*_\infty\mathcal{A}_Y$. It is a sheaf of graded commutative algebras on X, and the pair $(X, \mathfrak{P}^0_\infty\mathcal{A}_Y)$ is a graded manifold. This graded manifold is the projective limit of the inverse system of graded jet manifolds

$$(X, \mathcal{A}_Y) \longleftarrow (X, \mathcal{A}_{J^1Y}) \longleftarrow \cdots (X, \mathcal{A}_{J^rY}) \longleftarrow \cdots ,$$

and is called the graded infinite jet manifold. Then one can think of elements of the algebra $\mathcal{O}^*_\infty\mathcal{A}_Y$ as being graded exterior forms on the graded manifold $(X, \mathfrak{P}^0_\infty\mathcal{A}_Y)$.

There is the canonical splitting of

$$\mathcal{O}^*_\infty\mathcal{A}_Y = \bigoplus_{k,s} \mathcal{O}^{k,s}_\infty\mathcal{A}_Y , \qquad 0 \leq k , \qquad 0 \leq s \leq n ,$$

into $\mathcal{O}^0_\infty\mathcal{A}_Y$-modules $\mathcal{O}^{k,s}_\infty\mathcal{A}_Y$ of k-contact and s-horizontal graded forms. Accordingly, the graded exterior differential d on $\mathcal{O}^*_\infty\mathcal{A}_Y$ is split into the sum $d = d_H + d_V$, where d_H is the nilpotent graded horizontal differential

$$d_H(\phi) = dx^\lambda \wedge d_\lambda(\phi) : \mathcal{O}^{k,s}_\infty\mathcal{A}_Y \to \mathcal{O}^{k,s+1}_\infty\mathcal{A}_Y .$$

With respect to the BRST operator s, the graded exterior forms $\phi \in \mathcal{O}^*_\infty\mathcal{A}_Y$ are characterized by the ghost number

$$\mathrm{gh}(dc^a_\Lambda) = \mathrm{gh}(c^a_\Lambda) = \mathrm{gh}(c^a)$$

and one puts $\mathrm{s} \circ d_H + d_H \circ \mathrm{s} = 0$.

4. Even Physical Fields and Ghosts

In order to describe odd and even elements of the classical basis of field–antifield BRST theory on the same footing, we will generalize the notion of a graded manifold to graded commutative algebras generated by both odd and even elements.[10]

Let $Y = Y_0 \oplus Y_1$ be the Whitney sum of vector bundles $Y_0 \to X$ and $Y_1 \to X$. We regard it as a bundle of graded vector spaces with the typical fibre $V = V_0 \oplus V_1$. Let us consider the quotient of the tensor bundle

$$\otimes Y^* = \bigoplus_{k=0}^{\infty} \left(\bigotimes_X^k Y^* \right)$$

by the elements

$$y_0 y_0' - y_0' y_0 , \qquad y_1 y_1' + y_1' y_1 , \qquad y_0 y_1 - y_1 y_0$$

for all $y_0, y_0' \in Y_{0x}^*$, $y_1, y_1' \in Y_{1x}^*$, and $x \in X$. It is an infinite-dimensional vector bundle, further denoted by $\wedge Y^*$. Global sections of $\wedge Y^*$ constitute a graded commutative algebra $\mathcal{A}_Y(X)$, which is the product over $C^\infty(X)$ of the commutative algebra $\mathcal{A}_0(X)$ of global sections of the symmetric bundle $\vee Y_0^* \to X$ and the graded algebra $\mathcal{A}_1(X)$ of global sections of the exterior bundle $\wedge Y_1^* \to X$.

Let $\mathcal{A}$, $\mathcal{A}_0$ and $\mathcal{A}_1$ be the sheaves of germs of sections of the vector bundles $\wedge Y^*$, $\vee Y_0^*$ and $\wedge Y_1^*$, respectively. The pair $(X, \mathcal{A}_1)$ is a familiar simple graded manifold. Therefore, we agree to call $(X, \mathcal{A})$ the graded commutative manifold, determined by the characteristic graded vector bundle Y. Given a bundle coordinate chart $(U; x^\lambda, y_0^i, y_1^a)$ of Y, the local basis for $(X, \mathcal{A})$ is $(x^\lambda, c_0^i, c_1^a)$, where $\{c_0^i\}$ and $\{c_1^a\}$ are the fibre bases for the vector bundles Y_0^* and Y_1^*, respectively. Then a straightforward repetition of all the above constructions for a simple graded manifold provides us with the differential algebra $\mathcal{O}_\infty^* \mathcal{A}$ of graded commutative exterior forms on the graded commutative infinite jet manifold $(X, \mathfrak{P}_\infty^0 \mathcal{A})$. This is a $C^\infty(X)$-algebra generated locally by the elements

$$(1, c_{0\Lambda}^i, c_{1\Lambda}^a, dx^\lambda, \theta_{0\Lambda}^i, \theta_{1\Lambda}^a) , \qquad 0 \le |\Lambda| .$$

Its $C^\infty(X)$-subalgebra $\mathcal{O}_\infty^* \mathcal{A}_1$, generated locally by the elements $(1, c_{1\Lambda}^i, dx^\lambda, \theta_{1\Lambda}^i)$, is exactly the differential algebra of graded exterior forms on the graded manifold $(X, \mathfrak{P}^0 \mathcal{A}_1)$. The $C^\infty(X)$-subalgebra $\mathcal{O}_\infty^* \mathcal{A}_0$ of $\mathcal{O}_\infty^* \mathcal{A}$ generated locally by the elements $(1, c_{0\Lambda}^i, dx^\lambda, \theta_{0\Lambda}^i)$, $0 \le |\Lambda|$, is isomorphic to the polynomial subalgebra P_∞^* of the differential algebra $\mathcal{O}_\infty^*$ of exterior forms on the infinite jet manifold $J^\infty Y_0$ of the vector bundle $Y_0 \to X$ after its pull-back onto X.[6,7] The algebra $\mathcal{O}_\infty^*$ provides the differential calculus in classical field theory.

5. Antifields

The jet formulation of field–antifield BRST theory enables one to introduce antifields on the same footing as physical fields and ghosts. Let Φ^A be a collective symbol for physical fields and ghosts. Let E be the characteristic graded vector bundle of the graded commutative manifold, generated by Φ^A. Treated as source coefficients of BRST transformations, antifields Φ^*_A with the ghost number

$$\mathrm{gh}\,\Phi^*_A = -\mathrm{gh}\,\Phi^A - 1$$

are represented by elements of the graded commutative manifold, determined by the characteristic graded vector bundle $\overset{n}{\wedge} T^*X \otimes E^*$ (see Refs. 14 and 15 for the geometric model of antifields in functional BRST formalism). Then the total characteristic graded vector bundle of a graded commutative manifold for a classical basis of field–antifield BRST theory is

$$Y = E \oplus \left(\overset{n}{\wedge} T^*X \otimes E^*\right).$$

In particular, gauge potentials a^r_λ in Yang–Mills theory on a principal bundle $P \to X$ are represented by sections of the affine bundle $J^1P/G \to X$, modelled on the vector bundle $T^*X \otimes V_GP \to X$. Accordingly, the characteristic vector bundle for their odd antifields is

$$\overset{n}{\wedge} T^*X \otimes TX \otimes V^*_GP \to X.$$

As was mentioned above, the characteristic vector bundle for ghosts C^r in Yang–Mills theory is the Lie algebra bundle $V_GP \to X$. Then the characteristic vector bundle for their even antifields is

$$\overset{n}{\wedge} T^*X \otimes V^*_GP \to X.$$

Thus, the total characteristic graded vector bundle for BRST Yang–Mills theory is

$$\begin{aligned} Y = Y_0 \oplus Y_1 &= \left[(T^*X \otimes V_GP) \oplus \left(\overset{n}{\wedge} T^*X \otimes V^*_GP\right)\right] \\ &\oplus \left[V_GP \oplus \left(\overset{n}{\wedge} T^*X \otimes TX \otimes V^*_GP\right)\right]. \end{aligned}$$

The jets $\Phi^*_{A\Lambda}$ of antifields Φ^*_A are introduced similarly to jets Φ^A_Λ of physical fields and ghosts Φ^A.

6. The Variational Complex in BRST Theory

The differential algebra $\mathcal{O}^*_\infty\mathcal{A}$ gives everything for global formulation of Lagrangian field–antifield BRST theory on a manifold X. We restrict our consideration to the short variational complex

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{O}^0_\infty\mathcal{A} \xrightarrow{d_H} \mathcal{O}^{0,1}_\infty\mathcal{A} \cdots \xrightarrow{d_H} \mathcal{O}^{0,n}_\infty\mathcal{A} \xrightarrow{\delta} \operatorname{Im}\delta \to 0, \tag{6}$$

where δ is the variational operator such that $\delta \circ d_H = 0$. It is given by the expression

$$\delta(L) = (-1)^{|\Lambda|}\theta^a \wedge d_\Lambda(\partial^\Lambda_a L), \qquad L \in \mathcal{O}^{0,n}_\infty\mathcal{A},$$

with respect to a physical basis $(\zeta^a) = (\Phi^A, \Phi^*_A)$.

The variational complex (6) provides the algebraic approach to the antibracket technique, where one can think of elements L of $\mathcal{O}^{0,n}_\infty\mathcal{A}$ as being Lagrangians of fields, ghosts and antifields. Note that, to be well-defined, a global BRST Lagrangian should factorize through covariant differentials of physical fields, ghosts and antifields $D_\lambda\zeta^a = \zeta^a_\lambda - \tilde\gamma^a_\lambda$, where $\tilde\gamma$ is a connection on the graded commutative manifold $(X, \mathcal{A})$.

In order to obtain cohomology of the variational complex (6), let us consider the sheaf $\mathfrak{P}^*_\infty\mathcal{A}$ of germs of elements $\phi \in \mathcal{O}^*_\infty\mathcal{A}$ and the graded differential algebra $P^*_\infty\mathcal{A}$ of global sections of this sheaf. Note that $P^*_\infty\mathcal{A} \neq \mathcal{O}^*_\infty\mathcal{A}$. Roughly speaking, any element of $\mathcal{O}^*_\infty\mathcal{A}$ is of bounded jet order, whereas elements of $P^*_\infty\mathcal{A}$ need not be so.

We have the short variational complex of sheaves

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{P}^0_\infty\mathcal{A} \xrightarrow{d_H} \mathfrak{P}^{0,1}_\infty\mathcal{A} \cdots \xrightarrow{d_H} \mathfrak{P}^{0,n}_\infty\mathcal{A} \xrightarrow{\delta} \operatorname{Im}\delta \to 0\,. \tag{7}$$

Graded commutative exterior forms $\phi \in \mathcal{O}^*_\infty\mathcal{A}$ are proved to satisfy the algebraic Poincaré lemma, i.e. any closed graded commutative exterior form on the graded manifold $(\mathbb{R}^n, \mathfrak{P}^0_\infty\mathcal{A})$ is exact.[3,5] Consequently, the complex (7) is exact. Since $\mathfrak{P}^{0,*}_\infty\mathcal{A}$ are sheaves of $C^\infty(X)$-modules on X, they are fine and acyclic. Without inspecting the acyclicity of the sheaf $\operatorname{Im}\delta$, one can apply a minor modification of the abstract de Rham theorem[7] to the complex (7), and obtain that cohomology of the complex:

$$0 \longrightarrow \mathbb{R} \longrightarrow P^0_\infty\mathcal{A} \xrightarrow{d_H} P^{0,1}_\infty\mathcal{A} \cdots \xrightarrow{d_H} P^{0,n}_\infty\mathcal{A} \xrightarrow{\delta} \operatorname{Im}\delta \to 0 \tag{8}$$

is isomorphic to the de Rham cohomology of a manifold X.

Following suit of Theorem 7 in Ref. 7 (Theorem 9 in Ref. 6) and replacing exterior forms on $J^\infty Y$ with graded commutative forms on $(X, \mathfrak{P}^0_\infty\mathcal{A})$, one can show that cohomology of the short variational complex (6) is isomorphic to that of the complex (8) and, consequently, to the de Rham cohomology of X. Moreover, this isomorphism is performed by the natural monomorphism of the de Rham complex $\mathcal{O}^*$ of exterior forms on X to the complex (8). It follows that:

(i) every d_H-closed graded form $\phi \in \mathcal{O}^{0,m<n}_\infty\mathcal{A}$ is split into the sum $\phi = \varphi + d_H\xi$, where φ is a closed exterior m-form on X;

(ii) every δ-closed graded form $\phi \in \mathcal{O}^{0,n}_\infty\mathcal{A}$ is split into the sum $\phi = \varphi + d_H\xi$, where φ is an exterior n-form on X.

One should mention the important case of BRST theory where Lagrangians are independent of coordinates x^λ. Let us consider the subsheaf $\bar{\mathfrak{P}}^*_\infty\mathcal{A}$ of the sheaf $\mathfrak{P}^*_\infty\mathcal{A}$ which consists of germs of x-independent graded commutative exterior forms. Then we have the subcomplex

$$0 \longrightarrow \mathbb{R} \longrightarrow \bar{\mathfrak{P}}^0_\infty\mathcal{A} \xrightarrow{d_H} \bar{\mathfrak{P}}^{0,1}_\infty\mathcal{A} \cdots \xrightarrow{d_H} \bar{\mathfrak{P}}^{0,n}_\infty\mathcal{A} \xrightarrow{\delta} \operatorname{Im}\delta \to 0 \tag{9}$$

of the complex (7) and the corresponding subcomplex

$$0 \longrightarrow \mathbb{R} \longrightarrow \bar{P}^0_\infty \mathcal{A} \xrightarrow{d_H} \bar{P}^{0,1}_\infty \mathcal{A} \cdots \xrightarrow{d_H} \bar{P}^{0,n}_\infty \mathcal{A} \xrightarrow{\delta} \operatorname{Im}\delta \to 0 \tag{10}$$

of the complex (8). Clearly, $\bar{P}^{0,*}_\infty \mathcal{A} \subset \mathcal{O}^{0,*}_\infty \mathcal{A}$, i.e. the complex (10) is also a subcomplex of the short variational complex (6).

The key point is that the complex of sheaves (9) fails to be exact. The obstruction to its exactness at the term $\bar{\mathfrak{P}}^{0,k}_\infty$ consists of the germs of constant exterior k-forms on X.[4] Let us denote their sheaf by S^k_X. We have the short exact sequences of sheaves

$$0 \to \operatorname{Im} d_H \to \operatorname{Ker} d_H \to S^k_X \to 0\,, \qquad 0 < k < n\,,$$

$$0 \to \operatorname{Im} d_H \to \operatorname{Ker}\delta \to S^n_X \to 0$$

and the corresponding sequences of modules of their global sections

$$0 \to \operatorname{Im} d_H(X) \to \operatorname{Ker} d_H(X) \to S^k_X(X) \to 0\,, \qquad 0 < k < n\,,$$
$$0 \to \operatorname{Im} d_H(X) \to \operatorname{Ker}\delta(X) \to S^n_X(X) \to 0\,.$$

The latter are exact because $S^{k<n}_X$ and S^n_X are subsheaves of the sheaves $\operatorname{Ker} d_H$ and $\operatorname{Ker}\delta$, respectively. Therefore, the kth cohomology group of the complex (10) is isomorphic to the $\mathbb{R}$-module $S^k_X(X)$ of constant exterior k-forms, $0 < k \leq n$, on the manifold X. Consequently, any d_H-closed graded commutative k-form, $0 < k < n$, and any δ-closed graded commutative n-form ϕ, constant on X, are split into the sum $\phi = \varphi + d_H\xi$ where φ is a constant exterior form on X.

Thus, the obstruction to the exactness of variational complexes in the field–antifield BRST theory on an arbitrary manifold X reduces to closed exterior forms on X. Then one can follow the proof of Theorem 1 in Ref. 6 in order to justify the existence of global descent equations and the isomorphism between local and total BRST cohomology. It also follows that the topological ambiguity of a proper solution of the master equation in the Lagrangian BRST theory reduces to exterior forms on X.

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