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Large-time evolution of an electron in photon bath

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ABSTRACT

The problem of infrared divergence of the effective electromagnetic field produced by elementary charges is revisited using the model of an electron freely evolving in a photon bath. It is shown that for any finite travel time, the effective field of the electron is infrared-finite, and that at each order of perturbation theory the radiative contributions grow unboundedly with time. Using the Schwinger–Keldysh formalism, factorization of divergent contributions in multi-loop diagrams is proved, and summation of the resulting infinite series is performed. It is found that despite the unbounded growth of individual contributions to the effective field, their sum is bounded, tending to zero in the limit of infinite travel time. It is concluded that the physical meaning of infrared singularity in the effective field is the existence of a peculiar irreversible spreading of electric charges, caused by their interaction with the electromagnetic field. This spreading originates from the quantum electromagnetic fluctuations, rather than the electron–photon scattering, and exists in vacuum as well as at finite temperatures. It shows itself in a damping of the off-diagonal elements of the momentum-space density matrix of electron, but does not affect its momentum probability distribution. This effect is discussed in terms of thermalization of the electron state, and the asymptotic growth of its quantum entropy is determined. Relationship of the obtained results to the Bloch–Nordsieck theorem is established and considered from the standpoint of measurability of the electromagnetic field. The effect of irreversible spreading on the electron diffraction in the classic two-slit experiment is determined, and is shown to be detectable in principle by modern devices already at room temperature.

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1. Introduction

The low-energy behavior of quantum systems is an issue which is important in both fundamental and applied aspects of quantum field theory. Investigation of the low-energy properties of particle interactions constitutes an essential part in establishing the correspondence between classical and quantum theories, and finds numerous applications in all areas of quantum physics, from noise theory to quantum cosmology. As is well known, these properties essentially depend on the presence of massless particle states. In non-relativistic quantum mechanics, the existence of massless photons and gravitons shows itself in the form of long-range Coulomb and Newton forces between massive particles. Account of the radiation effects brings in, among other outcomes of the union of quantum theory with the relativistic principle, a factor that has no counterpart in the physics of systems involving only massive particles. Namely, the masslessness of photons and gravitons, together with the non-conservation of the particle number, imply that production of these quanta in any scattering process is beyond the experimental control. More precisely, finite sensitivity of any experimental setup does not allow one to distinguish scattering processes which involve different numbers of massless quanta with sufficiently small energy.

The role of this indistinguishability in the scattering theory is also well known. In the standard formulation using the S -matrix, scattering processes are considered formally on an infinite time interval, implying that the 4-momenta of virtual photons describing radiative corrections may take on arbitrarily small values. Integration over such momenta gives formally infinite results at every order of perturbation theory. On the other hand, the uncontrollable production of photons with arbitrarily small energy means that the observed scattering cross-section is actually an infinite sum of terms, each of which represents the given scattering process with a definite number of extra real soft photons, that is photons with energies below the sensitivity threshold.¹ Integration of the cross-section over the real soft photon momenta brings in another divergence which exactly cancels the divergence due to virtual photons, resolving thereby the infrared catastrophe of quantum electrodynamics [1–3]. For brevity, this result will be referred to below as the Bloch–Nordsieck theorem. Similar cancellations take place in quantum gravity [4], and in a more intricate way, in Yang–Mills theories [5–7].

These results, though resolve the infrared catastrophe in a physically adequate way, leave open the question about possible physical manifestations of the infrared singularities, other than mere explicit dependence of the cross-sections on the sensitivity threshold. That the Bloch–Nordsieck theorem does not exhaust the infrared problem can be seen from the standpoint of the measurement theory. A result of fundamental importance regarding measurability of the electromagnetic field, proved by Bohr and Rosenfeld eighty years ago, asserts consistency of the principal limitations imposed by quantum theory on realizability of field measurements with formal predictions of quantum electrodynamics [8]. More specifically, all statistical predictions following from the formal relations between electromagnetic field operators can be verified, with accuracy limited only by the uncertainty principle, using an appropriately designed *macroscopic* measuring device. The demonstration given by Bohr and Rosenfeld refuted objections against the possibility of such verification, raised earlier by Landau and Peierls [9], which were based on consideration of the field measurement using *single* test charge. It is the existence of uncontrollable radiation by the test charge under the influence of the field being measured that led the authors of [9] to the conclusion that quantum electrodynamics imposes additional limitations on the accuracy of field measurements, which turned out to be significantly more stringent than those following from the uncertainty principle. In particular, the use of single test charge restricts the accuracy of separate measurements of individual field components, in contradiction with the formal apparatus of quantum electrodynamics, which implies no such restriction. An important conclusion of the work [8] is that despite the principal impossibility to control the radiation produced by test bodies, the effect of this radiation on their motion can be compensated with the help of an appropriate experimental arrangement, but such compensation is possible only when the test bodies employed consist of sufficiently many elementary charges.²

¹ And with a total energy going into unobserved photons less than the uncertainty in the energy of “hard” particles, i.e., the particles being scattered.

² See also [10]. This result was carried over to the case of gravitational interaction by DeWitt [11].

Suppose now that we want to determine the electromagnetic field produced by a free electron in a given state. It follows from what was just said that scattering a test charge on the electron is not the best method for this purpose. From the theoretical point of view, this means that using the S -matrix (say, the two-particle scattering amplitude) to describe the electron field generally is not adequate, as it unavoidably misses part of information about this field, namely that hidden by the uncontrollable radiation. Instead, since all statistical properties of the electromagnetic field are encoded in its operator, a complete description of the electron field can be obtained in terms of expectation values of the products of field operators, evaluated over the given state. Then the results of [8] guarantee that predictions obtained this way are amenable to experimental verification. One of the most important quantities of this sort is the expectation value of the electromagnetic field itself, called also mean, or effective, field. Thus, the very definition of the effective field excludes from consideration the uncontrollable radiation produced by the test bodies (whereas the uncontrollable radiation from the system that produces the field being measured is fully taken into account by the effective field formalism, cf. Sections 2.1 and 3.1).

The purpose of the present paper is to investigate infrared properties of the effective electromagnetic field in the case when the field-producing electron is embedded in a photon bath at finite temperature. At zero temperature, this problem was considered already at the dawn of quantum field theory, to determine radiative corrections to the field of a classical point source (e.g., atomic nucleus) [12,13]. The restriction to classical, that is, sufficiently heavy source is necessary in the conventional formulation precisely because of the presence of infrared divergences: radiative corrections to the electromagnetic form factors of charged particles vanish in the large-mass limit, which gives a formal reason to put the question about their divergence aside. Another reason which is often used in the literature to discard infrared-divergent contributions to the effective field is that at every order of the perturbation theory, such contributions are local, in the sense that they vanish when considered within the long-range expansion with respect to the distance from the source.³ It is worth mentioning in this connection that despite numerous attempts [14–18], extension of the S -matrix formalism to finite temperatures is still an open question. Therefore, regardless of the fundamental reasons given above, the effective field is an indispensable means for studying temperature effects in quantum field theory.

The following circumstance is crucial for the discussion of infrared singularity in the effective field. The infrared divergences occur because evolution of the field-producing system is considered on an infinite time interval. An infinite temporal extent is required already by the procedure of adiabatic switching of the interaction, which is widely employed in quantum field calculations, in particular, in constructing the scattering matrix. The proper conclusion to be drawn from this fact is that in the presence of massless particles, the use of the notion of infinitely remote past requires special justification. The Bloch–Nordsieck theorem gives such justification in the case of the S -matrix, but in the effective field formalism this notion turns out to be physically inadequate. The point is that, as was demonstrated in Ref. [19], the infrared singularity in the effective field signifies the existence of a peculiar spreading of the source particle, which precludes preparation of a spatially localized particle state at finite times by operating with arbitrary free particle states in the remote past. This was shown by evaluating the effective electromagnetic field of an electron, regularized by means of the momentum cutoff method appropriately modified to allow factorization of the infrared contributions in multi-loop diagrams (called λ -regularization in Ref. [19]). Namely, the electron was assumed to be prepared in the remote past in such a way that it would be spatially localized at a finite time if the electron was noninteracting. Its effective field was found to vanish at any given spatial point in the limit of removed regularization, in a way that respects the total charge conservation. It was argued that the momentum cutoff can be endowed with a physical meaning as estimating the inverse duration of the measurement process, which made it possible to explicitly describe the electron evolution subjected to the irreversible spreading, and to estimate its possible observational effects.

Except for a modification of the regularization scheme, investigation carried out in Ref. [19] employs the conventional method of calculating the effective field, based on consideration of the

³ This is actually the only reason to get rid of infrared-divergent contributions to the effective gravitational field, for the radiative corrections to the gravitational form factors do not vanish in the large-mass limit.

system on an infinite time interval. This investigation is therefore incomplete in view of what has been said regarding the role of the temporal extent in studying the infrared singularity. In addition to that, the use of an auxiliary infrared regularization raises the question about scheme dependence of the obtained results. At last, it is necessary to establish an exact form of the time-dependence of the electron state affected by the infrared singularity. Thus, we have to consider the electron evolution on a finite time interval, and to determine the leading large-time contributions to the electron density matrix. Finiteness of the time interval renders Feynman integrals infrared-convergent, removing thereby the question of scheme dependence, as there is no need in auxiliary infrared regularization. The infrared singularity is now contained in the large-time asymptotic of the effective field, or equivalently, of the effective electromagnetic current of the electron, and the problem is to consistently extract this asymptotic. It turns out that this requires significant modification of the standard calculational scheme, because restricting the consideration to a finite interval raises the issue of initial conditions for the electromagnetic field, which in turn enforces using the canonical Coulomb gauge instead of the covariant one, to avoid violation of the Gauss law in the initial state. It will be shown below how these issues are interrelated, and why they do not arise in the standard formulation based on the adiabatic switching of the interaction. The paper is organized as follows. The main tools to be used to study electron evolution in a photon bath are described in Section 2.1. In Section 2.2 we identify the time scales characterizing two essentially different stages of the electron evolution—the infrared thermalization corresponding to the infrared singularity, and the usual relaxation of the electron momentum, described by a kinetic equation. Here we also give a heuristic derivation of the irreversible spreading. The complications introduced by finiteness of the time interval are discussed in detail in Section 2.3. It is shown, in particular, that the standard procedure of transition from a canonical gauge to the Lorentz gauge cannot be accomplished in the usual way: the gauge-non-invariance of the electron density matrix leads to appearance of a Lorentz-non-invariant term in the Lagrangian. The resulting modification of the Feynman rules in the Schwinger–Keldysh method is described in Section 3. The infrared thermalization is studied in Section 4. First of all, the set of diagrams contributing to the large-time asymptotic of the effective current is identified in Section 4.1; this set turns out to be different from that representing the effective current within the λ -regularization when the initial electron state is specified at $t = -\infty$. Factorization and summation of the infrared contributions are then performed in Section 4.2. Section 5 contains an alternative derivation of the main result, which uses specifics of the four-dimensional Feynman integrals to reduce the problem of extracting the large-time asymptotic to solving a differential equation for the electron density matrix. Some applications are given in Section 6: the physical meaning of the infrared singularity as representing an irreversible spreading of electric charges is illustrated in Section 6.1 by working out evolution of a Gaussian wavepacket. The effect of infrared thermalization on the electron diffraction in the classic two-slit experiment is determined in Section 6.2, and is shown to be detectable in a proper experimental arrangement already at room temperature. Finally, it is demonstrated in Section 7 how interaction of the electron with non-infrared photons leads to the usual relaxation of the electron momentum. This is shown by using the method of Section 5 to obtain a differential equation for the electron momentum distribution, which turns out to be the usual kinetic equation. Conclusions are drawn in Section 8. The paper has two appendices one of which discusses the role of the non-invariant term in the Lagrangian in the Lorentz gauge, and the other contains a derivation of the quantum entropy of electron described by a Gaussian wavepacket.

2. General formulation

2.1. The model

Consider a non-relativistic electron of mass m interacting with virtual and real photons in equilibrium at finite temperature⁴ $T \ll m$. The expectation (effective) value of any physical quantity

⁴ We use relativistic units $\hbar = c = 1$. Also, Minkowski metric is $\eta_{\mu\nu} = \text{diag}\{+1, -1, -1, -1\}$.

F that describes the system at time t can be found as

$$F^{\text{eff}}(t) = \langle F(t) \rangle, \quad (1)$$

where $F(t)$ is the corresponding gauge-invariant Heisenberg operator built of fermion and electromagnetic fields, and averaging over the given state of the system is denoted by angular brackets. This formula applies at finite temperatures as well as at $T = 0$, and according to the general rules of quantum theory, it yields the observable averaged over series of measurements in the given state. We shall deal with two closely related objects: the effective electromagnetic current of the electron, J_μ^{eff} , and the effective electromagnetic field $F_{\mu\nu}^{\text{eff}}$. The operators $J_\mu(x) = \bar{\psi}(x)\gamma_\mu\psi(x)$ and $F_{\mu\nu}[A(x)] = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$ are defined at every spacetime point x . In view of linearity of the field strength, one has $F_{\mu\nu}^{\text{eff}}(x) = \partial_\mu A_\nu^{\text{eff}}(x) - \partial_\nu A_\mu^{\text{eff}}(x)$, where $A_\mu^{\text{eff}}(x)$ is the effective electromagnetic potential, whereas linearity of the field equations implies that

$$\partial_\mu F_{\mu\nu}^{\text{eff}}(x) = eJ_\nu^{\text{eff}}(x). \quad (2)$$

Since the electron travels in a bath of infinite extent, it is never in equilibrium with photons, and to further specify the state of the system one has to introduce an initial condition. The simplest and most important in applications is the condition that the electron is spatially localized near a point \mathbf{x}_0 at some instant $t_0 < t$. It is convenient to assume that this state results from an auxiliary measurement performed on the electron at times $\leq t_0$, which is complete in that at the instant t_0 the electron is statistically independent of its surrounding. To put it differently, we assume that the effective electric field of the electron at t_0 is the Coulomb field, so that all radiative corrections to it arise as the result of entanglement of the electron with the electromagnetic field at later times. To express this mathematically, it is useful to go over to the Schrödinger picture. Then the condition of statistical independence at t_0 means that the initial density matrix of the system is the product of the electron and photon density matrices: $\varrho_0 e^{-\beta H_\phi} / \mathcal{N}$. Here ϱ_0 is the electron density matrix at the instant t_0 in the Schrödinger representation, the second factor is the density matrix of photons in equilibrium, with H_ϕ the Hamiltonian of free photons, $\beta = 1/T$, and \mathcal{N} is the normalization constant. Let $U(t, t_0)$ denote the evolution operator on the interval (t_0, t) . Then Eq. (1) specified to the given initial condition takes the form

$$F^{\text{eff}}(t) = \mathcal{N}^{-1} \text{Tr} \left(U(t_0, t) F U(t, t_0) \varrho_0 e^{-\beta H_\phi} \right), \quad (3)$$

where $F \equiv F(t_0)$, and the trace is over all electron and photon states.

According to the chosen initial condition, $U(t, t_0) \varrho_0 e^{-\beta H_\phi} U(t_0, t)$ is the density matrix of the system in the Schrödinger picture at arbitrary time $t > t_0$. Traced over all photon states, it reduces to the electron density matrix

$$\varrho(t) = \mathcal{N}^{-1} \text{Tr}_\phi \left(U(t, t_0) \varrho_0 e^{-\beta H_\phi} U(t_0, t) \right). \quad (4)$$

Indeed, since creation of the electron–positron pairs is negligible under condition $T \ll m$, the initial state is carried by $U(t, t_0)$ into a one-electron state. Expression (4) is thus nothing but the electron density matrix at time t . It is to be noted that as defined, this matrix takes into account all radiation effects due to the electron–photon interaction, including scattering of the heat bath photons on the electron, and photon emission by the electron (uncontrollable radiation). In fact, the trace in Eq. (4) is over all possible photon states. $\varrho(t)$ plays an important role in subsequent considerations, and naturally appears in the formalism as follows. Consider the case when F is built of the fermion fields only (e.g., the electromagnetic current). Any state vector of the electron interacting with photons can be expanded in the direct products of states describing a free electron, $|e\rangle$, and states describing arbitrary number of free photons, $|\phi\rangle$, all with definite momenta and spin/polarizations. One then has $F|\phi\rangle|e\rangle = |\phi\rangle F|e\rangle$, and using completeness of the products $|\phi\rangle|e\rangle$ the effective value of F at time t can be written as

$$F^{\text{eff}}(t) = \mathcal{N}^{-1} \sum_{\phi, e} \langle e | \langle \phi | U(t, t_0) \varrho_0 e^{-\beta H_\phi} U(t_0, t) F | \phi \rangle | e \rangle = \sum_e \langle e | \varrho(t) F | e \rangle. \quad (5)$$

The electron density matrix helps to expose a certain complementarity between the effective field and another important object—transition probability. Consider the particular case when electron

is initially in a pure state $|\psi_0\rangle$. The initial probability of the photon state ϕ with energy E_ϕ is $w(\phi) = e^{-E_\phi/T} / \sum_\phi e^{-E_\phi/T}$. That is, the initial density matrix of the system takes the form $\sum_{\phi_0} w(\phi_0) |\psi_0\rangle |\phi_0\rangle \langle \phi_0| \langle \psi_0|$; hence the probability to find the system in a state $|\phi\rangle |e\rangle$ at time $t > t_0$ reads

$$\sum_{\phi_0} w(\phi_0) \langle e | \langle \phi | U(t, t_0) | \phi_0 \rangle | \psi_0 \rangle \langle \psi_0 | \langle \phi_0 | U(t_0, t) | \phi \rangle | e \rangle.$$

Being interested solely in the electron evolution, we sum over all final photon states. This yields the probability of transition of the electron into a state $|e\rangle$

$$\sum_{\phi, \phi_0} w(\phi_0) \langle e | \langle \phi | U(t, t_0) | \psi_0 \rangle | \phi_0 \rangle \langle \phi_0 | \langle \psi_0 | U(t_0, t) | e \rangle | \phi \rangle = \langle e | \varrho(t) | e \rangle, \quad (6)$$

where $\varrho(t)$ is given by Eq. (4) with $\varrho_0 = |\psi_0\rangle \langle \psi_0|$.

Thus, the inclusive transition probability and the effective quantities are expressed via the same object—the electron density matrix. An important difference is that in the first instance we deal only with diagonal elements of the density matrix, whereas the effective fields depend also on off-diagonal elements. From the point of view of the infrared problem, this means that the cancellation of infrared singularities in the S -matrix, asserted by the Bloch–Nordsieck theorem and its generalizations to $T \neq 0$ [14–16], implies finiteness of only diagonal elements of the electron density matrix in the limit $t_0 \rightarrow -\infty, t \rightarrow +\infty$. In other words, consideration of the effective field gives an important piece of information about the system, which is not contained in the S -matrix. In fact, the existence of an irreversible charge spreading, revealed by consideration of the effective electromagnetic field, is related precisely to the off-diagonal elements of the electron density matrix.

2.2. Stages of electron thermalization

Our goal is to infer physical consequences of the infrared singularity in the effective electromagnetic field of the electron, or equivalently, in its effective current. Accordingly, we will be interested in the large-time asymptotics of these quantities. There are two essentially different types of contributions which diverge for $t \rightarrow \infty$, when treated within the perturbation theory. They describe different stages of the relaxation process in the system, which are characterized by significantly different time scales. Namely, the faster process is the infrared thermalization described in [19] within the λ -regularization, which takes place in vacuum as well as at finite temperature, and which shows itself in a damping of the off-diagonal elements of the momentum-space density matrix of the electron. Consideration of this process is the main subject of the present paper. To better expose its physical meaning, we give here a highly simplified derivation of the irreversible electron spreading which constitutes essential part of the infrared thermalization. The derivation employs the well-known heuristic description of the relativistic radiative effects, used initially to estimate the Lamb shift. This description is based on the view that the interaction of electron with each of the infinite number of electromagnetic modes, empty or occupied, causes the electron to “jiggle” with respect to the nonrelativistic wavefunction treated as a background, which in the present case is the free-electron wavepacket subject to the usual quantum-mechanical spreading. Assuming that the electron jiggling can be described by a function $\mathbf{r}(t)$, and restoring for a moment the ordinary units, one has for the Fourier component of the “position” $\mathbf{r}(t)$ in the field of a plane wave

$$m \frac{d^2 \mathbf{r}_\omega}{dt^2} = e \mathbf{E}_\omega \sin \omega t,$$

from which one finds the contribution of the given mode to the electron position variance

$$\langle \mathbf{r}_\omega^2 \rangle = \frac{e^2 \mathbf{E}_\omega^2}{2m^2 \omega^4},$$

where angular brackets denote time averaging. The field amplitude $|\mathbf{E}_\omega|$ can be found by noting that the energy of n_ω photons in the given mode is

$$\hbar\omega \left(n_\omega + \frac{1}{2} \right) = \frac{\mathbf{E}_\omega^2}{2} V,$$

where $V \rightarrow \infty$ is the volume occupied by the system. Next, by the time-energy uncertainty relation, only the modes with $\omega \gtrsim \hbar/\tau$ are able to affect the electron during its travel for a time $\tau = t - t_0$. Therefore, substituting $n_\omega = [\exp(\hbar\omega/T) - 1]^{-1}$, and $V\omega^2 d\omega/\pi^2 c^3$ for the number of modes in the interval $(\omega, \omega + d\omega)$, the total position variance takes the form

$$\langle \mathbf{r}^2 \rangle \sim \frac{e^2 \hbar}{\pi^2 c^3 m^2} \int_{1/\tau}^{\infty} \frac{d\omega}{\omega} \left[\frac{1}{e^{\hbar\omega/T} - 1} + \frac{1}{2} \right].$$

The upper limit in this integral is actually $\sim mc^2/\hbar$, as set by the ultraviolet renormalization.

It is seen that in the case of electron freely moving in vacuum ($T = 0$), the radiative effects add

$$\frac{e^2 \hbar}{\pi^2 c^3 m^2} \ln(\tau mc^2/\hbar) \quad (7)$$

to the usual nonrelativistic spreading. At finite temperatures, the corresponding contribution is

$$\delta \langle \mathbf{r}^2 \rangle \sim \frac{e^2 T \tau}{\pi^2 c^3 m^2}. \quad (8)$$

We thus have the following estimate for the characteristic time of the infrared thermalization

$$\tau_1 \sim \frac{m^2 c^2 r^2}{\hbar \alpha T},$$

where $\alpha = e^2/(4\pi\hbar c)$ is the fine-structure constant, and r is the characteristic length of the problem, *e.g.*, the distance between electron and the point of observation of its field, or the fringe spacing of an interference pattern in the two-slit experiment (see Section 6.2), *etc.* This estimate was obtained in [19] by interpreting an expression for the λ -regularized effective electromagnetic field of the electron. An exact expression for $\delta \langle \mathbf{r}^2 \rangle$ will be found below.

The other (slower) stage of the electron thermalization is the usual relaxation of the electron momentum due to its collisions with thermal photons, which occurs only at $T \neq 0$. Its rate can be estimated by noting that the cross-section of the low-energy electron–photon scattering is of order $(\alpha\hbar/mc)^2$, whereas the photon density is $\sim (T/\hbar c)^3$. Hence, the electron mean free time is

$$\tau_2 \sim \frac{\hbar m^2 c^4}{\alpha^2 T^3}.$$

This process is described by the usual kinetic equation for the electron momentum probability distribution (that is, the diagonal elements of the electron density matrix), and will be considered in Section 7.

The ratio of the two time scales,

$$\frac{\tau_1}{\tau_2} \sim \alpha \left(\frac{rT}{\hbar c} \right)^2, \quad (9)$$

is to be considered small within the perturbation theory. Less formally,

$$\frac{\tau_1}{\tau_2} \sim \frac{1}{137} (4.36 r T)^2,$$

where T is assumed to be expressed in kelvins, and r in centimeters, so that the ratio is small in microscopic processes ($r \sim 10^{-8}$ cm) for all practically important temperatures.

Below, the two stages of the electron evolution are considered separately, the reason being that, as was already mentioned, they deal with different elements of the electron density matrix.

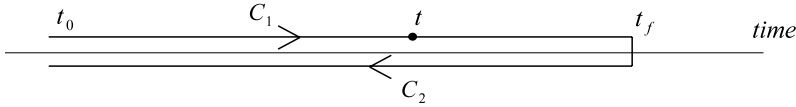


Fig. 1. Time-integration contour in Eq. (10).

2.3. Gauge fixing on a finite time interval

To evaluate F^{eff} , we shall use the Schwinger–Keldysh formalism [20,21], according to which Eq. (3) can be written in the interaction picture as

$$F^{\text{eff}}(t) = \mathcal{N}^{-1} \text{Tr} \left(T_c \left[\exp i \left(\int_C d^4 y L_I(y) \right) F(t) \right] e^{-\beta H_\phi} \rho_0 \right), \quad (10)$$

where L_I is the interaction Lagrangian, y^0 -integration is along the time-contour $C = C_1 \cup C_2$ running from t_0 to $t_f \geq t$ and back as shown in Fig. 1, and T_c denotes operator ordering along this contour. It is conventional to take the limit $t_f \rightarrow \infty$ in this formula, but the choice $t_f = t$ is more appropriate for our purposes.

To completely specify the scheme of calculations, one has to fix gauge invariance of the theory, which can be done by including a gauge-fixing term into Lagrangian density in Eq. (10). The amount of computational labor essentially depends on the gauge choice, and covariant gauges are well-known to be of great advantage over canonical ones in this respect, but their use requires special justification. A mere reference to gauge-independence of the effective electromagnetic current is not sufficient: one has to prove that the use of covariant rules gives the same results as the original canonical method. It turns out that Eq. (10) is the case where covariant techniques are not applicable, the failure being directly related to finiteness of the initial instant t_0 . To explain the point, it will be convenient to follow the general Faddeev–Popov method [22] of transition to covariant gauges, despite the fact that the gauge group in the present case is only Abelian.

We choose to start with the Coulomb gauge, as it admits canonical quantization wherein the gauge condition can be regarded as an operator relation (i runs over 1, 2, 3)

$$\partial_i A^i = 0. \quad (11)$$

According to the standard quantization procedure,⁵ Eq. (10) written in terms of the functional integral takes the form

$$F^{\text{eff}}(t) = \mathcal{N}^{-1} \int d\psi d\bar{\psi} \langle \psi^1 | \rho_0 | \psi^2 \rangle \int dA_\nu \langle A^1 | e^{-\beta H_\phi} | A^2 \rangle B_C \delta(\partial_i A_i) \\ \times \exp i \left(\int_C d^4 y L(y) \right) F(t). \quad (12)$$

Here $d\psi d\bar{\psi}$ is shorthand for the fermion functional integral measure,

$$\prod_{\mathbf{x}, t' \in C_1} d\psi^1(\mathbf{x}, t') d\bar{\psi}^1(\mathbf{x}, t') \prod_{\mathbf{x}, t' \in C_2} d\psi^2(\mathbf{x}, t') d\bar{\psi}^2(\mathbf{x}, t'),$$

and similarly for the A -field; the superscripts 1, 2 distinguish fields belonging to the corresponding branch of the time contour C ; integration is over all fields satisfying $\psi^1(t) = \psi^2(t)$, $\bar{\psi}^1(t) = \bar{\psi}^2(t)$, $A^1(t) = A^2(t)$. Next, $|\psi^{1,2}\rangle$, $|A_i^{1,2}\rangle$ are eigenvectors of the field operators at t_0 , e.g., $\hat{\psi}(t_0, \mathbf{x}) |\psi^1\rangle = \psi^1(t_0, \mathbf{x}) |\psi^1\rangle$, $\hat{A}_i(t_0, \mathbf{x}) |A_i^1\rangle = A_i^{1\text{tr}}(t_0, \mathbf{x}) |A_i^1\rangle$, etc., where $A_i^{1,2\text{tr}}$ are the transversal components of $A_i^{1,2}$ ($\hat{A}_i \equiv \hat{A}_i^{\text{tr}}$ in the Coulomb gauge). Finally, the factor B_C (a formally infinite constant) is defined by

$$B_C \int d\omega \delta(\partial_i A_i^\omega) = 1, \quad (13)$$

⁵ Details of canonical quantization in the Coulomb gauge can be found in Refs. [23,24].

where integration is over the gauge group, and A_v^ω is the result of action of the gauge group element ω on the field A_v . Being built of the transversal components of \mathbf{A} , vector $|A_i\rangle$ is independent of the non-dynamical variables— A_0 and the longitudinal component of \mathbf{A} . In fact, integration over A_0 yields the Coulomb law

$$A_0 = -\Delta^{-1}J_0, \quad (14)$$

which is thus also treated as an operator relation. Now, to go over to a covariant (Lorentz) gauge, we use the following trick [25]. Let us define B_L (another infinite constant) by

$$B_L \int d\omega \delta(\partial^\mu A_\mu^\omega(x) - a(x)) = 1,$$

with $a(x)$ an arbitrary function of spacetime coordinates, and multiply (12) by unity:

$$F^{\text{eff}}(t) = \mathcal{N}^{-1} \int d\psi d\bar{\psi} \langle \psi^1 |_{\mathcal{Q}_0} | \psi^2 \rangle \int dA_v \langle A_i^1 | e^{-\beta H_\phi} | A_i^2 \rangle B_C \delta(\partial_i A_i) \\ \times B_L \int d\omega \delta(\partial^\mu A_\mu^\omega - a) \exp i \left(\int_C d^4 y L(y) \right) F(t). \quad (15)$$

A change of integration variables $\psi^\omega \rightarrow \psi$, $\bar{\psi}^\omega \rightarrow \bar{\psi}$, $A^\omega \rightarrow A$, $\omega \rightarrow \omega^{-1}$ gives

$$F^{\text{eff}}(t) = \mathcal{N}^{-1} \int d\omega \int d\psi d\bar{\psi} \langle \psi^{1\omega} |_{\mathcal{Q}_0} | \psi^{2\omega} \rangle \int dA_v \langle A_i^{1\omega} | e^{-\beta H_\phi} | A_i^{2\omega} \rangle B_C \delta(\partial_i A_i^\omega) \\ \times B_L \delta(\partial^\mu A_\mu - a) \exp i \left(\int_C d^4 y L(y) \right) F(t, \mathbf{x}),$$

where gauge invariance of the Lagrangian L and of the operator F was taken into account. Moreover, since the photon state vectors are defined by transversal components of \mathbf{A} , they are gauge-invariant, $|A_i^{1,2\omega}\rangle = |A_i^{1,2}\rangle$. But this is not true of the fermion states $|\psi^{1,2}\rangle$. Therefore, $\langle \psi^{1\omega} |_{\mathcal{Q}_0} | \psi^{2\omega} \rangle \neq \langle \psi^1 |_{\mathcal{Q}_0} | \psi^2 \rangle$, for \mathcal{Q}_0 is not gauge-invariant. Therefore, integration over ω leads to the following expression for the effective current in the Lorentz gauge

$$F^{\text{eff}}(t) = \mathcal{N}^{-1} \int d\psi d\bar{\psi} \int dA_v \langle e^{ie\alpha(A)} \psi^1 |_{\mathcal{Q}_0} | e^{ie\alpha(A)} \psi^2 \rangle \langle A_i^1 | e^{-\beta H_\phi} | A_i^2 \rangle \\ \times B_L \delta(\partial^\mu A_\mu - a) \exp i \left(\int_C d^4 y L(y) \right) F(t),$$

where $\alpha(A) = -\Delta^{-1} \partial_i A_i$. If the limit $t_0 \rightarrow -\infty$ is taken, and the interaction is negligible in the remote past, the function $\langle e^{ie\alpha(A)} \psi^1 |_{\mathcal{Q}_0} | e^{ie\alpha(A)} \psi^2 \rangle$ can be replaced by $\langle \psi^1 |_{\mathcal{Q}_0} | \psi^2 \rangle$, for then the interaction can be adiabatically switched off in the remote past,

$$e \rightarrow 0, \quad t \rightarrow -\infty,$$

which kills the phase factor $e^{ie\alpha(A)}$. This is exactly what happens in the S -matrix formalism, and is realized there through the use of in- and out-states. At finite t_0 , one can get rid of the phase factor by making the inverse change of the fermion integration variables $\psi \rightarrow e^{-ie\alpha(A)} \psi$, $\bar{\psi} \rightarrow e^{ie\alpha(A)} \bar{\psi}$, but then an additional term appears in the Lagrangian

$$F^{\text{eff}}(t) = \mathcal{N}^{-1} \int d\psi d\bar{\psi} \int dA_v \langle \psi^1 |_{\mathcal{Q}_0} | \psi^2 \rangle \langle A_i^1 | e^{-\beta H_\phi} | A_i^2 \rangle \\ \times B_L \delta(\partial^\mu A_\mu - a) \exp i \left(\int_C d^4 y [L(y) - e \bar{\psi} \gamma^\mu \psi \Delta^{-1} \partial_\mu \partial_i A_i] \right) F(t).$$

Finally, independence of a allows one to average this equation with a weight $\exp(-i \int d^4 y a^2(y)/2\xi)$, $\xi = \text{const}$, to obtain

$$F^{\text{eff}}(t) = \mathcal{N}^{-1} \int d\psi d\bar{\psi} \int dA_v \langle \psi^1 |_{\mathcal{Q}_0} | \psi^2 \rangle \langle A_i^1 | e^{-\beta H_\phi} | A_i^2 \rangle \\ \times B_L \exp i \left(\int_C d^4 y L_i(y) \right) F(t), \quad (16)$$

where

$$L_l = L - (\partial^\mu A_\mu)^2 / 2\xi - e J^\mu \Delta^{-1} \partial_\mu \partial_i A_i. \quad (17)$$

The averaging affects only the normalization factor denoted \mathcal{N} , as before. The Lagrangian L_l generates Feynman rules to be used to compute the effective field in the Lorentz gauge. It is seen that in contrast to the S -matrix formalism (based on taking the limit $t_0 \rightarrow -\infty$, $t_f \rightarrow +\infty$), these rules are not Lorentz-covariant: explicit Lorentz invariance is broken by the last term in Eq. (17), which largely deprives Lorentz gauge of its advantage over the Coulomb gauge. That the non-invariant term in L_l cannot be discarded is demonstrated in [Appendix A](#) where it is shown that doing so would violate the Gauss law already in the tree approximation.⁶

At last, it is worth mentioning that

$$\partial^\mu A_\mu^{\text{eff}}(x) = \text{const}. \quad (18)$$

Indeed, it follows from Eq. (16) that A_μ^{eff} satisfies

$$\square A_\mu^{\text{eff}}(x) - \left(1 - \frac{1}{\xi}\right) \partial_\mu \partial^\nu A_\nu^{\text{eff}}(x) = J_\mu^{\text{eff}}(x).$$

On the other hand, Eq. (2) is still in force, implying that $\partial_\mu \partial^\nu A_\nu^{\text{eff}}(x) = 0$.

3. Perturbation theory

3.1. Feynman rules

Perturbation expansion of Eq. (10) generates expressions of the form

$$\text{Tr}_\phi \text{Tr}_e \left(T_c \left[A_\mu(x_1) J^\mu(x_1) A_\nu(x_2) J^\nu(x_2) \dots \right] e^{-\beta H_\phi} \varrho_0 \right)$$

which on account of commutativity of the interaction picture operators factorize to

$$\text{Tr}_\phi \left(T_c \left[A_\mu(x_1) A_\nu(x_2) \dots \right] e^{-\beta H_\phi} \right) \text{Tr}_e \left(T_c \left[J^\mu(x_1) J^\nu(x_2) \dots \right] \varrho_0 \right).$$

The resulting Green functions can be further expanded in the products of particle propagators using the real-time techniques [26,27]. As usual, the T_c -ordering promotes the field propagators into 2×2 matrices, e.g.,

$$D^{(ij)}(x-y) = i \text{Tr}_\phi \left(T_c \left[A_\mu^{(i)}(x) A_\nu^{(j)}(y) \right] e^{-\beta H_\phi} \right)$$

for the electromagnetic field, and similarly for the fermion field; the matrix indices i, j take the value 1(2) for fields on the forward (backward) branch of the contour C . In momentum space, the electron propagator has the form

$$\begin{aligned} D^{(11)}(q) &= -\tilde{D}^{(22)}(q) = \frac{\not{q} + m}{m^2 - q^2 - i0}, \\ D^{(21)}(q) &= 2\pi i \theta(q_0) \delta(m^2 - q^2) (\not{q} + m), \\ D^{(12)}(q) &= 2\pi i \theta(-q_0) \delta(m^2 - q^2) (\not{q} + m), \end{aligned} \quad (19)$$

where the tilde symbolizes an operation of complex conjugation with respect to which the Dirac matrices are real. The photon propagator reads

$$\begin{aligned} D_{\mu\nu}^{(11)}(k) &= -\tilde{D}^{(22)}(k) = \left[\frac{1}{k^2 + i0} - 2\pi i n(\mathbf{k}) \delta(k^2) \right] d_{\mu\nu}(k), \\ D_{\mu\nu}^{(21)}(k) &= -2\pi i [\theta(k_0) + n(\mathbf{k})] \delta(k^2) d_{\mu\nu}(k), \\ D_{\mu\nu}^{(12)}(k) &= -2\pi i [\theta(-k_0) + n(\mathbf{k})] \delta(k^2) d_{\mu\nu}(k), \quad n(\mathbf{k}) = \frac{1}{e^{\beta|\mathbf{k}|} - 1}, \end{aligned} \quad (20)$$

⁶ Note that the charge conservation $\partial_\mu J^\mu = 0$ does not entail vanishing of the non-invariant contribution: integration by parts in Eq. (16) yields an integral of $e(J_0^2 - J_0^2) \Delta^{-1} \partial_i A_i$ over the hyperplane $y^0 = t_0$.

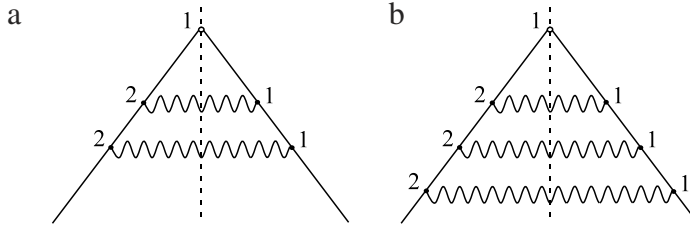


Fig. 2. Diagrams contributing to the effective electron current. (a) A fourth-order diagram describing the effect of electron–photon scattering. (b) A sixth-order diagram describing the effect of electron–photon scattering accompanied by a photon emission by the electron. Vertical broken lines show unitarity cuts to be made to relate these diagrams to the scattering cross-sections.

where

$$d_{\mu\nu}(k) = \eta_{\mu\nu} - \frac{k_0 k_\mu \eta_\nu + k_0 k_\nu \eta_\mu - k_\mu k_\nu}{k^2}, \quad \eta^\mu = (1, \mathbf{0}) \quad (21)$$

in the Coulomb gauge, whereas in the Lorentz gauge

$$d_{\mu\nu}(k) = \eta_{\mu\nu} + (\xi - 1) \frac{k_\mu k_\nu}{k^2}. \quad (22)$$

The interaction vertices are generated by the Lagrangian L_I , and are assigned indices 1 or 2, depending on the branch of the contour C to which the given vertex belongs, with an extra factor (-1) for each type-2 vertex. The point of observation x is assigned index 1. Each propagator connects vertices of the types assigned to the propagator ends, and integration in the vertices is over all space and over the time interval (t_0, t) . Diagrammatically, the electron and photon propagators will be depicted by straight and wavy lines, respectively. The operators being averaged ($A_\mu(x)$, $J^\mu(x)$, etc.) will be collectively denoted by an open circle, while the interaction vertices by full circles.

The diagrammatic representation of the effective field in the Schwinger–Keldysh technique has its specifics when compared to the usual S -matrix diagrammatics. As far as the single electron problem is considered, this primarily concerns representation of the photons present in the system. The effect of the heat bath photons is taken into account by the term proportional to $n(\mathbf{k})$ in Eq. (20). As to the photons produced by the electron, their effect is also encoded in the internal photon lines, in contrast to the S -matrix case where photons emitted by charges (in particular, the uncontrollable radiation) are represented by external lines attached to the charged particle propagators. It is integration of the scattering cross-sections over momenta of these photons that cancels the infrared divergences due to virtual photons appearing in the loops. The difference in the graphical representation arises because in the S -matrix case, the photons emitted by charges are not present in the *in*-state, and appear only in the *out*-state as a result of the scattering, whereas the effective field is evaluated entirely over the given *in*-state, and no *out*-state ever appears in the formalism. In this respect, the effective field diagrammatics bears some resemblance with that of the scattering cross-sections transformed using the unitarity relations. This is illustrated by Fig. 2(a) which represents one of diagrams describing the effect of the electron–photon scattering on the electron current. The vertical broken line shows the cut to be made to relate this diagram with the cross-section of the Compton scattering (note that all photon momenta in this diagram are on the mass shell). This effect will be considered in detail in Section 7, where it will be shown explicitly that the expression for the electron–photon scattering cross-section is contained in diagrams similar to that in Fig. 2(a), appearing in the kinetic equation for the electron momentum probability distribution. Analogously, shown in Fig. 2(b) is one of diagrams describing the effect of photon emission by the electron, which accompanies the scattering of heat-bath photons by the electron, etc.

Because of finiteness of the time contour C , energy is not conserved in the interaction vertices. As a result, the usual δ -functions expressing energy conservation in the S -matrix theory become smeared,

so that the vertex factor takes the form, in the Coulomb gauge,

$$-ie\gamma^\mu \bar{\Delta}_t(v), \quad \bar{\Delta}_t(v) = \frac{e^{iv^0 t} - e^{iv^0 t_0}}{iv_0} (2\pi)^3 \delta^{(3)}(\mathbf{v}), \quad (23)$$

where v is the sum of outgoing 4-momenta, to be referred to in what follows as the *residual momentum*. In the Lorentz gauge, the non-invariant term in L_l modifies the vertex to

$$-ie\gamma^\alpha g_\alpha^\mu(k) \bar{\Delta}_t(v), \quad g_\alpha^\mu(k) = \delta_\alpha^\mu + \frac{k_\alpha (k^\mu - \eta^\mu k_0)}{\mathbf{k}^2}. \quad (24)$$

It is to be noted that the gauge invariance of F implies ξ -independence of F^{eff} , for $g_\alpha^\mu(k)k_\mu \equiv 0$, so that $d_{\mu\nu}$ in the Lorentz gauge can be replaced by $\eta_{\mu\nu}$. Moreover, regarding the analytic structure of Feynman diagrams, the Lorentz gauge is completely equivalent to the Coulomb gauge when F does not involve A_μ . In that case, there are only internal photon lines each of which has its ends contracted with the tensor $g_\alpha^\mu(k)$:

$$g_\alpha^\mu(k) \eta_{\mu\nu} g_\beta^\nu(k).$$

A simple calculation shows that the latter is exactly the expression (21) for $d_{\alpha\beta}$ in the Coulomb gauge.

As a result of the smearing of the energy δ -function, the effective electromagnetic field and the effective current become infrared finite, so that no special regularization such as the λ -regularization introduced in [19] is needed in investigating the infrared effects.

3.2. Pole prescriptions in Feynman integrals

Regarding pole prescriptions symbolized by $\pm i0$ in Eqs. (19), (20), it is worth to make the following technical remark. As we shall see below, evaluation of multiple Feynman integrals is often facilitated by the use of residual momenta as integration variables. However, this requires some care to avoid appearance of ambiguities when changing the order of integration. As an illustration, consider the diagram in Fig. 3. Its contribution is proportional to

$$\int \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{\bar{\Delta}_t(q_2 - q_1 - k) \bar{\Delta}_t(q_1 - q + k)}{m - \not{q}_2 - i0} \gamma^\alpha \frac{D_{\alpha\beta}^{(11)}(k)}{m - \not{q}_1 - i0} \gamma^\beta. \quad (25)$$

The imaginary infinitesimals ($-i0$) specify contours of integration over q_1^0 and q_2^0 , whose independence allows interchanging the order of integration. In order to go over to integration with respect to residual momenta v_1, v_2 , we first change the integration variable $q_1 \rightarrow v_1$ according to $q_1 = q - k + v_1$, and then change $q_2 \rightarrow v_2$ using the relation $q_2 = q + v_1 + v_2$, in which v_1 is treated as a complex parameter. Eq. (25) thus takes the form

$$\int \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 v_1}{(2\pi)^4} \int_{v_1} \frac{d^4 v_2}{(2\pi)^4} \frac{\bar{\Delta}_t(v_1) \bar{\Delta}_t(v_2)}{m - \not{q} - \not{v}_1 - \not{v}_2 - i0} \gamma^\alpha \frac{D_{\alpha\beta}^{(11)}(k)}{m - \not{q} + \not{k} - \not{v}_1 - i0} \gamma^\beta,$$

where the subscript v_1 indicates that the position of integration contour over v_2 , specified by the left pole factor in the integrand, depends also on v_1 which runs the contour specified by the other pole factor. Interchanging the order of integration with respect to v_1 and v_2 now is not legitimate, as it leads to ambiguity in integrating the pole $1/[m^2 - (q + v_1 + v_2)^2]$ with respect to q_1^0 . But the last formula and its generalizations turn out to be useful even when they do not admit changing the order of integration.

We will have more to say on this issue in Sections 4.1 and 4.2.

4. Infrared thermalization

We now proceed to the calculation of the infrared-singular contribution to the effective electromagnetic current. The estimates (7) and (8) show that there are two types of contributions

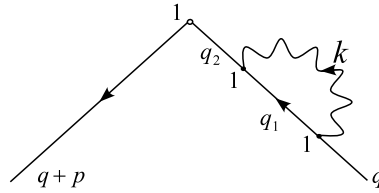


Fig. 3. A one-loop diagram contributing to the effective current.

growing with time: a linear (leading) that exists only at finite temperatures, and a logarithmic. To find the latter requires direct summation of the diagrams to all orders of perturbation theory, to be done in the present section, whereas the former can be found also in a simpler way to be described in Section 5.

Upon transition to the momentum space, Eq. (5) with $F = J_\mu$ takes the form

$$J_\mu^{\text{eff}}(\mathbf{x}, t) = \sum_{\sigma, \sigma'} \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \frac{d^3 \mathbf{p}}{(2\pi)^3} \varrho_{\sigma\sigma'}(t; \mathbf{q}, \mathbf{q} + \mathbf{p}) \frac{\bar{u}_{\sigma'}(\mathbf{q} + \mathbf{p}) \gamma_\mu u_\sigma(\mathbf{q})}{\sqrt{2\varepsilon_{\mathbf{q}+\mathbf{p}}} \sqrt{2\varepsilon_{\mathbf{q}}}} e^{-i\mathbf{p}\mathbf{x}}, \quad (26)$$

where $\varepsilon_{\mathbf{q}} = \sqrt{m^2 + \mathbf{q}^2}$, and $\varrho(t; \mathbf{q}, \mathbf{q}')$ is the electron density matrix in the momentum space representation; by definition,

$$\varrho_{\sigma\sigma'}(t; \mathbf{q}, \mathbf{q}') = \langle e | \varrho(t) | e' \rangle, \quad (27)$$

where \mathbf{q}, σ are the electron momentum and spin in the state $|e\rangle$, and \mathbf{q}', σ' same for $|e'\rangle$. This matrix is normalized on unity,

$$\sum_{\sigma} \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \varrho_{\sigma\sigma}(\mathbf{q}, \mathbf{q}) = 1. \quad (28)$$

Finally, the bispinor amplitudes $u_\sigma(\mathbf{q})$ are also normalized on unity, $\bar{u}_\sigma u_\sigma = 1$, and satisfy

$$(\not{q} - m)u_\sigma(\mathbf{q}) = 0.$$

The effective electromagnetic field can be found from Eq. (2). Note that in the Coulomb gauge the scalar potential reads simply

$$A_0^{\text{eff}}(\mathbf{x}, t) = e \sum_{\sigma, \sigma'} \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \frac{d^3 \mathbf{p}}{(2\pi)^3} \varrho_{\sigma\sigma'}(t; \mathbf{q}, \mathbf{q} + \mathbf{p}) \frac{\bar{u}_{\sigma'}(\mathbf{q} + \mathbf{p}) \gamma_0 u_\sigma(\mathbf{q})}{\sqrt{2\varepsilon_{\mathbf{q}+\mathbf{p}}} \sqrt{2\varepsilon_{\mathbf{q}}}} \frac{e^{-i\mathbf{p}\mathbf{x}}}{p^2}. \quad (29)$$

Using this expression and taking into account the gauge condition $\partial_i A_i = 0$, it is not difficult to verify that the effective electromagnetic field satisfies Gauss law in the infinite space.

The contributions we are interested in come from integration over small virtual and residual momenta. The results of [19] imply that these contributions diverge in the limit $t_0 \rightarrow -\infty$, or equivalently $t \rightarrow \infty$, as $\ln(mt)$ in the vacuum case, and as Tt at finite temperature. Typical infrared-singular diagram is shown in Fig. 4. Its internal structure is further specified by Fig. 5, where k, κ denote virtual momenta, and v, u the residual momenta; henceforth, we set $t_0 = 0$, without restricting generality. The factor $\exp -i(\sum_i v_i^0 + \sum_j u_j^0)t$ is brought in by the current operator in the interaction picture, whereas factors $(1 - \exp iv_0 t)$, $(1 - \exp iu_0 t)$ come from the vertex functions $\bar{\Delta}_t$. It is easily verified that the product of all these factors is invariant under the following replacement

$$e^{-i(\sum_i v_i^0 + \sum_j u_j^0)t} \rightarrow 1, \quad \bar{\Delta}_t(v) \rightarrow \Delta_t(v) = i \frac{e^{-iv_0 t} - 1}{v_0} (2\pi)^3 \delta^{(3)}(\mathbf{v}), \quad (30)$$

a fact that will be used below.

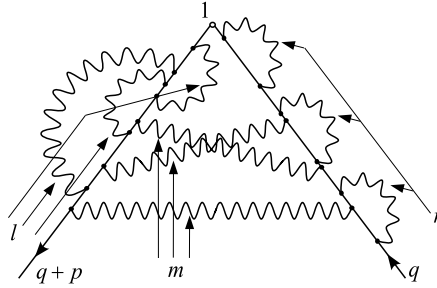


Fig. 4. General infrared-divergent contribution to the effective current. q and $q + p$ are the electron 4-momenta.

4.1. Approximations

Since we are interested in the low-energy properties of the effective current, and since we have assumed nonrelativistic conditions for the electron, in what follows we shall systematically neglect radiative corrections to the electron and photon self-energy as well as to their interaction, but only if they represent *finite relative* corrections to the effective field, that is, if they give rise to factors $[1 + O(|\mathbf{q}|/m)]$ or $[1 + O(|\mathbf{p}|/m)]$ in the leading term. Also, we shall use the condition $T \ll m$ underlying our model to omit terms proportional to T/m , unless such term is leading. In this approximation, the momentum space density matrices at the instants t and t_0 are related by

$$\varrho_{\sigma\sigma'}(t; \mathbf{q}, \mathbf{q}') = \varrho_{\sigma\sigma'}(t_0; \mathbf{q}, \mathbf{q}') R(t; \mathbf{q}, \mathbf{q}') e^{ip_0 t}, \quad (31)$$

where $p_0 = \varepsilon_{\mathbf{q}+\mathbf{p}} - \varepsilon_{\mathbf{q}}$, and the scalar factor $R(t; \mathbf{q}, \mathbf{q}')$ incorporates radiative corrections ($R = 1$ in the tree approximation). Indeed, the γ -matrix structure of a N -loop infrared-divergent diagram contributing to J_μ^{eff} is (see Fig. 5)

$$\begin{aligned} & \sum_{\sigma, \sigma'} \varrho_{\sigma\sigma'}(t_0; \mathbf{q}, \mathbf{q}') \bar{u}_{\sigma'}(\mathbf{q}') \gamma^{\beta_1} (\not{q}' - \not{k}_1 - \not{p}_1 + m) \cdots \gamma^{\beta_f} \left(\not{q}' - \sum_j [\not{k}_j + \not{p}_j] + m \right) \gamma_\mu \\ & \times (\not{q} - \not{k}_1 + \not{p}_1 + m) \gamma^{\alpha_1} \cdots \left(\not{q} - \sum_i [\not{k}_i - \not{p}_i] + m \right) \gamma^{\alpha_s} u_\sigma(\mathbf{q}), \end{aligned}$$

where $s + f = 2N$. At zero temperature, the Feynman integrals diverge logarithmically, so that \not{k} , \not{p} , etc. in this expression give rise to infrared-finite terms. For $T \neq 0$, the divergence is linear, and the corresponding subleading contribution diverges logarithmically. On dimensional grounds, its relative order is $O(T/m)$, so that this contribution can be omitted. Thus, the above expression can be replaced by

$$\begin{aligned} & \sum_{\sigma, \sigma'} \varrho_{\sigma\sigma'}(t_0; \mathbf{q}, \mathbf{q}') \bar{u}_{\sigma'}(\mathbf{q}') \gamma^{\beta_1} (\not{q}' + m) \cdots \gamma^{\beta_f} (\not{q}' + m) \gamma_\mu (\not{q} + m) \gamma^{\alpha_1} \cdots (\not{q} + m) \gamma^{\alpha_s} u_\sigma(\mathbf{q}) \\ & = \sum_{\sigma, \sigma'} \varrho_{\sigma\sigma'}(t_0; \mathbf{q}, \mathbf{q}') 2q'^{\beta_1} \cdots 2q'^{\beta_f} 2q^{\alpha_1} \cdots 2q^{\alpha_s} \bar{u}_{\sigma'}(\mathbf{q}') \gamma_\mu u_\sigma(\mathbf{q}), \end{aligned}$$

from which Eq. (31) follows. This relation will be derived in a more direct way in Section 5. The above reasoning also implies a similar simplification in denominators of the electron propagators

$$\frac{1}{m^2 - (q - \sum k_i + \sum v_i)^2 \pm i0} \rightarrow \frac{1}{2q (\sum k_i - \sum v_i) \pm i0}, \quad \text{etc.}$$

These simplifications of the vertex factors and propagators are quite similar to those underlying classic treatments of the infrared catastrophe in the scattering theory [2–4]. We might go further and adopt the Bloch–Nordsieck model assumption that the electron propagators can be taken in the form

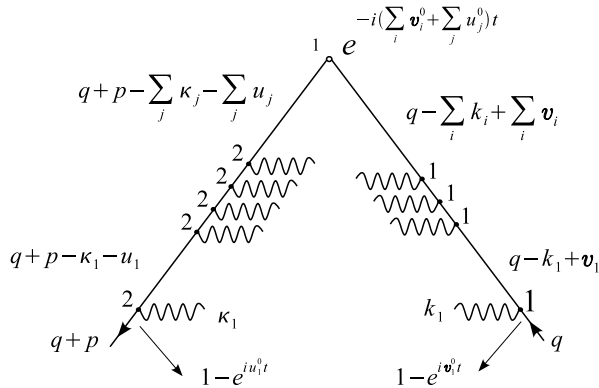


Fig. 5. Detailed structure of diagrams contributing to the effective current. The horizontal photon lines are supposed to be arbitrarily paired.

$D^{(11)}(q) = (m - u^\mu q_\mu)^{-1}$ etc., which is often used in the modern analyses even in cases $T > m$ (see, e.g., [28]). Although with certain qualifications regarding the choice of the vector u^μ on the incoming and outgoing electron lines, this replacement can be justified in the present case too, we prefer not to use it, and to prove factorization of the infrared contributions following the lines of Refs. [2–4]. It should be emphasized in this connection that the effects we are interested in cannot be described within the strict eikonal approximation in which quantum fluctuations of the electron wavefunction, caused by the radiative effects, are completely neglected. These fluctuations are just what was qualitatively described in Section 2.2 as the electron jiggling.

Next, we note that graphs with a 1-vertex appearing to the left of a 2-vertex can be omitted, because they involve the function $D^{(12)}(q) \sim \theta(-q_0)$, and therefore do not contribute at small loop momenta. In particular, all vertices on the incoming electron line (the right slope of the diagram in Fig. 5) must be type-1. Less trivial is the fact that all vertices on the outgoing electron line (the left slope of the diagram in Fig. 5) are to be of type-2 to give rise to a nonvanishing contribution. To see this, imagine for a moment that the rightmost vertex on the outgoing electron line is of type-1. Then following the recipe formulated in Section 3.2, we first perform integration over u_f , and find that

$$\int \frac{du_f^0}{2\pi i} \frac{e^{-iu_f^0 t} - 1}{u_f^0} \frac{1}{2q \left(\sum_j \kappa_j + \sum_{j \neq f} u_j + u_f \right) - i0} = 0,$$

because the contour of integration can be closed in the lower half-plane of complex u_f^0 (recall that $t > 0$). Therefore, the rightmost vertex on the left slope must be type-2, which proves the assertion, for as was shown before, vertices to the left of a 2-vertex must be type-2.

4.2. Factorization of infrared contributions

To separate contributions singular in the limit $t \rightarrow \infty$, we use the condition $T \ll m$ to introduce a momentum threshold Λ such that $T \ll \Lambda \ll m$, which identifies the photons with $0 < |\mathbf{k}| < \Lambda$ as “soft”. As was already mentioned in Section 3.1, our model requires no special infrared regularization, because restriction to a finite time interval renders all Feynman integrals convergent at small momenta. As to the ultraviolet divergences, they are supposed to be regularized using some conventional means, say, the Pauli–Villars technique. If, as usual, the corresponding masses are chosen larger than the electron mass; then the ultraviolet divergences can be isolated and subtracted without affecting infrared properties of the theory, and we will assume that this has been done. Using the

momentum threshold, the perturbation series for the function $R(t; \mathbf{q}, \mathbf{q} + \mathbf{p})$ can be written as

$$R(t; \mathbf{q}, \mathbf{q} + \mathbf{p}) = I_A(\mathbf{p}, \mathbf{q}) \sum_{N=0}^{\infty} (e^2)^N I_N(\mathbf{p}, \mathbf{q}, \Lambda), \quad (32)$$

where the factor $I_A(\mathbf{p}, \mathbf{q})$ is the contributions of photons with momenta $|\mathbf{k}| > \Lambda$, and $I_N(\mathbf{p}, \mathbf{q}, \Lambda)$ are the infrared-singular parts of diagrams of the type shown in Fig. 4. Introducing abridged notation $Q_{rs}(k) = q_r^\mu q_s^\nu D_{\mu\nu}^{(rs)}(k)$, where $q_1 = q$, $q_2 = q + p$, and r, s take on values 1, 2, the functions I_N take the form

$$I_N(\mathbf{p}, \mathbf{q}, \Lambda) = \frac{1}{i^N} \sum_{r=0}^N \sum_{l=0}^{N-r} \int^\Lambda \prod_{i=1}^m \frac{d^4 k_i}{(2\pi)^4} Q_{21}(k_i) \frac{F_{1r}^m(k_1 \dots k_m) F_{2l}^m(k_1 \dots k_m)}{r! 2^r l! 2^l m!}, \quad (33)$$

where N is the number of virtual photon lines, of which r (l) reside on the incoming (outgoing) electron line, while the remaining $m \equiv N - r - l$ connect the two electron lines; the symbol \int^Λ indicates that all loop integrals are cut off at $|\mathbf{k}| = \Lambda$; finally, the functions F_{2l}^m and F_{1r}^m incorporate electron propagators and vertex factors. They include sums over all permutations of vertices residing on the outgoing and incoming electron line, respectively, the factor $r! 2^r l! 2^l$ accounting for the redundant permutations. To put them in a form admitting factorization of multi-loop diagrams, we proceed as explained in Section 3.2 and go over to integration with respect to residual momenta. To be specific, consider the lowest order diagram shown in Fig. 6. Writing

$$2\pi i \delta(m^2 - q^2) = \frac{1}{m^2 - q^2 - i0} - \frac{1}{m^2 - q^2 + i0}$$

for the factor brought in by $D^{(21)}$, we see that the first term does not contribute (cf. discussion at the end of Section 4.1), whereas the second term gives

$$F_{21}^0 = \int^\Lambda \frac{d^4 k}{(2\pi)^4} Q_{22}(k) \int_k \frac{d^4 u_1}{(2\pi)^4} \frac{\Delta_t(u_1)}{(k + u_1)q_2 + i0} \int_{u_1} \frac{d^4 u_2}{(2\pi)^4} \frac{\Delta_t(u_2)}{(u_1 + u_2)q_2 + i0} + (u_1 \leftrightarrow u_2),$$

where $(u_1 \leftrightarrow u_2)$ denotes the first term with u_1, u_2 interchanged. We observe that the contours of integration with respect to u_1^0, u_2^0 can be chosen so as to meet the pole prescriptions in both terms simultaneously. Namely, the u_1^0 -contour must go above the poles $\pm kq_2/q_2^0, -u_1^0$, and not intersect the u_2^0 -contour going above the poles $\pm kq_2/q_2^0, -u_1^0$, as shown in Fig. 7. Taking into account that the function $Q_{22}(k)$ is even, and introducing new variables $w_1 = u_1 + k, w_2 = u_2 - k$ yields

$$\begin{aligned} F_{21}^0 &= \int^\Lambda \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 u_1}{(2\pi)^4} \int \frac{d^4 u_2}{(2\pi)^4} Q_{22}(k) \frac{\Delta_t(u_1) \Delta_t(u_2)}{(u_1 + u_2)q_2 + i0} \\ &\quad \times \left[\frac{1}{(k + u_1)q_2 + i0} + \frac{1}{(-k + u_2)q_2 + i0} \right] \\ &= \int^\Lambda \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 w_1}{(2\pi)^4} \int \frac{d^4 w_2}{(2\pi)^4} Q_{22}(k) \frac{\Delta_t(w_1 - k)}{w_1 q_2 + i0} \frac{\Delta_t(w_2 + k)}{w_2 q_2 + i0}. \end{aligned}$$

This consideration is readily extended to all l, m . The factor F_{2l}^m can be written in general as $(w_1 + \dots + w_k \equiv W_k)$

$$\begin{aligned} F_{2l}^m(k_1, \dots, k_m) &= \int \frac{d^4 w_1}{(2\pi)^4} \dots \frac{d^4 w_{2l+m}}{(2\pi)^4} \int^\Lambda \prod_{i=1}^l \frac{d^4 k_{m+i}}{(2\pi)^4} Q_{22}(k_{m+i}) \\ &\quad \times \Delta_t(w_{2i-1} - k_{m+i}) \Delta_t(k_{m+i} + w_{2i}) \Delta_t(w_{2l+1} - k_1) \dots \Delta_t(w_{2l+m} - k_m) \\ &\quad \times \sum_{\text{perm}} \left[\frac{1}{W_1 q_2 + i0} \frac{1}{W_2 q_2 + i0} \dots \frac{1}{W_{2l+m} q_2 + i0} \right], \end{aligned}$$

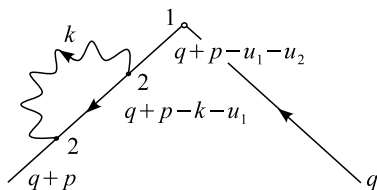


Fig. 6. The one-loop diagram determining the function F_{21}^0 .

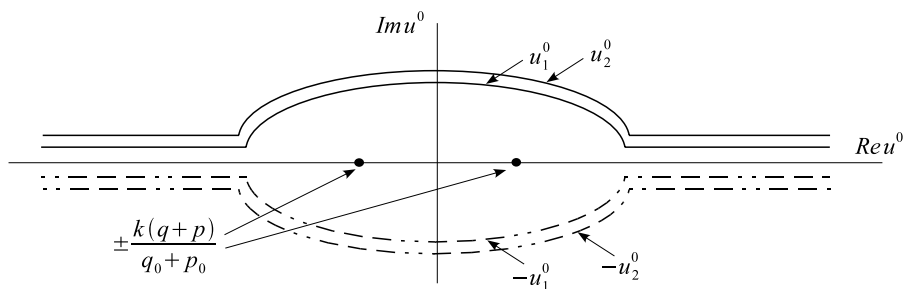


Fig. 7. Pole structure and contours of integration in the complex u^0 -plane in the integral representing F_{21}^0 .

where the sum is over all permutations of indices $1, 2, \dots, 2l + m$. That this sum appears under the sign of integral is precisely because the imaginary infinitesimals in all its terms are of the same sign. Therefore, this sum can be factorized using the well-known formula

$$\sum_{\text{perm}} \left[\frac{1}{W_1 q + i0} \frac{1}{W_2 q + i0} \cdots \frac{1}{W_m q + i0} \right] = \frac{1}{w_1 q + i0} \cdots \frac{1}{w_m q + i0}, \quad (34)$$

which is easily proved by induction for all integers m . As in the S -matrix case, applicability of this formula is crucial for the proof of factorization of infrared contributions. It is worth mentioning in this connection that when the infrared problem is treated on an infinite time interval, the existence of this or a similar formula depends on the choice of infrared regularization. For instance, as discussed in Ref. [19], no relation replacing Eq. (34) can be written if the finite pole shift is used to regularize the on-shell electron propagators, which precludes the factorization.

In a similar way, we find

$$\begin{aligned} F_{1r}^m(k_1, \dots, k_m) &= \int \frac{d^4 w_1}{(2\pi)^4} \cdots \frac{d^4 w_{2r+m}}{(2\pi)^4} \\ &\times \int^\Lambda \prod_{i=1}^r \frac{d^4 k_{m+i}}{(2\pi)^4} Q_{11}(k_{m+i}) \Delta_t(k_{m+i} - w_{2i-1}) \Delta_t(-k_{m+i} - w_{2i}) \\ &\times \Delta_t(k_1 - w_{2r+1}) \cdots \Delta_t(k_m - w_{2r+m}) \\ &\times \sum_{\text{perm}} \left[\frac{1}{W_1 q_1 - i0} \frac{1}{W_2 q_1 - i0} \cdots \frac{1}{W_{2r+m} q_1 - i0} \right], \end{aligned}$$

where the sum is over all permutations of indices $1, 2, \dots, 2r + m$, and factorization is accomplished using the complex conjugate of Eq. (34). We thus obtain

$$\begin{aligned} F_{2l}^m(k_1, \dots, k_m) &= \left[\int^\Lambda \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 w_1}{(2\pi)^4} \frac{d^4 w_2}{(2\pi)^4} Q_{22}(k) \frac{\Delta_t(w_1 - k) \Delta_t(w_2 + k)}{(w_1 q_2 + i0)(w_2 q_2 + i0)} \right]^l \\ &\times \prod_{i=1}^m \int \frac{d^4 w_{2n+i}}{(2\pi)^4} \frac{\Delta_t(w_{2l+i} - k_i)}{w_{2l+i} q_2 + i0}, \end{aligned}$$

$$F_{1r}^m(k_1, \dots, k_m) = \left[\int^\Lambda \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 w_1}{(2\pi)^4} \frac{d^4 w_2}{(2\pi)^4} Q_{11}(k) \frac{\Delta_t(-w_1 + k) \Delta_t(-w_2 - k)}{(w_1 q_1 - i0)(w_2 q_1 - i0)} \right]^r \\ \times \prod_{i=1}^m \int \frac{d^4 w_{2n+i}}{(2\pi)^4} \frac{\Delta_t(k_i - w_{2r+i})}{w_{2r+i} q_1 - i0}.$$

Finally, substitution into Eq. (33) gives

$$I_N(\mathbf{p}, \mathbf{q}, \Lambda) = \sum_{r=0}^N \sum_{l=0}^{N-r} \frac{g_{11}^r}{r! 2^r} \frac{g_{22}^l}{l! 2^l} \frac{g_{21}^m}{m!} = \frac{g^N}{N! 2^N}, \quad g \equiv g_{11} + g_{22} + 2g_{21},$$

where

$$g_{rs} = -i\eta_r \eta_s \int^\Lambda \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 w_1}{(2\pi)^4} \frac{d^4 w_2}{(2\pi)^4} Q_{rs}(k) \frac{\Delta_t(w_1 - k) \Delta_t(w_2 + k)}{(w_1 q_r + i0)(w_2 q_s + i0)}, \\ \eta_1 = 1, \eta_2 = -1.$$

Thus,

$$R(t; \mathbf{q}, \mathbf{q} + \mathbf{p}) = I_\Lambda(\mathbf{p}, \mathbf{q}) \exp \frac{e^2 g}{2}. \quad (35)$$

Using the formula

$$\int \frac{d^4 w}{(2\pi)^4} \frac{\Delta_t(w - k)}{w q + i0} = \int \frac{dw_0}{2\pi i} \frac{e^{-i(w_0 - k_0)t} - 1}{w_0 - k_0} \frac{1}{w_0 q^0 - \mathbf{kq} + i0} = \frac{e^{itkq/q^0} - 1}{kq},$$

we find

$$g_{rs} = i\eta_r \eta_s \int^\Lambda \frac{d^4 k}{(2\pi)^4} \frac{Q_{rs}(k)}{(kq_r)(kq_s)} \left[e^{itkq_r/q_r^0} - 1 \right] \left[e^{-itkq_s/q_s^0} - 1 \right].$$

Inserting explicit expressions (20) for the photon propagator, it can be put in the form

$$g_{21} = - \int^\Lambda \frac{d^4 k}{(2\pi)^3} [\theta(k_0) + n(\mathbf{k})] \delta(k^2) \frac{d_{\mu\nu}(k) q_2^\mu q_1^\nu}{(kq_2)(kq_1)} \\ \times \left[1 + e^{it(kq_2/q_2^0 - kq_1/q_1^0)} - e^{itkq_2/q_2^0} - e^{-itkq_1/q_1^0} \right], \\ g_{rr} = \int^\Lambda \frac{d^4 k}{(2\pi)^3} \left[-\frac{1}{2\pi i} \frac{\eta_r}{k^2 + i\eta_r 0} + n(\mathbf{k}) \delta(k^2) \right] \frac{d_{\mu\nu}(k) q_r^\mu q_r^\nu}{(kq_r)(kq_r)} \left[2 - e^{itkq_r/q_r^0} - e^{-itkq_r/q_r^0} \right].$$

Each of these expressions consists of a vacuum and a temperature part: $g_{rs} = g_{rs}^0 + g_{rs}^T$, where g_{rs}^0 is the limit of g_{rs} for $T \rightarrow 0$ and t fixed. The contribution to g_{21} growing with time turns out to be real because the integrand is on the photon mass-shell. On the contrary, off-shell photons do contribute to the vacuum part of g_{rr} , producing an imaginary term unbounded at $t \rightarrow +\infty$. To see this, we close the contour of k^0 -integration in the lower (upper) half-plane in that part of the expression which is proportional to e^{-itkq_r/q_r^0} (e^{itkq_r/q_r^0} or 2), and find

$$\text{Im } g_{rr}^0 = i\eta_r t \int^\Lambda \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{m^2}{(\mathbf{kq}_r)^2 - (q_r^0)^2 \mathbf{k}^2},$$

whereas the real parts of the large-time asymptotics of g_{rs} can be combined in a single expression

$$\text{Re } g_{rs} = \eta_r \eta_s \int^\Lambda \frac{d^4 k}{(2\pi)^3} [\theta(k_0) + n(\mathbf{k})] \delta(k^2) \frac{d_{\mu\nu}(k) q_r^\mu q_s^\nu}{(kq_r)(kq_s)} \\ \times \left[1 + e^{it(kq_r/q_r^0 - kq_s/q_s^0)} - e^{-it\eta_r kq_r/q_r^0} - e^{-it\eta_s kq_s/q_s^0} \right], \quad t \rightarrow \infty.$$

The combination $g = g_{11} + g_{22} + 2g_{21}$ in various limiting cases reads (assuming $\mathbf{q} \ll m$, $\mathbf{p} \ll m$)

$$\operatorname{Re} g^0 = \begin{cases} 0, & mt \ll 1, \\ -\frac{1}{3\pi^2} \frac{\mathbf{p}^2}{m^2} \ln \Lambda t, & mt \gg 1, |\mathbf{p}|t \ll 1, \\ -\frac{1}{3\pi^2} \frac{\mathbf{p}^2 + \mathbf{q}^2 + \mathbf{p}\mathbf{q}}{m^2} \ln \Lambda t, & |\mathbf{p}|t \gg 1, \end{cases} \quad (36)$$

$$\operatorname{Re} g^T = \begin{cases} 0, & Tt \ll 1, \\ \frac{1}{3\pi^2} \frac{\mathbf{p}^2}{m^2} \ln Tt - \frac{1}{3\pi} \frac{\mathbf{p}^2}{m^2} Tt, & Tt \gg 1, T|\mathbf{p}|t/m \ll 1, \\ \frac{1}{3\pi^2} \frac{\mathbf{p}^2 + \mathbf{q}^2 + \mathbf{p}\mathbf{q}}{m^2} \ln Tt - \frac{1}{3\pi} \frac{\mathbf{p}^2}{m^2} Tt, & T|\mathbf{p}|t/m \gg 1, \end{cases} \quad (37)$$

$$\operatorname{Im} g = i \frac{\Lambda t}{3\pi^2} \left[\frac{\mathbf{q}^2}{m^2} - \frac{(\mathbf{q} + \mathbf{p})^2}{m^2} \right]. \quad (38)$$

Substitution of Eq. (38) in Eq. (31) brings the electron density matrix to the form

$$\begin{aligned} \varrho_{\sigma\sigma'}(t; \mathbf{q}, \mathbf{q} + \mathbf{p}) &= \varrho_{\sigma\sigma'}(t_0; \mathbf{q}, \mathbf{q} + \mathbf{p}) I_\Lambda \\ &\times \exp \left[i \left(\frac{1}{2m} - \frac{e^2 \Lambda}{6\pi^2 m^2} \right) ((\mathbf{q} + \mathbf{p})^2 - \mathbf{q}^2) t + \frac{\operatorname{Re} g}{2} \right]. \end{aligned}$$

It follows that the imaginary contribution to g merely renormalizes the electron mass $m \rightarrow m - e^2 \Lambda / 3\pi^2 + o(e^2)$. Hence, the function R takes the form

$$R(t; \mathbf{q}, \mathbf{q} + \mathbf{p}) = I_\Lambda(\mathbf{q}, \mathbf{p}) \exp \frac{\operatorname{Re} g}{2}$$

where $\operatorname{Re} g$ is given by Eqs. (36), (37). Dependence on Λ must cancel in the product of $I_\Lambda(\mathbf{p}, \mathbf{q})$ with the factor contributed by the term proportional to $\ln \Lambda t$ in Eq. (36). Since $\Lambda \ll m$, within the logarithmic accuracy this amounts to replacing $\ln \Lambda t \rightarrow \ln mt$. Conditions $T \ll m$ and $Tt \gg 1$ then imply that the logarithmic term is negligible in comparison with the term proportional to (Tt) . Thus, we arrive at the following expression for the large-time asymptotic of the function $R(t; \mathbf{q}, \mathbf{q} + \mathbf{p})$

$$R(t; \mathbf{q}, \mathbf{q} + \mathbf{p}) = \exp \left(-\frac{e^2}{6\pi} \frac{\mathbf{p}^2}{m^2} Tt \right), \quad Tt \gg 1. \quad (39)$$

We see that despite infrared contributions to $R(t; \mathbf{q}, \mathbf{q} + \mathbf{p})$ with $\mathbf{p} \neq 0$ grow with time at every order of perturbation theory, their sum is actually bounded, tending to zero in the limit $t \rightarrow \infty$. On the other hand, $R(t; \mathbf{q}, \mathbf{q})$ is independent of t , being finite at every order. This means that the infrared effects do not change the probability distribution of the electron momentum, whereas the off-diagonal components of the electron density matrix vanish in the limit $t \rightarrow \infty$ (see Eq. (31)). The vanishing of the infrared-divergent contribution at $\mathbf{p} = 0$ is actually a consequence of Eqs. (26), (31), and of the total charge conservation. In view of Eq. (6), it can also be considered as a special case of the Bloch–Nordsieck theorem.

The asymptotic expression for the effective electromagnetic current reads

$$J_\mu^{\text{eff}}(\mathbf{x}, t) = \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \int \frac{d^3 \mathbf{p}}{(2\pi)^3} e^{ip_0 t - i\mathbf{p}\mathbf{x}} \exp(-\Theta \mathbf{p}^2 t) J_\mu(\mathbf{q}, \mathbf{p}), \quad (40)$$

where

$$\Theta = \frac{2\alpha T}{3m^2}, \quad (41)$$

$\alpha = e^2/4\pi$ is the fine structure constant, and

$$J_\mu(\mathbf{q}, \mathbf{p}) = \sum_{\sigma, \sigma'} \varrho_{\sigma\sigma'}(t_0; \mathbf{q}, \mathbf{q} + \mathbf{p}) \frac{\bar{u}_{\sigma'}(\mathbf{q} + \mathbf{p}) \gamma_\mu u_\sigma(\mathbf{q})}{\sqrt{2\varepsilon_{\mathbf{q}+\mathbf{p}}} \sqrt{2\varepsilon_{\mathbf{q}}}}.$$

5. Alternative derivation of Eq. (39)

The exponential time dependence of the main result expressed by Eq. (39) suggests that it can be derived from consideration of the density matrix evolution in infinitesimal form. We give this derivation below in order to emphasize the role of the approximations made, and to demonstrate a very important fact that the infrared properties of currents can be determined without having to sum the perturbation series explicitly.

Let us first establish a general relation between the matrices $\varrho(t)$ and ϱ_0 in the momentum representation. It follows from Eq. (4) that

$$\begin{aligned}\varrho(t; \mathbf{q}, \mathbf{q}') &= \sum_{\phi, \phi_0} w(\phi_0) \langle e | \langle \phi | U(t) | \phi_0 \rangle \varrho_0 \langle \phi_0 | U^\dagger(t) | \phi \rangle | e' \rangle \\ &= \sum_{\phi, \phi_0, e_0, e'_0} w(\phi_0) \langle \phi | \langle e | U(t) | e_0 \rangle | \phi_0 \rangle \langle e_0 | \varrho_0 | e'_0 \rangle \langle \phi_0 | \langle e'_0 | U^\dagger(t) | e' \rangle | \phi \rangle,\end{aligned}\quad (42)$$

where $U(t) \equiv U(t, 0)$, and

$$\langle e_0 | \varrho_0 | e'_0 \rangle = \varrho(0; \mathbf{q}_0, \mathbf{q}'_0).$$

For simplicity, the electron is assumed henceforth unpolarized, $\varrho_{\sigma\sigma'} \sim \delta_{\sigma\sigma'}$, and the spin indices are omitted. By virtue of momentum conservation, the evolution operator has nonvanishing matrix elements only between states satisfying

$$\mathbf{q}'_0 = \mathbf{q}' + \mathbf{Q}, \quad \mathbf{q}_0 = \mathbf{q} + \mathbf{Q}$$

where \mathbf{Q} denotes the difference of total momenta of photons in states $|\phi\rangle$ and $|\phi_0\rangle$. It follows that

$$\mathbf{q}'_0 - \mathbf{q}_0 = \mathbf{q}' - \mathbf{q} \equiv \mathbf{p}.$$

Hence, the two matrices are related by

$$\varrho(t; \mathbf{q}, \mathbf{q} + \mathbf{p}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} K(t; \mathbf{q}, \mathbf{p}, \mathbf{k}) \varrho(0; \mathbf{q} + \mathbf{k}, \mathbf{q} + \mathbf{p} + \mathbf{k}). \quad (43)$$

The kernel $K(t; \mathbf{q}, \mathbf{p}, \mathbf{k})$ can be computed using the general formulas given in Sections 2 and 3.1, for which purpose it is useful to rewrite definition (27) as

$$\varrho(t; \mathbf{q}, \mathbf{q}') = \int d^3\mathbf{x} \int d^3\mathbf{x}' \sqrt{2\varepsilon_{\mathbf{q}'}} \sqrt{2\varepsilon_{\mathbf{q}}} e^{i\mathbf{q}'\mathbf{x}' - i\mathbf{q}\mathbf{x}} \langle \bar{\psi}(t, \mathbf{x}') u(\mathbf{q}') \bar{u}(\mathbf{q}) \psi(t, \mathbf{x}) \rangle. \quad (44)$$

To determine the large-time behavior of $\varrho(t)$, one may proceed as in Section 4.2, prove factorization of the infrared contributions, and then sum the perturbation series. However, the leading term of the large-time asymptotic can be found more directly. As was shown in Section 4.2, this term is produced by the interaction of the electron with equilibrium photons. To extract this term, we note that the property of being in equilibrium implies that the effect of such photons is homogeneous in time, but with one important qualification. The photon cloud surrounding the electron can be said to be in equilibrium at a given temperature T only when considered on time intervals $\gg 1/T$. This is because at lesser times ($\lesssim 1/T$), the quantum indeterminacy in the photon energy becomes of order of the photon mean energy, and in the presence of such large fluctuations it is evidently impossible to speak about time homogeneity. Thus, we introduce a time τ_0 satisfying

$$T\tau_0 \gg 1, \quad (45)$$

and restrict consideration to time intervals $\delta t \gtrsim \tau_0$. This condition justifies omission of the logarithmic contributions due to vacuum photon–electron interaction, making thereby the photon impact on the electron evolution homogeneous in time. This implies that the electron density matrices at arbitrary instants t and $t + \delta t$ are related by the same kernel as in Eq. (43), viz.,

$$\varrho(t + \delta t; \mathbf{q}, \mathbf{q} + \mathbf{p}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} K(\delta t; \mathbf{q}, \mathbf{p}, \mathbf{k}) \varrho(t; \mathbf{q} + \mathbf{k}, \mathbf{q} + \mathbf{p} + \mathbf{k}), \quad (46)$$

where K is independent of t . Whether or not this equation can be reduced to a differential equation depends on the structure of the kernel $K(\delta t; \mathbf{q}, \mathbf{p}, \mathbf{k})$. The point is that in general, one cannot differentiate it with respect to δt at $\delta t = 0$, because this would violate condition (45). To put it differently, if both sides of this equation are expanded in powers of δt , the question is whether the terms $O(\delta t^2)$ can be neglected in comparison with the linear term. That this is legitimate in the case under consideration is suggested by the results of the preceding section, and is confirmed by the subsequent computation. Thus, we expand $K(\delta t; \mathbf{q}, \mathbf{p}, \mathbf{k})$ up to the first order in δt , and obtain

$$\frac{\partial \varrho(t; \mathbf{q}, \mathbf{q} + \mathbf{p})}{\partial t} = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} K(\mathbf{q}, \mathbf{p}, \mathbf{k}) \varrho(t; \mathbf{q} + \mathbf{k}, \mathbf{q} + \mathbf{p} + \mathbf{k}), \quad (47)$$

where

$$K(\mathbf{q}, \mathbf{p}, \mathbf{k}) \equiv \left. \frac{\partial}{\partial t} K(t; \mathbf{q}, \mathbf{p}, \mathbf{k}) \right|_{t=0}. \quad (48)$$

In view of the smallness of the coupling constant, $e^2 \ll 1$, the function $K(\mathbf{q}, \mathbf{p}, \mathbf{k})$ can be found in the second order approximation. Indeed, if we use Eq. (44) to relate $\varrho(t + \delta t)$ and $\varrho(t)$, where $\delta t \sim \tau_0$, then the loop expansion of the integral kernel $K(\delta t; \mathbf{q}, \mathbf{p}, \mathbf{k})$ is a power series in $e^2 T \tau_0$, since each loop divergences linearly, with the dominant contribution coming from the integration over photon momenta $k \sim 1/\tau_0$. Therefore, contributions of higher order in e are negligible, provided that τ_0 satisfies

$$e^2 T \tau_0 \ll 1. \quad (49)$$

It is because of the smallness of the coupling constant that this condition is consistent with Eq. (45). In zeroth order in the coupling (free electron evolution), $\varrho(t; \mathbf{q}, \mathbf{q} + \mathbf{p}) \sim e^{ip_0 t}$, implying that

$$K(\mathbf{q}, \mathbf{p}, \mathbf{k})|_{e=0} = ip_0 (2\pi)^3 \delta^{(3)}(\mathbf{k}).$$

Diagrams representing the second-order term in $K(\mathbf{q}, \mathbf{p}, \mathbf{k})$ are shown in Fig. 8. Using the definition (48), this term can be written as

$$\begin{aligned} K(\mathbf{q}, \mathbf{p}, \mathbf{k})|_{e^2} = & -2 \frac{e^2}{\tau_0} \int \frac{dk^0}{2\pi} n(\mathbf{k}) \delta(k^2) \frac{d_{\mu\nu}(k) q_2^\mu q_1^\nu}{(kq_2)(kq_1)} \\ & \times \left[e^{i\tau_0 k q_2 / q_2^0} - 1 \right] \left[e^{-i\tau_0 k q_1 / q_1^0} - 1 \right] \\ & + \frac{e^2}{\tau_0} \delta^{(3)}(\mathbf{k}) \int \frac{d^4 k'}{2\pi} n(\mathbf{k}') \delta(k'^2) \sum_{r=1,2} \frac{d_{\mu\nu}(k') q_r^\mu q_r^\nu}{(k'q_r)(k'q_r)} \\ & \times \left[e^{i\tau_0 k' q_r / q_r^0} - 1 \right] \left[e^{-i\tau_0 k' q_r / q_r^0} - 1 \right]. \end{aligned}$$

The leading large-time contribution in this integral comes from $\mathbf{k} \sim 1/\tau_0$, and counting the powers of \mathbf{k} in Eq. (47) readily shows that this contribution is independent of τ_0 , as expected. It is to be emphasized in this connection that the four-dimensionality of spacetime is essential to reach this conclusion. In a three-dimensional spacetime, for instance, the leading term in the function $K(\delta t; \mathbf{q}, \mathbf{p}, \mathbf{k})$ is quadratic in δt , and transition from Eqs. (46) to (47) is not legitimate. We note, finally, that under condition (45), $\mathbf{k} \sim 1/\tau_0 \ll T$ can be neglected in the arguments of ϱ in Eq. (47), so that in effect, $K(\mathbf{q}, \mathbf{p}, \mathbf{k}) \sim \delta^{(3)}(\mathbf{k})$. A computation similar to that performed at the end of Section 4.2 then gives, after the mass renormalization,

$$K(\mathbf{q}, \mathbf{p}, \mathbf{k})|_{e^2} = -\Theta \mathbf{p}^2 (2\pi)^3 \delta^{(3)}(\mathbf{k}).$$

Substituting this into Eq. (47), we find

$$\frac{\partial}{\partial t} \varrho(t; \mathbf{q}, \mathbf{q} + \mathbf{p}) = [ip_0 - \Theta \mathbf{p}^2] \varrho(t; \mathbf{q}, \mathbf{q} + \mathbf{p}),$$

and therefore,

$$\varrho(t; \mathbf{q}, \mathbf{q} + \mathbf{p}) = \exp(-\Theta \mathbf{p}^2 t) e^{ip_0 t} \varrho(0; \mathbf{q}, \mathbf{q} + \mathbf{p}), \quad (50)$$

in agreement with Eq. (39).

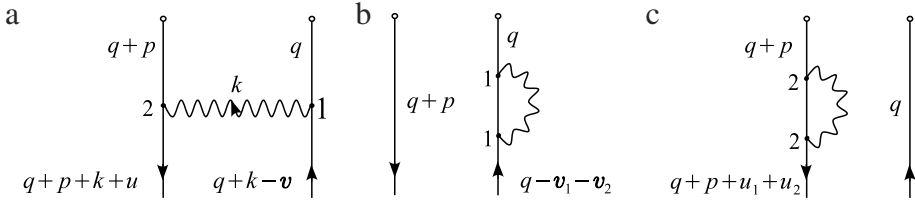


Fig. 8. Diagrams representing the second-order contribution to the integral kernel $K(\mathbf{q}, \mathbf{p}, \mathbf{k})$ in Eq. (47).

6. Physical manifestations of the infrared singularity

In this section, we discuss some physical applications of the main result expressed by Eq. (50).

6.1. Irreversible spreading of Gaussian wavepacket

To illustrate the role of the irreversible charge spreading, consider an unpolarized electron prepared at $t_0 = 0$ in a pure state described in the momentum space by a Gaussian wavefunction

$$\phi_0(\mathbf{q}) = (4\pi l^2)^{3/4} \exp\{-l^2 \mathbf{q}^2/2\}, \quad l = \text{const.}$$

This wavefunction represents the electron spatially localized in a region of characteristic size l near the origin. The corresponding density matrix is $\varrho_0(\mathbf{q}, \mathbf{q}') = \tilde{\phi}_0(\mathbf{q}')\phi_0(\mathbf{q})$. Substitution of this expression in Eq. (40) and evaluation of the Gaussian integrals yields the following expression for the electron density at arbitrary instant t

$$J_0^{\text{eff}}(\mathbf{x}, t) = \frac{1}{\pi^{3/2} l_t^3} \exp\left\{-\frac{\mathbf{x}^2}{l_t^2}\right\}, \quad l_t = \left(l^2 + \frac{t^2}{m^2 l^2} + 4\Theta t\right)^{1/2}. \quad (51)$$

This result shows that in addition to the usual quantum spreading described by the term $t^2/m^2 l^2$ in l_t , there is a spreading due to interaction of the electron with the heat-bath photons. It is easy to see that in the setting considered, the latter effect is dominated by the usual quantum-mechanical spreading. Indeed, for a given t , the minimum of the sum $(l^2 + t^2/m^2 l^2)$ is $2t/m$, which is very large compared to $4\Theta t \sim \alpha T t/m^2$, since $T \ll m$, $\alpha \ll 1$. But it is not difficult to give an example where the relation between the two effects is opposite. Namely, let the electron be prepared at $t_0 = 0$ in a pure state which is supposed to describe this electron localized at a later instant $t = \tau$ in a region of the size l near a point \mathbf{x}_0 . Then an appropriate momentum-space amplitude in nonrelativistic quantum mechanics would be

$$\phi(\mathbf{q}) = \phi_0(\mathbf{q}) e^{-i\mathbf{q}\mathbf{x}_0 + i\varepsilon \mathbf{q}\tau},$$

where $\phi_0(\mathbf{q})$ is real; the factor $e^{i\mathbf{q}\mathbf{x}_0}$ realizes the free electron evolution backward in time from $t = \tau$ to $t = 0$. Taking ϕ_0 as before, the effective electron density at $t = \tau$ now is

$$J_0^{\text{eff}}(\mathbf{x}, \tau) = \frac{\exp\left\{-\frac{(\mathbf{x}-\mathbf{x}_0)^2}{(l^2+4\Theta\tau)}\right\}}{\pi^{3/2} (l^2 + 4\Theta\tau)^{3/2}},$$

which demonstrates that the actual size of the wave packet at $t = \tau$ is $l_\tau = (l^2 + 4\Theta\tau)^{1/2}$. This simple example shows that the minimal uncertainty in the position of an electron evolving freely for a time τ is $\sim \sqrt{\Theta\tau}$. Incidentally, this fact justifies the term “irreversible spreading”. It also implies that in the conventional approach based on the notion of infinitely remote past, the uncertainty is formally infinite, and the effective charge density (hence, the effective field) is zero at any given spatial point:

$$\lim_{\tau \rightarrow \infty} J_0^{\text{eff}}(\mathbf{x}, \tau) = 0. \quad (52)$$

The value of the minimal uncertainty is actually independent of the initial electron state, and can be obtained by using Eq. (40) to write the electron position variance as

$$\langle \mathbf{x}^2 \rangle = \int d^3 \mathbf{x} J_0^{\text{eff}}(\mathbf{x}, t) \mathbf{x}^2 = - \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \frac{\partial^2}{\partial \mathbf{p}^2} [J_0(\mathbf{q}, \mathbf{p}) e^{ip_0 t}] \Big|_{\mathbf{p}=0} + 6\Theta t.$$

The first term on the right represents the usual quantum-mechanical spreading, whereas the second—the minimum of $\langle \mathbf{x}^2 \rangle$ imposed by the infrared radiative effects.

The linearity of the particle position variance with respect to the travel time is characteristic of diffusive processes, which suggests that the irreversible spreading of the electron wavepacket can be viewed as a kind of Brownian motion of the electron. However, this analogy cannot be taken literally, because the irreversible spreading is not driven by the electron–photon collisions. This is evident from the fact that it does not lead to relaxation of the electron momentum—as is seen from Eq. (50), the momentum probability distribution, $\varrho(t; \mathbf{q}, \mathbf{q})$, is unaffected by the infrared singularity. Furthermore, the irreversibility of this effect shows itself also as an increase of quantum [29] entropy of the electron state, $S = -\text{Tr}_e \varrho \ln \varrho$. Namely, it was shown in [19] that S grows asymptotically as

$$S = \frac{3}{2} \ln(\Theta t \Delta_q^2), \quad t \gg (\Delta_q^2 \Theta)^{-1}, \quad (53)$$

where Δ_q^2 is the electron momentum variance. An exact expression for the entropy in the special case of a Gaussian wavepacket is obtained in Appendix B. This growth of entropy is directly related to the fact that the infrared singularity damps the off-diagonal elements of the electron density matrix. But in contrast to the ordinary statistics, this thermalization takes place both at $T \neq 0$ and $T = 0$, though in vacuum⁷ the process is much slower (time dependence of the effective electron current is exponential at finite temperature, whereas in vacuum it is a power law). These circumstances emphasize the specifically quantum nature of the irreversible spreading, which has no proper analog in nonrelativistic physics.

How the usual relaxation of the electron momentum due to its collisions with the heat-bath photons is described in the present formalism will be shown in Section 7.

Conditions considered in the last example can be experimentally realized using a magnetic lens to focus an electron beam, but it is more advantageous to detect the effect of infrared thermalization as a decoherence of the electron waves in the classic two-slit experiment considered in the next section.

6.2. Electron diffraction

Consider the electron diffraction in the two-slit experiment shown schematically in Fig. 9. The wavefunction of diffracted electrons is a superposition of two outgoing cylindric waves, which at sufficiently large distance from the slits has the form

$$\psi = \frac{A}{\sqrt{2}} \left[\frac{e^{ikr_1}}{\sqrt{r_1}} + \frac{e^{ikr_2}}{\sqrt{r_2}} \right],$$

where r_1 (r_2) is the distance between the first (second) slit and the point of observation, A is a bispinor amplitude independent of the spatial coordinates x, z (z being directed along the incident beam), and $k = |\mathbf{k}|$ is the incident electron momentum. Let $2d$ denote the slit spacing, and $L \gg d$ the distance between a slit and the screen at which we observe the interference pattern. In a vicinity of the detector ($x, z \ll L$, assuming that the origin of the coordinate system is at the screen), the wavefunction can be written as

$$\psi(x, z) = \frac{\psi_0}{\sqrt{2}} e^{ikz} \left[e^{i\kappa(x+d)^2/4d} + e^{i\kappa(x-d)^2/4d} \right], \quad \psi_0 = \frac{A}{L} e^{ikL}, \quad \kappa = \frac{2kd}{L}.$$

⁷ It is to be emphasized that the described irreversibility is the property solely of the electron subsystem, rather than the total system electron + electromagnetic field, whose evolution is, of course, unitary. The electron density matrix exhibits an irreversible behavior because information about phase correlations between electron states with different momenta is lost as a result of tracing over photon degrees of freedom.

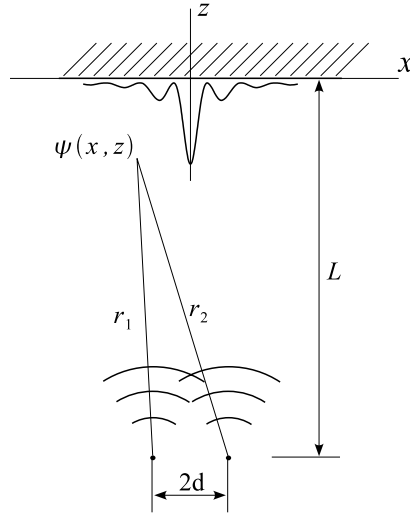


Fig. 9. Schematics of the two-slit experiment. The incident plane electron wave propagates upward; the slits are symbolized by small circles producing secondary electron waves.

Then in the absence of the photon bath, the electron density is given by

$$J_0(x) = \psi_0^\dagger \psi_0 [1 + \cos(\kappa x)].$$

To determine how this expression changes in the presence of thermal photons, we have to Fourier-expand the wavefunction

$$\psi(x, z) = \sqrt{\frac{2\pi id}{\kappa}} \psi_0 e^{ikz} \int_{-\infty}^{+\infty} \frac{dq}{2\pi} e^{-idq^2/\kappa} [e^{iq(x+d)} + e^{iq(x-d)}].$$

The corresponding expression for the electron density reads

$$J_0(x) = \frac{8\pi d}{\kappa} \psi_0^\dagger \psi_0 \iint_{-\infty}^{+\infty} \frac{dq dq'}{(2\pi)^2} \exp i[d(q^2 - q'^2)/\kappa - (q' - q)x] \cos(q'd) \cos(qd).$$

Now, inclusion of the infrared effect of thermal photons gives for the electron density, according to Eqs. (26), (50),

$$J_0^{\text{eff}}(x) = \frac{8\pi d}{\kappa} \psi_0^\dagger \psi_0 \iint_{-\infty}^{+\infty} \frac{dq dq'}{(2\pi)^2} \exp [-\Theta \tau (q' - q)^2 + id(q^2 - q'^2)/\kappa - i(q' - q)x] \times \cos(q'd) \cos(qd),$$

where Θ is defined in Eq. (41), and $\tau = mL/k$ is the electron travel time between the slits and the detector. Integrating back over q, q' , we find

$$J_0^{\text{eff}}(x) = \psi_0^\dagger \psi_0 \left[1 + \exp \left(-\frac{2\alpha TL}{3mk} \kappa^2 \right) \cos(\kappa x) \right].$$

The exponential factor in this formula describes the decoherence caused by the infrared electron thermalization. To determine conditions under which this effect is appreciable, we note that $\kappa \sim 1/r$, where r is the fringe spacing in the interference pattern. The interference is destroyed when the expression in the exponent becomes of order unity, or

$$\frac{TL}{\sqrt{\varepsilon} r^2} \sim 10^{20} \text{ K/cm eV}^{1/2}, \quad (54)$$

where T is to be expressed in kelvins, and the electron energy ε in electronvolts.

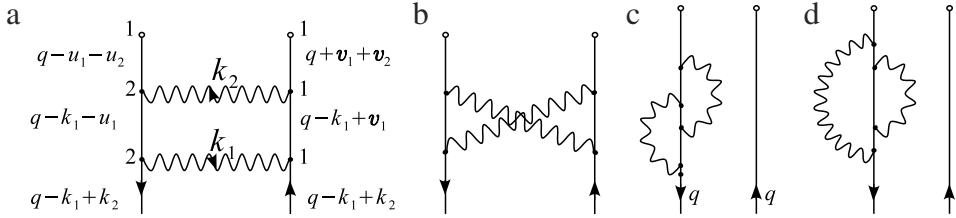


Fig. 10. Diagrams describing the effect of electron–photon scattering on the electron evolution. Open circles symbolize ψ -operators in the expression (44) for the electron density matrix.

It follows that the effect of infrared thermalization on the electron diffraction is normally negligible. For instance, in the classic experiment by Davisson and Germer [30], electrons with energies ~ 50 eV were scattered by a crystal of nickel at room temperature. A maximum in their azimuthal distribution was detected at the scattering angle 50° ; with the distance from the target to electron collector ~ 1 cm, this corresponds to $r \sim 1$ cm. Substituting this in Eq. (54) one finds that the infrared thermalization would be noticeable in this setting only at temperatures $T \sim 10^{20}$ K.

However, current technologies allow experimenting at much smaller scales than those of Ref. [30]. Resolution of the modern electron detectors employing magnetic lenses is a few angstrom. Therefore, for electron energy ~ 10 eV and $L \sim 1$ m, the effect is detectable already at $T \sim 100$ K.

7. Relaxation of the electron momentum

To clarify the role played by the infrared thermalization, and to better expose its distinction from the usual thermalization, we shall now show how interaction of the electron with non-infrared photons realizes relaxation in the system, i.e., how the electron momentum distribution tends to the equilibrium distribution. To this end, we have to consider evolution of the diagonal elements of the electron density matrix, $\varrho(t; \mathbf{q}, \mathbf{q})$, which is described in the lowest order by diagrams shown in Fig. 10. As in Section 5, the leading contribution we are interested in turns out to be linear in time, so that $\partial \varrho(t; \mathbf{q}, \mathbf{q}) / \partial t$ can be written as a functional of $\varrho(t; \mathbf{q}, \mathbf{q})$ at the same instant, as in Eq. (47), with an integral kernel given exactly by diagrams in Fig. 10. A slight change of notation in this figure is to be noted, namely, the electron momentum q is now off the mass shell, as it is associated with the ψ -operators in Eq. (44), symbolized by open circles in Fig. 10. On the contrary, the momentum $(q - k_1 + k_2)$ is on the mass shell, as it is associated with the external lines representing the initial density matrix. To express this fact, we write $q = \{q^0(k_1, k_2), \mathbf{q}\}$, where $q^0(k_1, k_2) = \varepsilon_{q-k_1+k_2} + k_1^0 - k_2^0$. As we shall see, the leading contribution comes from integration over finite k_1, k_2 , such that momentum q is near the mass-shell. The two conditions $(q - k_1 + k_2)^2 = m^2$ and $q^2 = m^2$ are clearly consistent, since the equation $\varepsilon_{q-k_1+k_2} = \varepsilon_q - k_1^0 + k_2^0$ has a continuum of non-trivial solutions with respect to $k_1 \neq k_2$ satisfying $k_1^2 = k_2^2 = 0$. The latter requirement follows from the fact that all vertices on the outgoing (incoming) electron line are of type 2 (1), by the same reason as in Section 4.1. It is to be noted also that k_1^0 and k_2^0 are of the same sign, as the opposite would allow for a double-photon emission by a free electron (formally, the relations $\varepsilon_{q-k_1+k_2} = \varepsilon_q - k_1^0 + k_2^0$, $q^2 = m^2$, $k_1^2 = k_2^2 = 0$ are inconsistent for $k_1^0 k_2^0 < 0$). Hence, no singularity of the type considered presently arise at zero temperature, as the vacuum contribution is proportional to $\theta(k_1^0) \theta(-k_2^0)$. Evidently, contributions given by the disconnected diagrams in Fig. 10(c), (d) do not change electron momentum, so that Eq. (47) can be written as

$$\frac{\partial \varrho(t, \mathbf{q})}{\partial t} = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} C(\mathbf{q}, \mathbf{k}) \varrho(t, \mathbf{q} + \mathbf{k}) + D(\mathbf{q}) \varrho(t, \mathbf{q}), \quad (55)$$

where $\varrho(t, \mathbf{q}) \equiv \varrho(t; \mathbf{q}, \mathbf{q})$, $C(\mathbf{q}, \mathbf{k})$ is the connected part of the integral kernel, represented by diagrams in Fig. 10(a), (b), and $D(\mathbf{q})$ is the disconnected part. The latter need not be calculated explicitly, since there is a simple relation between the two parts, following from the normalization

condition (28). Namely, in order for this condition be satisfied by all solutions of Eq. (55), it is necessary that

$$D(\mathbf{q}) = - \int \frac{d^3 \mathbf{k}}{(2\pi)^3} C(\mathbf{q} + \mathbf{k}, -\mathbf{k}),$$

and therefore,

$$\frac{\partial \varrho(t, \mathbf{q})}{\partial t} = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} [C(\mathbf{q}, \mathbf{k}) \varrho(t, \mathbf{q} + \mathbf{k}) - C(\mathbf{q} + \mathbf{k}, -\mathbf{k}) \varrho(t, \mathbf{q})]. \quad (56)$$

This is nothing but the usual kinetic equation for an electron in the photon bath, with $C(\mathbf{q}, \mathbf{k})$ playing the role of the probability of transition (per unit time) in which the electron goes from a state with momentum $(\mathbf{q} + \mathbf{k})$ over to a state with momentum \mathbf{q} . To determine this function, we have to evaluate diagrams in Fig. 10(a), (b). It is easy to see that the momenta $q - k_1$, $q + k_2$ are off the mass shell whenever $q - k_1 + k_2$ is on, and the light-like vectors k_1 , k_2 are nonzero. Therefore, the residual momenta can be neglected in the internal electron propagators when extracting the leading contribution:

$$\frac{\not{q} - \not{k}_1 + \not{v}_1 + m}{m^2 - (q - k_1 + v_1)^2 + i0} \rightarrow \frac{\not{q} - \not{k}_1 + m}{m^2 - (q - k_1)^2}, \quad \text{etc.}$$

Next, integration with respect to k_1^0 , k_2^0 yields four terms: each diagram contributes two terms—one with $k_1^0 = +|\mathbf{k}_1|$, $k_2^0 = +|\mathbf{k}_2|$, and the other with $k_1^0 = -|\mathbf{k}_1|$, $k_2^0 = -|\mathbf{k}_2|$. Changing $\mathbf{k}_{1,2} \rightarrow -\mathbf{k}_{1,2}$ in the latter case, the connected contribution to the variation of the density matrix takes the form

$$\begin{aligned} \delta_C \varrho(t, \mathbf{q}) &= \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \int \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \\ &\times \int \frac{[du]_{\delta t} [dv]_{\delta t} (2/\pi) \varepsilon_q^2 w(\mathbf{q}, \mathbf{k}_1, \mathbf{k}_2) n(\mathbf{k}_1) [1 + n(\mathbf{k}_2)] \varrho(t, \mathbf{q} + \mathbf{k})}{[m^2 - (q + v_1 + v_2)^2 - i0][m^2 - (q - u_1 - u_2)^2 + i0]}, \end{aligned} \quad (57)$$

where

$$\begin{aligned} w(\mathbf{q}, \mathbf{k}_1, \mathbf{k}_2) &= \frac{\pi e^4 d_{\mu\alpha}(k_1) d_{\nu\beta}(k_2)}{2\varepsilon_q 2\varepsilon_{q+k} 2|\mathbf{k}_1| 2|\mathbf{k}_2|} \\ &\times \text{tr} \left\{ (\not{q} + \not{k} + m) \left[\gamma^\beta \frac{\not{q} - \not{k}_1 + m}{m^2 - (q - k_1)^2} \gamma^\alpha + \gamma^\alpha \frac{\not{q} + \not{k}_2 + m}{m^2 - (q + k_2)^2} \gamma^\beta \right] \right. \\ &\times (\not{q} + m) \left[\gamma^\mu \frac{\not{q} - \not{k}_1 + m}{m^2 - (q - k_1)^2} \gamma^\nu + \gamma^\nu \frac{\not{q} + \not{k}_2 + m}{m^2 - (q + k_2)^2} \gamma^\mu \right] \Big\}, \end{aligned} \quad (58)$$

$$[du]_t \equiv \frac{d^4 u_1}{(2\pi)^4} \frac{d^4 u_2}{(2\pi)^4} \Delta_t(u_1) \Delta_t(u_2), \quad k_1^0 = |\mathbf{k}_1|, \quad k_2^0 = |\mathbf{k}_2|, \quad k = k_2 - k_1.$$

It is understood that $q = q(k_1, k_2)$ in these formulas, the arguments of q being suppressed for brevity. To extract the leading large-time contribution, we rewrite Eq. (57) as

$$\begin{aligned} \delta_C \varrho(t, \mathbf{q}) &= \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \int \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \int_{-\infty}^{+\infty} d\xi_z \int [du]_{\delta t} [dv]_{\delta t} \delta(k_{1z} - k_{1z}^*) \\ &\times \frac{(2/\pi) \varepsilon_q^2 w(\mathbf{q}, \mathbf{k}_1 + \boldsymbol{\xi}, \mathbf{k}_2) n(\mathbf{k}_1) [1 + n(\mathbf{k}_2)] \varrho(t, \mathbf{q} + \mathbf{k})}{[m^2 - (q(k_1 + \boldsymbol{\xi}, k_2) + v_1 + v_2)^2 - i0][m^2 - (q(k_1 + \boldsymbol{\xi}, k_2) - u_1 - u_2)^2 + i0]}, \end{aligned} \quad (59)$$

where $\boldsymbol{\xi} = (0, \boldsymbol{\xi})$, $\boldsymbol{\xi} = (0, 0, \xi_z)$, and k_{1z}^* is the root of $q^2(k_1, k_2) = m^2$ with respect to k_{1z} . Indeed, a shift $\xi_z \rightarrow \xi_z - k_{1z}$ followed by integration over k_{1z} removes the $\delta(k_{1z} - k_{1z}^*)$, bringing us back to Eq. (57). The leading term comes from integration near $\boldsymbol{\xi} = 0$. Therefore, when extracting this term, one can set $\boldsymbol{\xi} = 0$ in the numerator of the integrand in Eq. (59). The function $w(\mathbf{q}, \mathbf{k}_1, \mathbf{k}_2)$

given by Eq. (58) [in which $q(k_1, k_2)$ is now on the mass shell] is then nothing but the probability of the scattering

$$\text{electron}(q) + \text{photon}(k_2) \rightarrow \text{electron}(q + k) + \text{photon}(k_1),$$

where “electron(q)” denotes the unpolarized electron with momentum q , and “photon(k)” a photon with momentum k in any of the two polarization states over which summation is done for the initial as well as final photons. In the non-relativistic approximation,

$$w(\mathbf{q}, \mathbf{k}_1, \mathbf{k}_2) = \frac{\pi e^4}{2m^2 |\mathbf{k}_1| |\mathbf{k}_2|} \left[1 + \frac{(\mathbf{k}_1 \mathbf{k}_2)^2}{k_1^2 k_2^2} \right].$$

In the denominator, we expand $q(k_1 + \xi, k_2)$ with respect to ξ to the first order

$$m^2 - (q(k_1 + \xi, k_2) + v_1 + v_2)^2 \rightarrow -2q^0(k_1, k_2) \left(v_1^0 + v_2^0 + \xi_z \frac{\partial q^0(k_1, k_2)}{\partial k_{1z}} \right).$$

Introducing a new integration variable $\zeta = \xi_z \partial q^0(k_1, k_2) / \partial k_{1z}$, and using

$$\delta(k_{1z} - k_{1z}^*) = \left| \frac{\partial q^0(k_1, k_2)}{\partial k_{1z}} \right| \delta(q^0(k_1, k_2) - \varepsilon_{\mathbf{q}})$$

thus gives

$$\begin{aligned} \delta_C \varrho(t, \mathbf{q}) &= I(\delta t) \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \int \frac{d^3 \mathbf{k}_2}{(2\pi)^3} w(\mathbf{q}, \mathbf{k}_1, \mathbf{k}_2) n(\mathbf{k}_1) [1 + n(\mathbf{k}_2)] \\ &\quad \times \varrho(t, \mathbf{q} + \mathbf{k}) \delta(q^0(k_1, k_2) - \varepsilon_{\mathbf{q}}), \end{aligned} \quad (60)$$

where

$$I(t) = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} d\zeta \int \frac{[du]_t [dv]_t}{(v^0 + \zeta + i0)(u^0 - \zeta + i0)}.$$

Integrations are done with the help of the formulas

$$\begin{aligned} \int \frac{[dv]_t}{v^0 + \zeta + i0} &= \int \frac{dv_1^0}{2\pi i} \int \frac{dv_2^0}{2\pi i} \frac{e^{-iv_1^0 t} - 1}{v_1^0} \frac{e^{-iv_2^0 t} - 1}{v_2^0} \frac{1}{v_1^0 + v_2^0 + \zeta + i0} = \frac{1 - e^{i\zeta t}}{\zeta}, \\ \int_{-\infty}^{+\infty} d\zeta \frac{1 - e^{i\zeta t}}{\zeta} \frac{1 - e^{-i\zeta t}}{\zeta} &= 2\pi t. \end{aligned}$$

The result is $I(t) = t$. Substituting this into Eq. (60), and dividing by δt , we find the connected contribution to the derivative $\partial \varrho / \partial t$. Comparison with Eq. (56) now gives (we use $q^0(k_1, k_2) = \varepsilon_{\mathbf{q}-\mathbf{k}_1+\mathbf{k}_2} + |\mathbf{k}_1| - |\mathbf{k}_2|$, and change notation $\mathbf{k}_1 \rightarrow \mathbf{p}$, so that $\mathbf{k}_2 = \mathbf{p} + \mathbf{k}$)

$$C(\mathbf{q}, \mathbf{k}) = \int \frac{d^3 \mathbf{p}}{(2\pi)^3} w(\mathbf{q}, \mathbf{p}, \mathbf{k} + \mathbf{p}) \delta(\varepsilon_{\mathbf{q}+\mathbf{k}} + |\mathbf{p}| - |\mathbf{p} + \mathbf{k}| - \varepsilon_{\mathbf{q}}) n(\mathbf{p}) [1 + n(\mathbf{p} + \mathbf{k})].$$

Thus, Eq. (56) becomes

$$\begin{aligned} \frac{\partial \varrho(t, \mathbf{q})}{\partial t} &= \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \delta(\varepsilon_{\mathbf{q}+\mathbf{k}} + |\mathbf{p}| - |\mathbf{p} + \mathbf{k}| - \varepsilon_{\mathbf{q}}) w(\mathbf{q}, \mathbf{p}, \mathbf{k} + \mathbf{p}) \\ &\quad \times \{n(\mathbf{p}) [1 + n(\mathbf{p} + \mathbf{k})] \varrho(t, \mathbf{q} + \mathbf{k}) - n(\mathbf{p} + \mathbf{k}) [1 + n(\mathbf{p})] \varrho(t, \mathbf{q})\}. \end{aligned} \quad (61)$$

The right-hand side of this equation is the standard form of the collision integral as representing the difference of “gains” and “losses” of the electron due to its collisions with photons. Substituting Eq. (20) for the photon distribution, one finds that the equilibrium electron momentum distribution $\varrho(\mathbf{q}) \sim \exp(-\beta \varepsilon_{\mathbf{q}})$. It is to be noted that no quasi-classic condition on the electron state has been imposed in the above derivation. In fact, the existence of irreversible spreading implies that the large-time evolution of the electron cannot be considered quasi-classically.

8. Discussion and conclusions

As was explained in the introduction, the reason that makes the effective field formalism indispensable in quantum field theory is the principal incompleteness of the S -matrix regarding description of the field measurements. A general conclusion of our investigation is that the presence of infrared divergences in the field expectation values is not a sign of restricted validity of the effective field formalism, but rather an indication on inadequacy of the standard approach based on the assumption that the problem admits an infinite temporal extension. Specifically, we have shown that restricting consideration to a finite time interval makes the effective field infrared-finite, and that the presence of infrared divergences in the standard approach means that the radiative corrections to the classical field grow unboundedly with time at every order of perturbation theory. However, Eq. (39) shows that the sum of these contributions is bounded.

The physical meaning of infrared singularity in the effective electromagnetic field is the existence of irreversible spreading of electric charges. The simple example given in Section 6.1 demonstrates this irreversibility—the electron coordinate variance cannot be made less than $\sim \alpha T t / m^2$, if the electron travels for a time t in a photon bath at temperature T . The effect takes place also in vacuum, though it is much weaker in that case. This conclusion was formulated in Ref. [19] as a natural interpretation of the results obtained using the momentum-cutoff regularization in the standard approach, based on infinite temporal extension. Now that essentially the same results have been obtained directly from the fundamental principles of quantum field theory, without having to introduce an auxiliary infrared regularization and to use an *ad hoc* definition of the effective density matrix through the effective field, the existence of irreversible spreading is proved to be an unequivocal consequence of quantum electrodynamics.

It should be emphasized that the irreversible spreading does not affect the scattering cross-sections themselves. As long as the single electron evolution in a photon bath is considered, this follows directly from Eq. (6) and the infrared finiteness of the diagonal elements of the electron density matrix (cf. Section 4.2). But this result equally applies to any scattering process. The point is that the scattering amplitudes can be constructed entirely in terms of momenta (and polarizations) of free particles present in the initial and final states; the standard procedure is to formally replace particle wavepackets by infinitely wide homogeneous beams of identical particles, erasing thereby any information about spatial profiles of actual particle states. In other words, it is diagonal elements of the momentum density matrices of particles that only matter when computing the scattering cross-sections, and these are unaffected by the infrared singularity.

Next, some technical remarks are in order. The first concerns the role of the initial condition. Our assumption that the electron is initially statistically independent of the photon field is the simplest and computationally most convenient choice which is also perfectly suitable to study the non-equilibrium electron evolution, as it allows one to see how the electron becomes entangled with the electromagnetic field. It must be stressed, however, that the choice of the initial correlation in the system is immaterial for calculating the large-time asymptotic of the effective field. Indeed, for a given initial electron state (described by some density matrix), different correlations in the system differ by a number of non-equilibrium photons present initially, and the effect of these photons dies away with time in any given spatial point. More specifically, if at t_0 the electron is localized near \mathbf{x}_0 , and the electromagnetic field is measured at a later instant t near the point \mathbf{x} , the measurement is not affected by the additional photons, if $t - t_0 \gg |\mathbf{x} - \mathbf{x}_0|$. This is illustrated by an explicit calculation in the Appendix A. Apart from this, insensitivity of the large-time asymptotic to the choice of initial condition follows also from the fact that the infrared charge spreading irreversibly and unboundedly grows with time. Indeed, a similar growth of the effect produced by a change in the initial condition would mean that the system is unstable, while it is clear in advance that there can be no instability in the system under consideration.

Second, regarding the strength of the irreversible spreading, the role of four-dimensionality of spacetime must be emphasized. That this factor is crucial is evident already from the relation of this spreading to the infrared singularities of radiative corrections, but is particularly clear from comparison with the usual electron–photon scattering responsible for the electron momentum relaxation considered in Section 7. The latter is also described by the loop contributions which diverge

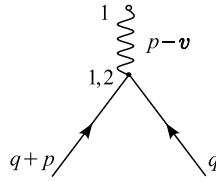


Fig. 11. The tree contribution to the effective electromagnetic field of electron.

for $t \rightarrow \infty$ (see Fig. 10), but by those only which are due to integration over finite photon momenta, and which are therefore insensitive to the spacetime dimensionality. Furthermore, it is the four-dimensionality of spacetime that eventually allows an infinitesimal treatment of the problem, given in Section 5. We note in this connection that the results of Section 5 provide an effective tool for investigating the infrared problem in non-Abelian gauge theories.

Comparison of the two stages of the electron evolution is also useful for interpreting the infrared singularity as a thermalization of the electron state. This thermalization essentially differs from that effected by scattering processes, in several respects. As is evident from the qualitative considerations of Section 2.2, the infrared thermalization is caused by the electron interaction with the electromagnetic field fluctuations, rather than by the electron–photon scattering. Though both processes damp the off-diagonal elements of the matrix $\varrho(t; \mathbf{q}, \mathbf{q}')$, the infrared effect is more complete, as it takes place for all $\mathbf{q}' \neq \mathbf{q}$, whereas the usual thermalization implies only vanishing of the matrix elements between states with different energy, that is, $\mathbf{q}'^2 \neq \mathbf{q}^2$. On the other hand, the infrared thermalization does not change the diagonal elements of the electron density matrix. This is directly related to the fact that it changes only the quantum (von Neumann) entropy, but not the thermodynamic one. Thus, one can say that there is no energy associated with the infrared thermalization. Another important difference is that the infrared thermalization is an $O(\alpha)$ -effect, whereas the electron–photon collision effects are $O(\alpha^2)$, which is reflected in the ratio of characteristic times of the two stages (see Eq. (9)). The role of temperature in these processes is also quite revealing: in contrast to the usual relaxation, the infrared thermalization takes place in vacuum as well as at $T \neq 0$. All these distinctions accentuate the peculiar nature of this phenomenon which has no proper analogy in nonrelativistic physics. Although its possible manifestations are rather weak, the estimates given in Section 6 show that they are detectable in principle by the modern instrumentation.

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Appendix A. Gauss law in the Lorentz gauge

We saw in Section 2.3 that the use of covariant techniques to compute the mean field is hindered by the fact that on a finite time interval, the usual procedure of adiabatic switching of the interaction is not applicable, which leads to appearance of a non-covariant term in the interaction Lagrangian (the last term in Eq. (17)). In this appendix, we demonstrate on a simple example that this complication is not a mere formality: omission of this term leads to violation of the Gauss law already in the tree approximation. In this approximation, the mean electromagnetic field is represented by the diagrams in Fig. 11, whose analytic expression is

$$A_{\mu}^{\text{eff}}(t, \mathbf{x}) = -e \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \int \frac{d^4 v}{(2\pi)^4} J^{\alpha}(\mathbf{p}) e^{i(p_0 - v_0)t - i\mathbf{p}\mathbf{x}} \bar{\Delta}_t(v) g_{\alpha}^{\nu}(p - v) \times [D_{\mu\nu}^{(11)}(p - v) - D_{\mu\nu}^{(21)}(p - v)], \quad (\text{A.1})$$

where $p_0 = \varepsilon_{\mathbf{q}+\mathbf{p}} - \varepsilon_{\mathbf{q}}$, and

$$J^\alpha(\mathbf{p}) = \int \frac{d^3\mathbf{q}}{(2\pi)^3} \sum_{\sigma\sigma'} Q_{\sigma\sigma'}(\mathbf{q}, \mathbf{q} + \mathbf{p}) \frac{\bar{u}_{\sigma'}(\mathbf{q} + \mathbf{p}) \gamma^\alpha u_\sigma(\mathbf{q})}{\sqrt{2\varepsilon_{\mathbf{q}+\mathbf{p}}} \sqrt{2\varepsilon_{\mathbf{q}}}}.$$

As was explained in Section 3.1, we may replace here $d_{\mu\nu}$ by $\eta_{\mu\nu}$, so that

$$D_{\mu\nu}^{(11)}(k) - D_{\mu\nu}^{(21)}(k) = \left[\frac{1}{k^2 + i0} + 2\pi i \theta(k_0) \delta(k^2) \right] \eta_{\mu\nu}.$$

Substituting this into Eq. (A.1), performing the integration over v by closing the contour of v^0 -integration in the lower half-plane ($t > 0$), and using the identity $p_\mu J^\mu(\mathbf{p}) = 0$ we obtain

$$A_\mu^{\text{eff}}(t, \mathbf{x}) = e \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{-i\mathbf{p}\mathbf{x}} \left[-J_\mu(\mathbf{p}) \left\{ \frac{e^{ip_0 t}}{p^2} - \frac{1}{2|\mathbf{p}|} \left(\frac{e^{i|\mathbf{p}|t}}{p_0 - |\mathbf{p}|} - \frac{e^{-i|\mathbf{p}|t}}{p_0 + |\mathbf{p}|} \right) \right\} \right. \\ \left. + J^0(\mathbf{p}) \frac{p_0 \eta_\mu - p_\mu}{p^2} \frac{e^{i|\mathbf{p}|t} - e^{-i|\mathbf{p}|t}}{2|\mathbf{p}|} \right]. \quad (\text{A.2})$$

The term proportional to $J^0(\mathbf{p})$ is the contribution of the non-covariant term in L_I .

Setting $t = 0$ in Eq. (A.2) gives $A_\mu^{\text{eff}}(0, \mathbf{x}) = 0$, but the electric field at this instant is given by

$$\mathbf{E}^{\text{eff}}(0, \mathbf{x}) = \left. \frac{\partial \mathbf{A}_\mu^{\text{eff}}(t, \mathbf{x})}{\partial t} \right|_{t=0} = e \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{-i\mathbf{p}\mathbf{x}} J^0(\mathbf{p}) \frac{-i\mathbf{p}}{p^2} = -\nabla \int d^3\tilde{\mathbf{x}} \frac{eJ^0(0, \tilde{\mathbf{x}})}{4\pi|\mathbf{x} - \tilde{\mathbf{x}}|},$$

where

$$J^0(0, \mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{-i\mathbf{p}\mathbf{x}} J^0(\mathbf{p}),$$

that is, by the Coulomb law, as expected. This is in fact the exact expression for the effective electric field at $t = 0$, since the electron at this instant is assumed to be statistically independent of the photon field (higher order contributions to \mathbf{E}^{eff} vanish for $t = 0$ together with the interaction vertices, cf. Eq. (30)). Furthermore, differentiation of Eq. (A.2) shows that the Gauss law holds at every instant,

$$\partial^\mu A_\mu^{\text{eff}}(t, \mathbf{x}) = 0, \quad \square A_\mu^{\text{eff}}(t, \mathbf{x}) = eJ_\mu(t, \mathbf{x}), \quad \text{where } J_\mu(t, \mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\mathbf{x}} J_\mu(\mathbf{p}).$$

It can be noted also that for $t \rightarrow \infty$, Eq. (A.2) reproduces the more familiar Lorentz-covariant expression

$$A_\mu^{\text{eff}}(t, \mathbf{x}) = -e \int \frac{d^3\mathbf{p}}{(2\pi)^3} J_\mu(\mathbf{p}) \frac{e^{i(p_0 t - \mathbf{p}\mathbf{x})}}{p^2}, \quad \text{when } t \rightarrow \infty,$$

found in formulations admitting infinite temporal extent, e.g., in the theory of S -matrix potentials. Inspection of the integrand in Eq. (A.2) wherein the field-producing electron is initially localized near \mathbf{x}_0 shows that the practical meaning of the limit $t \rightarrow \infty$ is that t must be large compared to $|\mathbf{x} - \mathbf{x}_0|$. Since any electromagnetic disturbance propagates with the speed of light, this is just the condition that all effects related to the initial state of the system die away in the given point.

Suppose now that the non-covariant term is omitted (which is equivalent to replacing $g_\alpha^v \rightarrow \delta_\alpha^v$ in Eq. (A.1)). Then the equation $\square A_\mu^{\text{eff}} = eJ_\mu$ still holds, but

$$\partial^\mu A_\mu^{\text{eff}}(t, \mathbf{x}) = e \int \frac{d^3\mathbf{p}}{(2\pi)^3} J_0(\mathbf{p}) \frac{e^{-i\mathbf{p}\mathbf{x}} \sin |\mathbf{p}|t}{|\mathbf{p}|}. \quad (\text{A.3})$$

It is not difficult to see that the latter equation is inconsistent with the Gauss law. For instance, consider a heavy particle localized in a small vicinity of the origin, so that $\mathbf{J} \approx 0$, $J_0(t, \mathbf{x}) \approx \delta^{(3)}(\mathbf{x})$. Then integration of Eq. (A.3) and $\square A_\mu^{\text{eff}} = eJ_\mu$ gives

$$\mathbf{A}^{\text{eff}}(t, \mathbf{x}) = 0, \quad A_0^{\text{eff}}(t, \mathbf{x}) = \frac{e}{4\pi r} \theta(t - r), \quad (\text{A.4})$$

where r is the distance between the charge and the observation point. Thus, at any given distance r , the Gauss law holds only at times $t > r$. In general, this law is restored only asymptotically: it is seen from Eq. (A.3) that because of the factor $\sin |\mathbf{p}|t$, $\partial^\mu A_\mu^{\text{eff}}$ exponentially tends to zero as $t \rightarrow \infty$.

In the tree approximation, this difficulty with the charge conservation can be overcome by imposing some special conditions on the photon state. However, omission of the non-invariant term in Eq. (17) turns out to be much more harmful in higher orders of perturbation theory, namely, it leads to gauge-dependence of the effective current, which cannot be cured by modifying the photon state vector.

Appendix B. Quantum entropy of a Gaussian wavepacket

Expression (53) gives the asymptotic of the quantum entropy of an arbitrary electron state for $t \gg (\Delta_q^2 \Theta)^{-1}$. In the special case of a Gaussian wavepacket considered in Section 6.1, it is possible to obtain an expression for S , whose validity is conditioned only by that of Eq. (50), that is $Tt \gg 1$. It follows from Eq. (50) that the electron density matrix incorporating infrared effects has the form

$$\varrho(t; \mathbf{q}_1, \mathbf{q}_2) = (8\pi\lambda)^{3/2} \exp \left\{ -\lambda (\mathbf{q}_1^2 + \mathbf{q}_2^2) - \Theta t (\mathbf{q}_2 - \mathbf{q}_1)^2 \right\}.$$

Since $\ln \varrho$ is not expandable in powers of ϱ , we shall evaluate the entropy written as

$$S = \lim_{b \rightarrow 0} S_b, \quad (\text{B.1})$$

where $S_b = -\text{Tr} [\varrho \ln(\varrho + b)]$ will be found as the analytic continuation of the series

$$S_b = -\ln b - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{nb^n} \text{Tr} [\varrho^{n+1}] \quad (\text{B.2})$$

from its convergence domain. Evaluation of the Gaussian integrals gives

$$\begin{aligned} \varrho^n(t; \mathbf{q}_1, \mathbf{q}_2) &= (8\pi\lambda)^{3n/2} d_n \exp \left\{ -\lambda_n (\mathbf{q}_1^2 + \mathbf{q}_2^2) - \theta_n (\mathbf{q}_2 - \mathbf{q}_1)^2 \right\}, \\ \text{Tr} \varrho^n &= \left(\frac{\lambda^n}{\lambda_n} \right)^{3/2} d_n, \end{aligned} \quad (\text{B.3})$$

where the constants d_n , θ_n , λ_n are defined by the following recurrent equations

$$d_{n+1} = \frac{d_n}{[4\pi(\theta_n + \theta + \lambda_n + \lambda)]^{3/2}}, \quad (\text{B.4})$$

$$\theta_{n+1} = \frac{\theta_n \theta}{\theta_n + \theta + \lambda_n + \lambda}, \quad (\text{B.5})$$

$$\lambda_{n+1} = \lambda_n + \frac{(\lambda + \lambda_n)\theta_n}{\theta_n + \theta + \lambda_n + \lambda}, \quad (\text{B.6})$$

with the initial conditions $\theta_1 = \Theta t \equiv \theta$, $\lambda_1 = \lambda$, $d_1 = 1$. Solution of Eqs. (B.4)–(B.6) can be written as

$$d_n = \left[\frac{1}{(4\pi\theta)^{n-1}} \frac{\sinh \varphi}{\sinh n\varphi} \right]^{3/2}, \quad \theta_n = \theta \frac{\sinh \varphi}{\sinh n\varphi}, \quad \lambda_n = \lambda \frac{\tanh(n\varphi/2)}{\tanh(\varphi/2)},$$

where $\varphi > 0$ is defined by $\cosh \varphi = 1 + \lambda/\theta$. Substitution in Eq. (B.3) yields

$$\text{Tr} \varrho^n = \left(\frac{2\lambda}{\theta} \right)^{\frac{3(n-1)}{2}} \frac{\sinh^3(\varphi/2)}{\sinh^3(n\varphi/2)}, \quad (\text{B.7})$$

so that the series (B.2) takes the form

$$S_b = -\ln b - \Phi \left(\frac{1}{b} \left(\frac{2\lambda}{\theta} \right)^{3/2}, \frac{\varphi}{2} \right), \quad (\text{B.8})$$

where

$$\Phi(x, \vartheta) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{\sinh^3 \vartheta}{\sinh^3(n+1)\vartheta} x^n. \quad (\text{B.9})$$

Eq. (B.8) can be analytically continued to $b \rightarrow 0$ using the following identity

$$\begin{aligned} \Phi(x, \vartheta) - 3e^{-2\vartheta} \Phi(xe^{-2\vartheta}, \vartheta) + 3e^{-4\vartheta} \Phi(xe^{-4\vartheta}, \vartheta) - e^{-6\vartheta} \Phi(xe^{-6\vartheta}, \vartheta) \\ = (1 - e^{-2\vartheta})^3 \ln(1 + xe^{-3\vartheta}), \end{aligned}$$

which can be verified by expanding the logarithm into a Taylor series. With the help of this identity, it can be proved that the analytic continuation of $\Phi(x, \vartheta)$ from the convergence domain of the series (B.9), $x < e^{3\vartheta}$, to the region $x \geq e^{3\vartheta}$ satisfies

$$\lim_{x \rightarrow \infty} [\Phi(x, \vartheta) - \ln x] = -\frac{3\vartheta}{\tanh \vartheta}.$$

Thus, taking the limit $b \rightarrow 0$ in Eq. (B.8) we find

$$S = \frac{3}{2} \ln \frac{\theta}{2\lambda} + \frac{3}{2} \frac{\varphi}{\tanh(\varphi/2)},$$

or,

$$S = \frac{3}{2} \ln \frac{\theta t}{2\lambda} + \frac{3}{2} \sqrt{\frac{2\theta t + \lambda}{\lambda}} \ln \left[1 + \frac{\lambda}{\theta t} + \frac{\sqrt{\lambda(2\theta t + \lambda)}}{\theta t} \right].$$

The first term of this expression corresponds to the general asymptotic result (53).

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