The Applicability Problem and a Naturalistic Perspective on Mathematics

Vladislav A. Shaposhnikov
Faculty of Philosophy, Lomonosov Moscow State University
Lomonosovsky Prospekt, 27-4, Moscow, GSP-1, 119991, Russia
vladislavshap@gmail.com

Abstract: The paper outlines a philosophical account of the interplay between pure and applied mathematics. This account is argued to harmonize well with the naturalistic philosophy of mathematics. The autonomy of mathematics is considered as a transitional form between theological and naturalistic views of mathematics. From the naturalistic standpoint, it is natural to understand pure mathematics through applied mathematics but not vice versa. The proposed approach to mathematics is interpreted as a revival of Aristotle's philosophy of mathematics and owes a lot to James Franklin. Wigner's puzzle of applicability is explained away as a survival of the positivist philosophy of mathematics.

Keywords: philosophy of mathematics, naturalism, fictionalism, Platonism, Aristotelianism, applicability of mathematics, Wigner's problem, philosophy of applied mathematics, mixed mathematics, saving the phenomena, autonomy of mathematics, Wigner's gap, TCA-triangle, Theo-Cosmo-Anthropological Triangle

Introduction

It is worth noting that the recent tendency in the philosophy of mathematics is the formation of a new field of inquiry that deals especially with applied mathematics and the applicability of mathematics. The way of reference to this new field of inquiry has significantly not taken its final shape yet. The most popular term is *the applicability of mathematics* (Steiner, 1989; Steiner, 1998; Pincock, 2010; Bangu, 2012); *the philosophy of applied mathematics* is its rival (Brown, 1999; Colyvan, 2009; Pincock, 2009; Colyvan, 2012). Nevertheless, the new field has already achieved a place under the sun as it has found its way into reference books in the philosophy of mathematics (Steiner, 2005; Colyvan, 2009).

This tendency deserves thorough investigation. Moreover, it implies that a new perspective on mathematics as a whole is possible. Let me give two quotations that point us in the right direction. Answering the question "what are the most important open problems in the philosophy of mathematics and what are the prospects for progress?" a British mathematician E. Brian Davies stressed:

> The first problem is that for many philosophers mathematics means pure mathematics, with applied mathematics and its scientific applications regarded as being of lesser importance. The second problem is the depersonalization of the subject, which is frequently regarded not as an activity of mathematicians but as an absolute entity, to be studied without reference to its history or applications to the natural world. This is harmless as far as mathematicians are concerned, but it makes a philosophical analysis of the subject more or less impossible. (Hendricks & Leitgeb, 2008, p. 96)

No doubt Davies hit the nail on the head! Here we almost have a program for philosophy of mathematics in nuce.

Long before Davies, in 1978, Willard Quine similarly observed: “There has been a perverse tendency to think of mathematics primarily as abstract or uninterpreted and only secondarily as interpreted or applied, and then to philosophize about application” (Quine, 1981, p. 148). These words should surely be considered in the light of Quine’s holism and compared with his critique of an isolationist tendency among mathematicians diagnosed with the help of a neologism “mathematosis” (Quine, 1987, pp. 127-129).

Quine and Davies both clearly stated that applied mathematics should be on a par with pure mathematics for a philosopher. What made them stress this equality? Who dared to infringe on the rights of applied mathematics? Whom did they argue against? They challenged a still widespread logical positivists’ account of mathematics. For this account mathematics means pure mathematics as a formal science. There is no applied mathematics but rather an application of a pure formal theory in some factual science through interpretation of the normally uninterpreted basic terms of this formal theory. In such a way mathematics looks quite *autonomous*, i.e. independent of factual sciences. This very account constitutes a background understanding of mathematics in Eugene Wigner’s famous 1959 lecture on “unreasonable effectiveness of mathematics in the natural sciences” (Wigner, 1960). The numerously cited and discussed *Wigner’s problem of applicability* owes its birth to the positivist account of mathematics and will stand or fall by this account.

My point is even stronger than those of Quine and Davies. Applied mathematics, or mathematics-in-application, to put it better, should be granted philosophical priority over pure mathematics. This does not mean the negation of pure mathematics; it just means an appeal to change our philosophical standpoint. When this paper was almost finished, I came across a straightforward presentation of this point of view that I had overlooked before. It was made in the very beginning of the applicability boom by James Franklin, an Australian philosopher of mathematics. Here it is:

> In the meantime it would be appropriate for the philosopher to adopt a neutral stand on the pure/applied distinction. Perhaps it is correct after all to regard mathematics as a body of essential pure knowledge, whose relation of “application” to the world is to be explained. The “unreasonable effectiveness of mathematics” is then an outstanding problem. On the other hand, perhaps the correct view is the one suggested by history, that mathematics is in the first instance applied, and pure mathematics consists in the hard problems of applied mathematics, which, after resisting solution in a single lifetime, acquire a life of their own - this being understood in a purely sociological sense, in that they are worked on by people who have forgotten their motivation. (Franklin, 1988, p. 81)

This attitude is closely associated with naturalism that is taken in my paper as a background methodology rather than a metaphysical creed. The question of a possible interrelation between methodological naturalism and religious belief is a subtle one, and it will be slightly touched upon in the conclusion. The aforementioned recent boom of philosophical publications on applicability is, at least partly, due to a growing influence of naturalism in the contemporary philosophy of mathematics.

According to naturalism mathematics cannot boast the absolute reality but only the historical one. Being put in historical perspective the applicability problem reveals some important details. In sections 1 and 2, I discuss radical changes in the understanding of the applicability of mathematics from antiquity to the time of Galileo to nowadays. Special attention is paid to theological presuppositions of applicability and an attempt to manage without them. This attempt was two-phased: the first phase was to maintain the autonomy of mathematics; the second one was to substitute the autonomy for naturalistic grounds. James Franklin (1994) challenged his colleagues to reflect on the status in our system of knowledge of new areas of mathematization that have emerged recently. His challenge is met in section 3.

1. Theo-Cosmo-Anthropological Triangle and Its Naturalistic Substitution

While discussing the applicability of mathematics it is usual to recall Galileo's famous words: "the book of the universe is written in mathematical language" and to recall them with sympathy. Nevertheless, one hardly ever recalls now their true sense and context for it has changed considerably since the first half of the 17th century. Galileo was a proponent of mathematical description of the world as an absolutely true picture of primary qualities being a part of the divine enterprise. His opponents, such as Cardinal Bellarmine, shared a fictionalist's account of mathematics as a human enterprise that provides us with a wide range of intellectual instruments capable of "saving the appearances" but never with the true nature of things. Our contemporaries' position on mathematics is usually closer to Cardinal Bellarmine's than to Galileo's.

The famous Galileo's words under discussion are taken from his book *Il Saggiatore* [*The Assayer*], published in Rome, in 1623. At the time he argued on the nature of comets with Orazio Grassi (pen name – Lotario Sarsi), a Jesuit mathematician from Collegio Romano. Let us look at these words as they are:

> Furthermore, I seem to detect in Sarsi the firm belief that in philosophizing one must rely upon the opinions of some famous author, so that if our mind does not marry the thinking of someone else, it remains altogether sterile and fruitless. Perhaps he thinks that philosophy is the creation of a man, a book like the *Iliad* or *Orlando Furioso*, in which the least important thing is whether what is written in them is true. Mr. Sarsi, that is not the way it is. Philosophy is written in this all-encompassing book that is constantly open before our eyes, that is the universe; but it cannot be understood unless one first learns to understand the language and knows the characters in which it is written. It is written in mathematical language, and its characters are triangles, circles, and other geometrical figures; without these it is humanly impossible to understand a word of it, and one wanders around pointlessly in a dark labyrinth. (Finocchiaro, 2008, p. 183)

According to Galileo, a mathematical description of the world was the *divine* and not a *human* enterprise. He contrasted his approach to the problem with Grassi's view of mathematics as a human fabrication, like mentioned in the quotation masterpieces of Homer and Ludovico Ariosto. In his later *Dialogue* (1632), he stressed this divine character of mathematical knowledge: within the boundaries of mathematics there is no difference between the human knowledge and the divine one (Galilei, 1967, p. 103). Galileo was not unique in this belief. For example, one of the most straightforward manifestations of the absolutist account of mathematics can be found in Johannes Kepler's *Harmonices Mundi* (1619):

> Geometry, which before the origin of things was coeternal with the divine mind and is God himself (for what could there be in God which would not be God himself?), supplied God with patterns for the creation of the world, and passed over to Man along with the image of God; … (Kepler, 1997, p. 304)

The key figures of the Scientific Revolution shared the kernel of this belief with Kepler and Galileo. René Descartes in his letter to Marin Mersenne (April 15, 1630) wrote:

> The mathematical truths which you call eternal have been laid down by God and depend on him entirely no less than the rest of his creatures. …Please do not hesitate to assert and proclaim everywhere that it is God who has laid down these laws in nature just as a king lays down laws in his kingdom. There is no single one that we cannot understand if our mind turns to consider it. They are all inborn in our minds just as a king would imprint his laws on the hurts of all his subjects if he had enough power to do so. (Descartes, 1970, p. 11)

The shared background conviction for Kepler's, Galileo's, and Descartes' accounts of mathematics can be summarized as a visual diagram (figure 1). Let me call it *The Theo-Cosmo-Anthropological Triangle (TCA-triangle).*

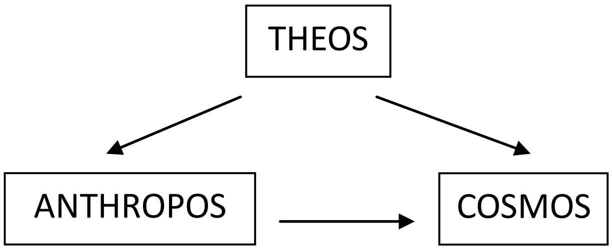

Figure 1: TCA-triangle

The first vertex of it, "God" (θεός), is connected by two arrows with other two vertices. These arrows mean "creation". The second vertex is "the world" (κόσμος) and the third one is "a human being" (ἄνθρωπος), while the horizontal arrow from the third to the second vertex means "cognition". This world and the human cognitive apparatus have been created by the only God. This God is not a deceiver (Descartes). This God is the true bedrock of the human knowledge. This God is the greatest mathematician, who "has arranged all things by measure and number and weight" (Wisdom 11:20). That is why any human being can imitate God by practicing mathematics and can obtain true knowledge of the world through mathematics.

The theological presuppositions played a great role in the projects of the modern science and of the Enlightenment. Thomas Reid successfully put it: as far as the constitution of the human mind is concerned "if we are deceived in it, we are deceived by him that made us, and there is no remedy" (Reid, 1769, p. 112). However, during the 19th century the situation had changed dramatically. Finally the change was diagnosed by Friedrich Nietzsche in the words "God is dead" (Heidegger, 2002).

The TCA-triangle was discredited. The rival account, which is of naturalism, eventually took its place. Naturalism presupposed getting without any explicit or implicit use of supernatural in our reasoning. The TCA-triangle was overturned and displaced by *the naturalistic triangle (N-triangle)*. "Nature" took the place of God (figure 2).

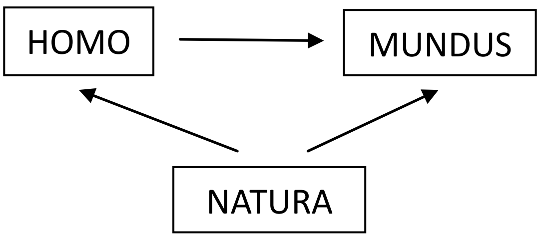

Figure 2: N-triangle

"Natura" in this diagram means a three-layer structure of biological, social and cultural evolutionary mechanisms (Schaeffer, 2007). These mechanisms ensure successful fitting-in of the human cognition ("homo" on the diagram) and the world ("mundus"), in mathematical reasoning, for instance. The Living God is mighty enough to support a straightforward pre-established harmony between the structure of the world and the mathematical mind of a human being. By contrast, the Nature (as defined above) can secure only a more or less appropriate approximation of the former with the latter. Mathematical fictionalism fitted the naturalistic perspective better than theological realism of Galileo and others, so fictionalism was transformed into a more convenient variety which happened to be the idea of mathematical modeling.

Has the theological approach to mathematics and its applicability been completely supplanted by the naturalistic one by now? Shall we witness the restoration of a deliberate theological realism in the near future? These questions are open-ended…

2. From Mixed to Applied Mathematics

Now I will try to examine the interconnection between (theological) realism, naturalism, and fictionalism (as far as mathematics is concerned) in more detail. Since antiquity there have been three accounts of mathematics, akin to the controversy over realism, nominalism, and conceptualism in the Middle Ages. The Pythagoreans, Plato and their heirs maintained that mathematical reality was the true essence of the physical world or even something more real and prior to it; to understand the physical world meant to uncover the hidden mathematical regularities. Let us call this first position *the realist account of mathematics*. Sophists, Epicureans and Skeptics asserted, by contrast, that mathematicians dealt with something non-existent and hence their theories were practically useless work of imagination having nothing to do with the physical world. Let us call the second position *the fictionalist account of mathematics*. The intermediate third position was that of Aristotle, Peripatetics and perhaps of Stoics.

A great discovery of Aristotle was that not all mental fictions were useless and false. According to him, theoretical objects are abstract objects, i.e. they do not exist beyond the mind, and yet to deal with them is not to make a mistake but to find the only possible way to grasp the physical reality in thought (*Met.*, 1078a 21-31). Let us call this position *the mentalist account of mathematics*. It is important to stress that Aristotle's mentalist account was *ambiguous* from the outset for it could be interpreted in both directions: 1) as *realist mentalism* through his concept of the divine mind (*Met.* Λ); 2) as *fictionalist mentalism*, a theory of useful abstractions never losing their links with sense data and physical reality. This ambiguity gives us two different interpretations of the interplay between physics and mathematics within Aristotelianism.

The problem of gaps was also articulated by Aristotle. First of all, in Platonism there is a gap between the divine realm and our world (*the Platonic gap*), but the rupture (χωρισμός) is incomplete for the sensible *participates* in the intelligible, whatever participation (μέθεξις) means. Human mind is divine in its intellectual core and it can be successful in understanding of the physical world only through the partaking in the divine reality. Without the powerful backing from this reality for the human cognition, this cognition would lack adequacy. Now, according to fictionalist account, there is a gap between products of human reason, along with imagination, and the real world (*the Skeptic gap*). Aristotle was the one who tried to bridge these gaps using his theory of abstraction.

Physics and mathematics stand side-by-side in Aristotle's classification of theoretical disciplines according to a degree of abstraction. On the one hand, there is an important difference between physics and mathematics: the first discipline copes with the concrete, especially with motion or process (κίνησις) while the other stands higher in the hierarchy for it abstracts the motion away and leaves only "quantity and continuity (τὸ ποσὸν κὰι συνεχές)" (*Met.*, 1061a 28-35). On the other hand, their side-by-sideness in the hierarchy means an intimate interaction. This interaction Aristotle understood as mixing: along with true mathematics (arithmetic and geometry) and true physics there is *physico-mathematics* or *mixed mathematics*, to use the terms coined in the 17th century (Dear, 1995, pp. 151-179; Brown, 1991), – intermediate, combined or composite disciplines (scientiae mediae) –

astronomy, harmonics, optics, and mechanics (*Post.Anal.*, 78b 35-39; *Phys.*, 194a 7-8) (figure 3a).

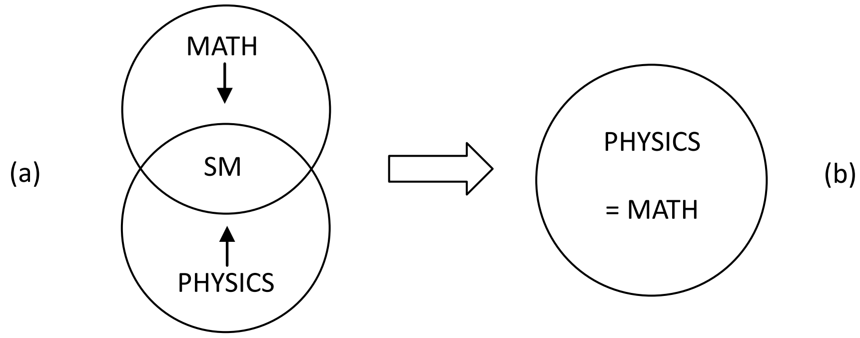

Figure 3: Scientiae mediae (SM) as a mixture of mathematics and physics (a) and the ideal of identification of physics with mathematics through the broadening of SM as physico-mathematics (b)

The 17th and 18th centuries added to this Aristotle's list acoustics (instead of harmonics), pneumatics (a study of gases) and the art of conjecture (probability theory), as well as significantly expanded the content of the old items (Daston, 1988, pp. 222-224; Brown, 1991, p. 89). Mechanics included hydromechanics, together with navigation and naval architecture; astronomy included chronology and cosmography; cosmography embraced geography, hydrography, aerometrics; military architecture and tactics should also be added here.

Aristotle's own example of such an interaction between physics and mathematics is the study of the rainbow (*Post.Anal.*, I, 13, 79a 10-13; *Meteorologica*, III, 2-6). Galileo and others elaborated the realistic interpretation of Aristotle's mentalist account which is so easily confused with the Platonic one (Lennox, 1986). However, there is some evidence that can be interpreted in favor of *instrumental* (i.e. as useful fictions) treatment of mathematical theories already in antiquity. Aristotle's discovery of useful fictions also gave birth to the methodology that I have already mentioned above in connection with the Cardinal Bellarmine vs. Galileo controversy. The case in question is the so called "saving of the appearances" (φαινόμενα σῴζειν), an approach represented mostly, but not exclusively, in astronomy. Pierre Duhem was the one to make the very expression and the fictionalist interpretation of the idea of *saving the phenomena* popular during the last century (Duhem, 1969).

This expression was used in antiquity by Plutarch while speaking on Aristarchus's heliocentrism:

> … in the style of Cleanthes, who thought it was the duty of Greeks to indict Aristarchus of Samos on the charge of impiety for putting in motion the Hearth of the Universe, this being the effect of his attempt to save the phenomena by supposing the heaven to remain at rest and the earth to revolve in an oblique circle, while it rotates, at the same time, about its own axis. (Heath, 1913, p. 304)

Aristarchus's case was quite similar to that of Galileo's. If Aristarchus's idea presupposed a genuine explanation of the real it could be treated as impiety; if it was just a useful fiction Cleanthes the Stoic was wrong. It is worth noting that Aristarchus was said to be studying under Strato of Lampsacus, who afterwards became the third scholarch of Aristotle's Lyceum, and who "liberated God from a big job", in the words of Cicero (Lucullus, 121). What was Aristarchus' original position between realist and fictionalist interpretations of Aristotelianism? We do not know.

Later Simplicius traced the problem of saving the phenomena in astronomy to Plato, as before him Geminus traced it to Pythagoreans, but in the same meaning: as the task of describing the apparent irregular planet motion through a combination of regular uniform circular motions. According to the realistic account this restriction to the mentioned regularity is due to the divine nature of celestial bodies while the irregularity of appearances (such as planetary stations and retrogradations) is ascribable to the illusive nature of sense experience. On the other hand, the fictionalist mentalism tends to interpret it as a lucky analysis and reduction of something complex to simple elements. The ancient Greeks were not alien to both interpretations. The ambiguity of Aristotle's philosophy of mathematics apparently had its decisive effect on the methodology of saving the phenomena (despite Geminus's and Simplicius's pieces of evidence). Lucio Russo was the one to advocate that this methodology was not restricted to astronomy in Hellenistic science; it "was used to some extent even in medicine, but in subjects such as geometry, optics, hydrostatics and astronomy, it reigned uniformly" (Russo, 2004, p. 189).

Aristotle viewed these mixed disciplines as subordinate to both physics and mathematics proper. In classic presentation by Geminus, astronomy, on the one hand, needs arithmetic and geometry, to deal with "quantity, size, and quality of form and shape". On the other, it needs physics to deal with essence or substance, creative forces, and causes. The astronomer "must go to the physicist for his first principles, namely that the movements of the stars are simple, uniform and ordered", while the astronomer by oneself "invents by way of hypothesis, and states certain expedients by the assumption of which the phenomena will be saved" (Heath, 1913, p. 275-276). This description can be applied, mutatis mutandis, to other mixed sciences.

This bunch of mixed disciplines "absorbed the greater part of mathematicians' energies" in the modern period (Daston, 1988, p. 224). Moreover, it pretended to overcome the Aristotelian demarcation line between physics and mathematics and to create a new science of a hybrid nature (figure 3b). This ideal was put into famous words by Kant in the preface to *Metaphysische Anfangsgründe der Naturwissenschaft*: "a doctrine of nature will contain only as much proper science as there is mathematics capable of application there [als Mathematik in ihr angewandt werden kann]" (Kant, 2004, p. 6; Kant, 1786, S. VIII-IX). Kant's *pure reason* and the French Revolution opened a new era. It is worth noting that the same word angewandt was used in 1826 in the title of the famous A.L. Crelle's *Journal für die reine und angewandte Mathematik*. Eventually, about the second half of the 19th century, the famous pair of *pure and applied mathematics* supplanted the terminology of mixed mathematics altogether. "By 1875 [mathematical] theories were no longer 'mixed' with experience, they were 'applied' to experience" (Brown, 1991, p. 102).

The new situation was put into words by Lorraine J. Daston in the following way:

> We believe that pure mathematics is conceptually and for the most part historically prior to and independent of applied mathematics. Indeed, the very term *applied mathematics* tells all: in order to be applied, the mathematics must already exist in its own right, just as theory is "applied" to practice. (Daston, 1988, p. 221)

Pure mathematics, according to Aristotle, is abstract mathematics. But abstraction initially did not mean real separation. It can be easily interpreted as never losing its ties with the world of experience. This way was chosen in d'Alembert's "Discours préliminaire" for the famous *Encyclopédie*, which is quite emblematic for the mid 18th century understanding of mathematics. Mathematical objects are abstract in the sense that they "have been systematically denuded of all those traits that normally accompany them in perception", but they are not "denaturated" (Daston, 1988, p. 222). For d'Alembert there was almost no gap between mathematics and the world of experience, but "a continuum along which mathematics was 'mixed' with sensible properties in varying proportions" (Daston, 1988, p. 223).

Until the end of the 19th century the way of understanding of pure mathematics changed radically. It was still abstract mathematics, but the very adjective *abstract* had changed its meaning. Now for an object to be abstract meant to be strictly separated from the world of concrete objects, to be *out there*, that is beyond space, time and physical causality. Such a position is closer to Plato's than to Aristotle's. The adjective *abstract* almost changed its meaning to something ideal, but while the Platonic *ideal* had been a name for ultimate reality, the modern *abstract* and *ideal* assimilated a shade of something fictitious. Moreover, there emerged a tendency to identify mathematics with pure mathematics and applied mathematics with application of pure mathematics. According to Penelope Maddy's apt turn of phrase even applied mathematics "became pure" (Maddy, 2008).

From this very point of view Eugene Wigner's "The Unreasonable Effectiveness of Mathematics in the Natural Sciences" treated mathematics (Wigner, 1960). That is why it is appropriate to call this gap between (pure) mathematics and physics (the latter is taken in Aristotle's broad sense) – *Wigner's gap* (figure 4).

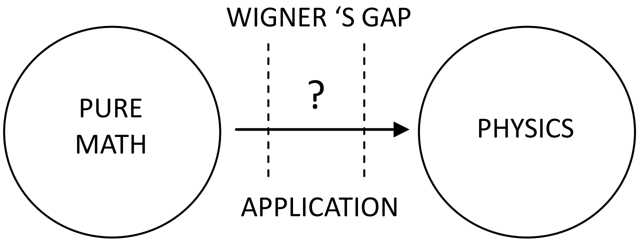

Figure 4: Wigner's problem: the gap between mathematics and physics makes applicability of mathematics a problem.

What caused such a revolution in relations between physics and mathematics, one should ask. Mathematics was posited by Aristotle as an intermediate between two other theoretical kinds of knowledge – the first philosophy (theology) and the second philosophy (physics). Taking this old scheme as a convenient starting point let us assume that mathematics could be seen in a threefold manner: 1) as theologically based; 2) as physically based; 3) as self based or autonomous.

In the 17-18th centuries a theologically based account of mathematics prevailed though with skeptical fictionalist opposition in the background. Then theological view of mathematics and TCA-triangle belief were continuously loosing people's credit (as a part of a more general process of secularization), while the rival physically based naturalistic account came into force only during the 20th century. In the transition period (the end of the 18th century – the beginning of the 20th century) the autonomy of mathematics seemed to be the most promising view and predominated.

This quest for autonomy went hand in hand with growing specialization of mathematics and in mathematics. Mathematicians ceased to be philosophers, theologians and naturalists at the same time, and then mathematics itself was split into minor specialties. Fictionalism got a new life for "not only imaginary elements, but also imaginary theories, theories-fictions" (Perminov, 1997, p. 15) were eventually legalized. Imaginary numbers and infinitesimals were replaced by theories of complex and hyperreal numbers, and numerous inhabitants of the realm of abstract algebra emerged. The legalization of non-Euclidean geometries called for a complete revision of the very concept of mathematics in the spirit of functional and formalist accounts. Pursuit of rigor and absolute intrinsic foundation for mathematics manifested the same trend towards autonomy.

Wigner's gap is a natural successor both to the Platonic and the Skeptic gaps. Pure mathematics of the 19-20th centuries preserved some features of the Platonic true reality as well as of a pure human fiction. Most of mathematicians, on the one hand, considered mathematics to be infallible, consistent, rigorous, certain, necessary and universal, and moreover, applicable to the world. On the other hand, they considered it as a free manifestation of the human mind. "Das Wesen der Mathematik liegt gerade in ihrer Freiheit [the essence of mathematics lies exactly in its freedom]", as Georg Cantor neatly put it

(Cantor, 1932, p. 182); “le but unique de la science, c’est l’honneur de l’esprit humain [the sole end of science is the honor of the human mind]”, according to Carl Jacobi (Jacobi, 1881, p. 454-455). Nevertheless, the status of pure mathematics remained highly obscure (from Kant’s pure reason to Frege’s third realm). The Aristotelian attempt at bridging the gaps was inherited and preserved in *the empiricism about mathematics* (e.g. J.S. Mill), but this position was too weak and uncompetitive at the time. Naturalism later turned out to be a more successful adversary for the theories of autonomy for mathematics.

3. Towards a Revival of Mixed Mathematics

The hope to acquire autonomy for mathematics gradually started ceasing after 1930. Gödel’s incompleteness theorems can be taken as a symbolic watershed. Early attempts at developing a naturalist philosophy of mathematics one can find in Konrad Lorenz (1941) and Leslie White (1947). Nevertheless, mathematical naturalism only got really influential after the Second World War; Yehuda Rav (1989) can serve as a convenient introduction to the subject.

This tendency should inevitably bring us to a revision of the interplay between physics and mathematics. For naturalism it is quite natural to consider mathematics as *physically based*. A naturalistic understanding of phenomena usually goes *upwards* from the human habitat and biological peculiarities of our species to social processes and psychology of human cognition to mathematics and spirituality as parts of human culture. In contrast with a naturalistic one, a theological understanding goes *downwards* from God as the ultimate reality to human spirituality to human cognitive activity and our place in the world. That is why for a theological approach, the genuine form of mathematics is the one oriented towards God rather than towards the world. No wonder that this genuine form usually is pure mathematics. For a naturalistic approach it is, on the contrary, applied mathematics or mathematics-in-application. Such a connection is not something indispensable, but it is rather common and natural, at least as far as the theological understanding is a Platonist one and the naturalistic understanding is an anti-Platonist one. Nevertheless, we cannot exclude the possibility that a theological approach happen to be a voluntarist and anti-Platonist rather than an intellectualist one. A naturalistic approach may tend to be strictly anti-reductionist one that emphasizes autonomy (though restricted) of the highest cultural layer. Furthermore, we also cannot entirely exclude the possibility that some sort of compromise between Platonism and naturalism will be eventually found.

The radical reconsideration of the demarcation line between physics and mathematics became popular in 1990s. The much discussed paper by Arthur Jaffe and Frank Quinn (1993) is a good example. The authors proposed to legitimize the so called *theoretical mathematics*, i.e. production of new mathematical ideas in a speculative manner and without rigorous proves almost indistinguishable from theoretical physics. Another representative of this trend is Vladimir Arnold’s widely known thesis that “mathematics is a part of physics” (Arnol’d, 1998, p. 229). Roland Omnès suggested an approach to mathematics which he called *physism*, according to it “mathematics belongs to the laws of nature, and most closely to those of physics” (Omnès, 2005, p. 199). Eric Zaslow (2005) proposed *physmatics* as a unity of pure mathematics and theoretical physics.

It is worth noting that a similar view of mathematics was proposed by Pavel Florensky (in some connection with the growing interest in the development of analog computers) in the early 1930s. He criticized logicism and formalism for a tendency to attribute human mathematics with absoluteness and mathematicians with omniscience and for denial of vital reliance of mathematical investigation on a wide range of empirical intuitions.

> The more conscious and wide is the life basis of mathematics the more luxurious will be the blossoming of its creativity. Mathematics served and serves science and technology; but let the latter two serve mathematics back. Let various physical factors underlie the development of mathematical [automation] devices; let mathematics take from engineering, from physics, from science, in an open and free gesture, what it has the right to take and what it was always taking from there, but by stealth. Physical models, physical and perhaps chemical devices, biological and psychological aids should be introduced into mathematics. (Florensky, 1932, p. 46)

Florensky's words remind someone of the so called *experimental mathematics*, which became widely discussed only in 1980s-1990s. True, before Florensky there was Oliver Heaviside who called mathematics "an experimental science" in 1893 and stated "rigorous mathematics is narrow, physical mathematics bold and broad" a few years later (Nahin, 2002, pp. 217, 222). However, they were both gifted mavericks.

In my opinion, one of the most challenging works of the last decade of the 20th century that is still of high interest for the philosophy of mathematics is James Franklin's 1994 paper. The author calls attention of philosophers to a host of new disciplines that emerged in the course of the 20th century and whose status is quite obscure. They are operations research, cybernetics, cluster analysis, network analysis, game theory, the theory of self-organizing systems, theoretical computer science, etc. The issue is to propose a general philosophical framework for discussing the status of these new disciplines.

Franklin uses for them a positivist term *formal sciences* (Carnap, 1953), i.e. adds the disciplines in question to logic and mathematics as opposed to factual (natural and social) sciences. On the one hand, he states a similarity between the formal sciences and engineering, "but the formal sciences, though they arose in most cases out of engineering requirements, are sciences, and can be pursued without reference to applications". On the other hand, they are similar to applied mathematics, but there are "some reasons for regarding the formal sciences as something beyond applied mathematics", for "in almost all cases the mathematics had to be created" before being applied (Franklin, 1994, p. 24). He also formulates a task for philosophers: "It would be desirable to have a unified theory that covered mathematics, pure and applied, as well as the formal sciences, and explained both their affinity and their differences" (Franklin, 1994, p. 25).

To my mind the very division of sciences into formal and factual is debatable and rather outdated. Franklin's challenge can be met by return to the term *mixed mathematics* though the concept of mixed mathematics should be modified. In Aristotelian approach mathematics and physics was something primary while mixed sciences was something secondary. The major modification consists in interchange of their roles. Let us test a perspective in which the mixed or hybrid zone is a point of departure while mathematics and physics proper are something derived (figure 6).

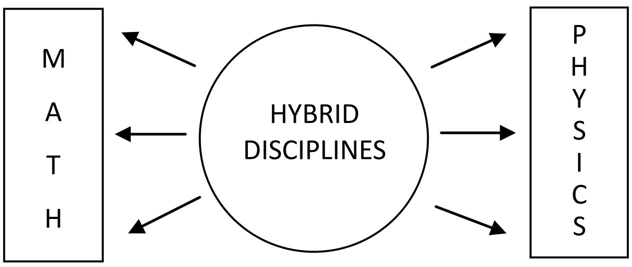

Figure 6: Physico-mathematical hybrid disciplines as a source of new ideas both for mathematics and physics proper.

This perspective should be historically oriented. Mathematics, as well as physics, was isolated by the Ancient Greeks from some previous undifferentiated form of knowledge. Later on, the boundaries of mathematics were constantly changing (for there are no absolute confines of mathematics from a naturalistic point of view). Events on the frontier usually played a decisive role in those changes. A hybrid layer is dividing as well as connecting mathematics with non-mathematics (figure 7).

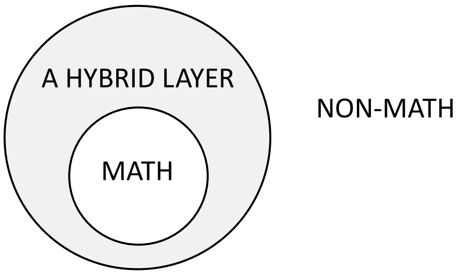

Figure 7: The historically changeable demarcation line between mathematics and non-mathematics is usually blurred into a hybrid layer.

Wigner's gap can be closed through a change of our vision. My above use of the adjective *hybrid* has something to do with Bruno Latour's concept of modernity (Latour, 1993). Wigner's gap is a product of modernity, a product of *the work of purification* that establishes the strict dichotomy between non-human nature and human culture, between the world and mathematics, between pure mathematics and physics. It is high time to bring it into correlation with *the work of translation* that creates *networks* of hybrids of nature and culture, hybrids of physics and mathematics (Latour, 1993, p. 11).

Franklin's new disciplines are new mixed sciences that emerged in the 20th century due to high tech expansion, especially the computer revolution, and military needs. They represent a new stage of interaction of mathematics and the world and are fraught with current and future transformations of mathematics as a distinct area of culture.

Conclusion

A special philosophical interest in the applicability of mathematics that has grown appreciably over the last fifteen years is apparently associated with ongoing changes in mathematics mainly due to its frontier and mixed areas. This interest is also associated with failed attempts to gain autonomy for mathematics and the growing influence of a naturalistic position.

It seems to me that an *applicability-oriented* philosophy of mathematics is an enterprise worth undertaking. Naturalism is an appropriate methodological platform for this enterprise, it is interesting to find out what could and what could not be attained on this route. Is mathematics human or divine in its nature after all? I do not dare to prejudge the issue. I am quite sure that mathematics is not a divine undertaking in some straightforward and primitive manner. Nevertheless, a tangled venture of evolution as a whole perhaps can turn out to be no more than a part in the fulfillment of a divine plan.

Platonism and Skeptical fictionalism, as well as the concept of autonomy, open a gap of one sort or another that necessitate strong positing. Aristotelian and naturalistic approaches try to stop these gaps by less strong means. Was there really an impassable gap between pure mathematics and physics in the culture of the 19th century? Mathematical physics testify against the claim. Let us recall, for example, Joseph Fourier's, Oliver Heaviside's, and Henri Poincaré's achievements. This gap was real but only in a popular philosophy of mathematics.

The extremes of Platonism (theological realism) and Skeptical fictionalism are both hardly viable as far as the philosophy of mathematics is concerned. The most feasible philosophy of mathematics happened to be Aristotelian mentalism, despite or rather thanks to its ambiguity. In the 19th and early 20th centuries' philosophy of mathematics, Aristotelianism had lost its ground and the extremes met (in a paradoxical manner) in the concept of autonomy of mathematical knowledge. It appears to me that now we witness a revival of Aristotelian (in the broad sense) approach to mathematics. It is no coincidence that James Franklin was the one to announce such a revival (Franklin, 2009; 2011; 2014). That is why a naturalistic interpretation and modernization of the concepts of mixed mathematics and of saving the phenomena, inspired principally by Aristotle's philosophy of mathematics, deserve thorough consideration.

References

Arnol'd, V. I. (1998). On teaching mathematics. *Russian Mathematical Surveys*, *53*, 229-236.

Bangu, S. (2012). *The applicability of mathematics in science: Indispensability and ontology.* Palgrave Macmillan.

Brown, G. I. (1991). The evolution of the term "mixed mathematics". *Journal of the History of Ideas*, *52*, 81-102.

Brown, J. R. (1999). *Philosophy of mathematics: An introduction to the world of proofs and pictures.* New York, NY: Routledge.

Cantor, G. (1932). Grundlagen einer allgemeinen Manigfaltigkeitslehre (Leipzig 1883) [Foundations of a general theory of aggregates]. In G. Cantor, *Gesammelte Abhandlungen mathematischen und philosophischen inhalts* (E. Zermelo, Ed.) (pp. 165-209). Berlin: Springer Verlag.

Carnap, R. (1953). Formal and factual sciences. In H. Feigl & M. Brodbeck (Eds.), *Readings in the philosophy of science* (pp. 123-128). New York, NY: Appleton-Century-Crofts, Inc.

Colyvan, M. (2009). Mathematics and the world. In A. D. Irvine (Ed.), *Philosophy of Mathematics. (Handbook of the philosophy of science)* (pp. 651-702). Amsterdam: North Holland/Elsevier.

Colyvan M. (2012). *An introduction to the philosophy of mathematics.* New York, NY: Cambridge University Press.

Daston, L. J. (1988). Fitting numbers to the world: The case of probability theory. In W. Aspray & P. Kitcher (Eds.), *History and philosophy of modern mathematics* (pp. 221-237). Minneapolis, MN: University of Minnesota Press.

Dear, P. (1995). *Discipline and experience: The mathematical way in the scientific revolution.* Chicago, IL: University of Chicago Press.

Descartes, R. (1970). *Descartes: Philosophical letters.* (A. Kenny, Ed. and trans.). Oxford: Clarendon Press.

Duhem, P. (1969). *To save the phenomena: An essay on the idea of physical theory from Plato to Galileo.* Chicago, IL: University of Chicago Press.

Finocchiaro, M. A. (Ed.). (2008). *The essential Galileo.* Indianapolis: Hackett Publishing, Inc.

Florensky, P. A. (1932). Fizika na sluzhbe matematiki [Physics in the service of mathematics]. *Socialisticheskaja Rekonstrukcija i Nauka*, no. 4, 43-63. [In Russian]

Franklin, J. (1988). Mathematics, the computer revolution and the real world. *Philosophica*, 42, 79-92.

Franklin, J. (1994). The formal sciences discover the philosopher's stone. *Studies in History and Philosophy of Science*, *25*, 513-533.

Franklin, J. (2009). Aristotelian realism. In A. D. Irvine (Ed.), *Philosophy of Mathematics. (Handbook of the philosophy of science)* (pp. 101-153). Amsterdam: North Holland/Elsevier.

Franklin, J. (2011). Aristotelianism in the philosophy of mathematics. *Studia Neoaristotelica, 8*, 3-15.

Franklin, J. (2014). *An Aristotelian realist philosophy of mathematics: Mathematics as the science of quantity and structure.* Palgrave Macmillan.

Galilei, G. (1967). *Dialogue concerning the two chief world systems* (2nd rev. ed.). Berkeley, CA: University of California Press.

Heath, T. (1913). *Aristarchus of Samos, the ancient Copernicus.* Oxford: Clarendon Press.

Heidegger, M. (2002). Nietzsche's word: "God is dead". In M. Heidegger, *Off the beaten track* (J. Young & K. Haynes, Eds. and trans.) (pp. 157-199). Cambridge, UK: Cambridge University Press.

Hendricks, V. F. & Leitgeb, H. (Eds.). (2008). *Philosophy of mathematics: 5 questions.* Automatic Press of Vince Inc. Press.

Jacobi, C. G. J. (1881). Correspondance mathématique avec Legendre [Mathematical correspondence with Legendre]. In *C. G. J. Jacobi's gesammelte Werke*, Band I (pp. 385-461). Berlin: Verlag von G. Reimer.
Jaffe, A. & Quinn F. (1993). „Theoretical mathematics": Toward a cultural synthesis of mathematics and theoretical physics. *Bulletin of the American Mathematical Society, 29*, 1-13.
Kant, I. (1786). *Metaphysische Anfangsgründe der Naturwissenschaft* [Metaphysical foundations of natural science]. Riga: J. F. Hartknoch.
Kant, I. (2004). *Metaphysical foundations of natural science* (M. Friedman, Ed. and trans.). Cambridge, UK: Cambridge University Press.
Kepler, J. (1997). *The harmony of the world*. Philadelphia, PA: American Philosophical Society.
Latour, B. (1993). *We have never been modern* (C. Porter, trans.). Cambridge, MA: Harvard University Press.
Lennox, J. G. (1986). Aristotle, Galileo, and "mixed sciences". In W. A. Wallace (Ed.), *Reinterpreting Galileo* (pp. 29-51). Washington, DC: The Catholic University of America Press.
Lorenz, K. (1941). Kants Lehre vom Apriorischen im Lichte gegenwärtiger Biologie [Kant's doctrine of the a priori in the light of contemporary biology]. *Blätter für Deutsche Philosophie, 15*, 94—125.
Maddy, P. (2008). How applied mathematics became pure. *The Review of Symbolic Logic, 1*, 16-41.
Nahin, P. J. (2002). *Oliver Heaviside: The life, work, and times of an electrical genius of the Victorian age*. Baltimore, MD: Johns Hopkins University Press.
Omnès, R. (2005). *Converging realities: Toward a common philosophy of physics and mathematics.* Princeton, NJ: Princeton University Press.
Perminov, V. Ya. (1997). The philosophical and methodological thought of N. I. Lobachevsky. *Philosophia Mathematica (3), 5*, 3-20.
Pincock, C. (2009). Towards a philosophy of applied mathematics. In O. Bueno & Ø. Linnebo (Eds.), *New waves in philosophy of mathematics*. Palgrave Macmillan.
Pincock, C. (2010). The Applicability of Mathematics. In *Internet Encyclopedia of Philosophy (IEP)*. Retrieved from http://www.iep.utm.edu/math-app/
Rav, Y. (1989). Philosophical problems of mathematics in the in the light of evolutionary epistemology. *Philosophica, 43*, 49-78.
Schaeffer, J. - M. (2007). *La fin de l'exception humaine* [The end of the human exception]. Paris: Gallimard.
Steiner, M. (1989). The application of mathematics to natural science. *The Journal of Philosophy, 86*, 449-480.
Steiner, M. (1998). *The applicability of mathematics as a philosophical problem.* Cambridge, MA: Harvard University Press.
Steiner, M. (2005). Mathematics – application and applicability. In S. Shapiro (Ed.), *The Oxford handbook of philosophy of mathematics and logic* (pp. 625-650). New York, NY: Oxford University Press, 2005.
Quine, W. V. (1981). Success and limits of mathematization. In W. V. Quine, *Theories and things* (pp. 148-155). Cambridge, MA: Belknap Press of Harvard University Press.
Quine, W. V. (1987). *Quiddities: An intermittently philosophical dictionary.* Cambridge, MA: Belknap Press of Harvard University Press.
Reid, T. (1769). *An inquiry into the human mind on the principles of common sense* (3rd ed.). London.
Russo, L. (2004). *The forgotten revolution: How science was born in 300 BC and why it had to be reborn.* Berlin: Springer.
White, L. A. (1947). The locus of mathematical reality: An anthropological footnote. *Philosophy of Science, 14*, 289-303.
Wigner, E. P. (1960). The unreasonable effectiveness of mathematics in the natural sciences. *Communications on Pure and Applied Mathematics, 13*, 1-14.
Zaslow, E. (2005). Physmatics. [The paper that accompanied the public lecture at the Fields Institute.] Retrieved from http://arxiv.org/abs/physics/0506153