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Necessary Conditions for an Extremum in a Mathematical Programming Problem

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Abstract—For minimization problems with equality and inequality constraints, first- and second-order necessary conditions for a local extremum are presented. These conditions apply when the constraints do not satisfy the traditional regularity assumptions. The approach is based on the concept of 2-regularity; it unites and generalizes the authors' previous studies based on this concept.

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1. INTRODUCTION

Let X and Y be Banach spaces. We consider a mathematical programming problem

$$f(x) \to \min, \qquad x \in D, \tag{1.1}$$

where

$$D = \bigl\{x \in X \mid F(x) = 0,\ g(x) \le 0\bigr\}, \tag{1.2}$$

$f\colon X \to \mathbb{R}$ is a smooth function, and $F\colon X \to Y$ and $g\colon X \to \mathbb{R}^m$ are smooth mappings.

The literature on the first- and second-order conditions for local extrema in problem (1.1), (1.2) is very extensive (see, e.g., [1, Ch. 1; 2, Sections 3.2, 3.4; 3, Ch. 2, Section 5.2.2] and the references therein). However, in the vast majority of cases (except for certain special cases, which are discussed in what follows), the extremum conditions given there apply only when the Mangasarian–Fromovitz constraint qualification (MFCQ) holds at a point $x_* \in D$ under consideration, that is, when $\operatorname{im} F'(x_*) = Y$ and there exists a $\bar\xi \in X$ such that

$$F'(x_*)\bar\xi = 0 \qquad \text{and} \qquad \langle g_i'(x_*), \bar\xi\rangle < 0 \quad \forall\, i \in I(x_*), \tag{1.3}$$

where $I(x_*) = \{i = 1, \dots, m \mid g_i(x_*) = 0\}$ is the set of indices of inequality constraints that are active at the point x_*.

Throughout the paper, we assume that the subspace $\operatorname{im} F'(x_*)$ is closed. The traditional studies of problem (1.1), (1.2) are based on the Lagrange principle and use the Lagrangian

$$L(x, \lambda_0, \lambda, \mu) = \lambda_0 f(x) + \langle \lambda, F(x)\rangle + \langle \mu, g(x)\rangle \tag{1.4}$$

of this problem; here $x \in X$, $\lambda_0 \in \mathbb{R}$, $\lambda \in Y^*$, and $\mu \in \mathbb{R}^m$. The classical first-order necessary extremum condition is as follows: If a point x_* is a local solution to problem (1.1), (1.2), then there

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exist multipliers $\lambda_0 \in \mathbb{R}$, $\lambda \in Y^*$, and $\mu \in \mathbb{R}^m$ such that they do not vanish simultaneously and

$$\frac{\partial L_0}{\partial x}(x_*, \lambda_0, \lambda, \mu) = 0, \qquad \lambda_0 \geq 0, \qquad \mu \geq 0, \qquad \langle \mu, g(x_*) \rangle = 0. \tag{1.5}$$

If the MFCQ holds at the point x_*, then the above necessary condition can be satisfied only for $\lambda_0 > 0$ and is therefore equivalent to the existence of multipliers $\lambda \in Y^*$ and $\mu \in \mathbb{R}^m$ such that

$$\frac{\partial L}{\partial x}(x_*, 1, \lambda, \mu) = 0, \qquad \mu \geq 0, \qquad \text{and} \qquad \langle \mu, g(x_*) \rangle = 0. \tag{1.6}$$

If the MFCQ is violated at x_*, then the necessary condition (1.5) holds automatically for $\lambda_0 = 0$ and some $\lambda \in Y^*$ and $\mu \in \mathbb{R}^m$ that do not vanish simultaneously; in this form, this condition is merely equivalent to the violation of the MFCQ at the point x_* and, accordingly, does not express anything else. Therefore, the construction of more delicate necessary extremum conditions that would be meaningful even when the MFCQ is violated is of great interest.

In [4, 5], the following generalized Lagrangian for problem (1.1), (1.2) was introduced:

$$L_2(x, h, \lambda_0, \lambda^1, \lambda^2, \mu^1, \mu^2) = \lambda_0 f(x) + \langle \lambda^1, F(x) \rangle + \langle \mu^1, g(x) \rangle + \langle \lambda^2, F'(x)h \rangle + \langle \mu^2, g'(x)h \rangle, \tag{1.7}$$

where $x, h \in X$, $\lambda_0 \in \mathbb{R}$, $\lambda^1, \lambda^2 \in Y^*$, and $\mu^1, \mu^2 \in \mathbb{R}^m$ (here λ_0, λ^1, λ^2, μ^1, and μ^2 play the role of Lagrange multipliers). Applying this function to the problem with equality constraints (i.e., with $m = 0$), E.R. Avakov obtained first- and second-order necessary conditions for a local extremum that remain meaningful in the irregular case; in [6], with the help of the same function, A.V. Arutyunov found second-order sufficient conditions closely related to these necessary conditions. In (1.7), h is a parameter that varies in a certain set determined by the first and second derivatives of F at the point x_*. All these constructions are based on the notion of 2-regularity; a generalization of this notion plays a central role in this paper as well.

The results of [4, 5] were developed in [7], where they were extended to problems of variational calculus, and in [8], where optimal control problems were considered. In [9], these results were extended to the case of weaker smoothness conditions, and in [10], to the case of nonclosed $\operatorname{im} F'(x_*)$. Similar ideas were used in [11, 12] for problems with inequality constraints. However, problems with both equality and inequality constraints in the case where the MFCQ is violated have not been studied so far. In the present paper, we fill this gap.

We use the following notation. Given a normed linear space U, we denote by U^* its dual and by $B_\varepsilon(\bar{u}) = \{u \in U \mid \|u - \bar{u}\| \leq \varepsilon\}$ the ball of radius $\varepsilon > 0$ centered at $\bar{u} \in U$. For a set $S \subset U$, we denote its interior by $\operatorname{int} S$, its closure by $\operatorname{cl} S$, its linear hull (i.e., the minimal linear subspace containing S) by $\operatorname{lin} S$, its affine hull (i.e., the minimal affine set containing S) by $\operatorname{aff} S$, its conical hull (the minimal cone containing S) by $\operatorname{cone} S$, and its annihilator by $S^\perp = \{l \in U^* \mid \langle l, u \rangle = 0 \; \forall\, u \in S\}$. The distance from a point $\bar{u} \in U$ to a set S is denoted by $\operatorname{dist}(\bar{u}, S) = \inf_{u \in S} \|u - \bar{u}\|$. If V is another normed linear space, then $\mathcal{L}(U, V)$ is the space of continuous linear operators from U to V. Given a linear operator $A\colon U \to V$, denote by $\operatorname{im} A$ its range (the set of values) and by $\ker A$ its kernel (the set of zeros).

2. AUXILIARY RESULTS

Below we need the following lemma, which is a consequence of Theorems 17 and 18 from [13].

Lemma 1. *Suppose that U and V are Banach spaces, $P \subset U$ is a closed convex set, $\Phi\colon U \to V$ is a mapping continuous in a neighborhood of a point $\bar{u} \in P$, and $\bar{v} = \Phi(\bar{u})$. Suppose also that there exist numbers $a > 0$, $\varepsilon_1 > 0$, and $\varepsilon_2 > 0$ such that*

$$\operatorname{dist}\bigl(u, \Phi^{-1}(v) \cap P\bigr) \leq a\|\Phi(u) - v\| \qquad \forall\, u \in B_{\varepsilon_1}(\bar{u}) \cap P, \quad \forall\, v \in B_{\varepsilon_2}(\bar{v}).$$

Then, there exist numbers $c = c(a) > 0$, $\delta_1 = \delta_1(a, \varepsilon_1, \varepsilon_2) > 0$, and $\delta_2 = \delta_2(a) > 0$ such that, for any mapping $\varphi\colon U \to V$ satisfying the Lipschitz condition with constant $l < 1/a$ on $B_{\delta_1}(\bar{u})$ and the condition $\|\bar{v} + \varphi(\bar{u})\| \le \delta_2$, there exists a $u \in P$ for which

$$\Phi(u) + \varphi(u) = 0 \qquad \text{and} \qquad \|u - \bar{u}\| \le c\|\bar{v} + \varphi(\bar{u})\|.$$

Lemma 2. *Suppose that U and V are Banach spaces, $A \in \mathcal{L}(U, V)$, the subspace $\operatorname{im} A$ is closed, $B \in \mathcal{L}(U, \mathbb{R}^s)$, $l \in U^*$, $a \in \mathbb{R}$, $v \in V$, and $w \in \mathbb{R}^s$. Suppose also that*

$$S = \big\{u \in U \mid Au + v = 0,\ Bu + w \le 0\big\} \ne \varnothing. \tag{2.1}$$

Then, the condition

$$\langle l, u\rangle + a \ge 0 \qquad \forall\, u \in S \tag{2.2}$$

is equivalent to the existence of $\lambda \in V^*$ and $\mu \in \mathbb{R}^s$ such that

$$l + A^*\lambda + B^*\mu = 0, \qquad \mu \ge 0, \qquad \text{and} \qquad a + \langle\lambda, v\rangle + \langle\mu, w\rangle \ge 0. \tag{2.3}$$

Proof. For finite-dimensional U and V, the implication (2.2) $\Rightarrow$ (2.3) was proved in [14, Ch. 3, Theorem 12.3]. The proof can be easily extended to the infinite-dimensional case. The idea is to reduce the inhomogeneous case to the homogeneous one. Namely, we set

$$\widetilde{S} = \big\{(u, \tau) \in U \times \mathbb{R}_+ \mid Au + \tau v = 0,\ Bu + \tau w \le 0\big\} \ne \varnothing$$

and claim that

$$\langle l, u\rangle + a\tau \ge 0 \qquad \forall\,(u, \tau) \in \widetilde{S}. \tag{2.4}$$

Indeed, for $\tau > 0$, this assertion follows immediately from (2.2). On the other hand, if there exists a $\widetilde{u} \in U$ for which

$$\langle l, \widetilde{u}\rangle < 0, \qquad A\widetilde{u} = 0, \qquad \text{and} \qquad B\widetilde{u} \le 0,$$

then, for any $u \in S$, we have $u + t\widetilde{u} \in S$ for all $t \ge 0$, and $\langle l, u + t\widetilde{u}\rangle + a < 0$ for any sufficiently large $t \ge 0$, which contradicts (2.2). This proves (2.4). Now, the required assertion follows easily from (2.4) and the dual cone lemma [2, Section 3.3.4].

The implication (2.3) $\Rightarrow$ (2.2) is verified directly. This completes the proof of the lemma.

Recall that a cone $K \subset V$ is said to be *finitely generated* if it is the conical hull of finitely many points in V.

Lemma 3. *Suppose that V is a Banach space, $W \subset V$ is a closed linear subspace, and $K \subset V$ is a finitely generated cone.*

Then, the cone $W + K$ is closed.

Proof. Let $K = \operatorname{cone}\{\bar{v}^1, \dots, \bar{v}^s\}$, where $\bar{v}^1, \dots, \bar{v}^s$ are given points. We prove the lemma by induction on s.

Let $s = 1$. If $\bar{v}^1 \in W$, then the set $W + K = W$ is closed. Suppose that $\bar{v}^1 \notin W$. Then the second separation theorem for convex sets (see, e.g., [15, p. 210]) implies the existence of a linear continuous functional $\nu \in W^\perp$ such that $\langle\nu, \bar{v}^1\rangle \ne 0$. Consider an arbitrary sequence $\{v^k\} \subset W + K$ converging to some $\bar{v} \in V$. For any k, the point v^k can be represented as $v^k = w^k + \theta_k\bar{v}^1$, where $w^k \in W$ and $\theta_k \ge 0$. Therefore,

$$\theta_k\langle\nu, \bar{v}^1\rangle = \langle\nu, v^k\rangle \to \langle\nu, \bar{v}\rangle \qquad \text{as} \quad k \to \infty;$$

this and the inequality $\langle\nu, \bar{v}^1\rangle \ne 0$ imply that the sequence $\{\theta_k\}$ is bounded. Passing to a subsequence if necessary, we achieve the convergence of $\{\theta_k\}$ to some $\bar{\theta}$. The equality $w^k = v^k - \theta_k\bar{v}^1$ and the convergence of $\{v^k\}$ to $\bar v$ imply $\{w^k\} \to \bar w = \bar v - \bar\theta \bar v^1$; since W is closed, it follows that $\bar w \in W$. Hence, $\bar v = \bar w + \bar\theta\bar v^1 \in W + K$, as required.

Let us prove the required assertion for $s = r$, assuming that it is true for $s = r - 1$. First, suppose that $\{-\bar v^1, \dots, -\bar v^r\} \subset W + K$. Let us show that

$$W + K = W + \operatorname{lin} K. \tag{2.5}$$

The inclusion $W + K \subset W + \operatorname{lin} K$ is obvious. On the other hand, since $\pm\bar v^i \in W + K$ for all $i = 1, \dots, r$ and $W + K$ is a convex cone, it follows that $\operatorname{lin} K = \operatorname{lin}\{\bar v^1, \dots, \bar v^r\} \subset W + K$, which (together with the obvious inclusion $W \subset W + K$) implies $W + \operatorname{lin} K \subset W + K$. This completes the proof of (2.5). In turn, (2.5) implies that $W + K$ is closed (because, as is known, the sum of a closed linear subspace and a finite-dimensional one is closed).

Now, suppose that $-\bar v^i \notin W + K$ for at least one $i = 1, \dots, r$, say, for $i = r$. The cone $\widetilde K = W + \operatorname{cone}\{\bar v^1, \dots, \bar v^{r-1}\}$ is closed by the induction hypothesis. Consider an arbitrary sequence $\{v^k\} \subset W + K$ converging to some $\bar v \in V$. For any k, the point v^k can be represented as $v^k = w^k + \theta_k \bar v^r$, where $w^k \in \widetilde K$ and $\theta_k \ge 0$. If the sequence $\{\theta_k\}$ is bounded, then, passing to a subsequence if necessary, we achieve the convergence of $\{\theta_k\}$ to some $\bar\theta$. The equality $w^k = v^k - \theta_k \bar v^r$ and the convergence of $\{v^k\}$ to $\bar v$ imply $\{w^k\} \to \bar w = \bar v - \bar\theta \bar v^r$, and the closedness of $\widetilde K$ implies $\bar w \in \widetilde K$. Therefore, $\bar v = \bar w + \bar\theta \bar v^r \in W + K$, as required.

If $\{\theta_k\} \to +\infty$ as $k \to \infty$, then the equality $v^k = w^k + \theta_k \bar v^r$ implies $\frac{1}{\theta_k} w^k = \frac{v^k}{\theta_k} - \bar v^r$, and the convergence of $\{v^k\}$ implies $\{w^k/\theta_k\} \to -\bar v^r$. Moreover, all members of this sequence belong to the closed cone $\widetilde K$, and so $-\bar v^r \in \widetilde K \subset W + K$, which contradicts the assumption $-\bar v^r \notin W + K$. This completes the proof of the lemma.

Recall that the codimension of a linear subspace L in a linear space V is said to be at most r if there exists a linear subspace $M \subset V$ such that $\dim M = r$ and $L + M = V$. This property is equivalent to the existence of (not necessarily continuous) linear functionals $\nu^i \colon V \to \mathbb{R}$, $i = 1, \dots, r$, such that $L = \bigcap_{i=1}^r \ker \nu^i$.

Below we need the following proposition, which is proved, for example, in [16, Ch. 3, Section 5, p. 203].

Proposition 1. *Suppose that U and V are Banach spaces, $\Lambda \in \mathcal{L}(U, V)$, and the codimension of* $\operatorname{im}\Lambda$ *in V is finite.*

Then $\operatorname{im}\Lambda$ *is closed.*

Lemma 4. *Suppose that U and V are Banach spaces, $A \in \mathcal{L}(U, V)$, the subspace* $\operatorname{im} A$ *is closed, and $B \in \mathcal{L}(U, \mathbb{R}^s)$.*

Then the subspace $M = \{(v, w) \in V \times \mathbb{R}^s \mid \exists\, u \in X$ such that $v = Au$ and $w = Bu\}$ is closed.

Proof. Since $\operatorname{im} A$ is closed, the subspace $\operatorname{im} A \times \mathbb{R}^s$ is closed in $V \times \mathbb{R}^s$ and can be regarded as a Banach space (with the induced norm). Consider the operator $\Lambda \in \mathcal{L}(U, \operatorname{im} A \times \mathbb{R}^s)$ defined by $\Lambda u = (Au, Bu)$. Obviously, the codimension of $\operatorname{im}\Lambda$ in $\operatorname{im} A \times \mathbb{R}^s$ is at most s. Therefore, according to Proposition 1, the subspace $M = \operatorname{im}\Lambda$ is closed. This completes the proof of the lemma.

Lemma 5. *Suppose that U and V are Banach spaces, $A, \Lambda \in \mathcal{L}(U, V)$, the subspace $\Lambda(\ker A)$ is closed, $B \in \mathcal{L}(U, \mathbb{R}^s)$, $\bar v \in V$, and $\bar w \in \mathbb{R}^s$. Suppose also that*

$$S = \bigl\{u \in U \mid Au + \bar v = 0,\ Bu + \bar w \le 0\bigr\} \ne \varnothing \tag{2.6}$$

and

$$\operatorname{cl}(\operatorname{lin}\Lambda(S)) = V. \tag{2.7}$$

Then

$$\operatorname{int}\Lambda(S) \ne \varnothing. \tag{2.8}$$

Proof. Without loss of generality, we can assume that $0 \in \Lambda(S)$ (otherwise, we replace $\Lambda(S)$ by $\Lambda(S) - v$ for an arbitrary fixed $v \in \Lambda(S)$ everywhere in the proof; such a change affects neither (2.7) nor (2.8)).

We set $L = \ker A \cap \ker B$. Obviously, $S + L = S$, which implies $L \subset \operatorname{lin} S$ and

$$\Lambda(L) \subset \operatorname{lin} \Lambda(S). \tag{2.9}$$

Take any element $\bar{u} \in U$ for which $A\bar{u} + \bar{v} = 0$. By virtue of (2.6), (2.7), and the closedness of $\Lambda(\ker A)$, we have

$$\begin{aligned} V = \operatorname{cl}(\operatorname{lin} \Lambda(S)) &\subset \operatorname{cl}(\operatorname{lin} \Lambda(\ker A + \bar{u})) \subset \operatorname{cl}(\Lambda(\ker A) + \operatorname{lin}\{\Lambda\bar{u}\}) \\ &\subset \operatorname{cl} \Lambda(\ker A) + \operatorname{cl}(\operatorname{lin}\{\Lambda\bar{u}\}) = \Lambda(\ker A) + \operatorname{lin}\{\Lambda\bar{u}\}. \end{aligned} \tag{2.10}$$

Obviously, the codimension of L in $\ker A$ is at most s; i.e., there exists a linear subspace M in $\ker A$ such that $\dim M = s$ and $L + M = \ker A$. It follows from (2.10) that

$$V = \Lambda(L + M) + \operatorname{lin}\{\Lambda\bar{u}\} = \Lambda(L) + \Lambda(M) + \operatorname{lin}\{\Lambda\bar{u}\}, \tag{2.11}$$

which implies that the codimension of $\Lambda(L)$ in V is at most $s + 1$. Therefore, by Proposition 1, $\operatorname{im} \Lambda(L)$ is closed. It easily follows from (2.9) and (2.11) that

$$\operatorname{lin} \Lambda(S) = \Lambda(L) + \big(\Lambda(M) + \operatorname{lin}\{\Lambda\bar{u}\}\big) \cap \operatorname{lin} \Lambda(S);$$

therefore, $\operatorname{lin} \Lambda(S)$ is closed, being the sum of a closed linear subspace and a finite-dimensional one. By virtue of (2.7), we have

$$\operatorname{lin} \Lambda(S) = V. \tag{2.12}$$

Since $\Lambda(L)$ is a closed linear subspace of finite codimension in V, there exists a finite-dimensional subspace $\widetilde{V}$ in V such that $V = \Lambda(L) \oplus \widetilde{V}$. Let π denote the projector onto $\widetilde{V}$ along $\Lambda(L)$, i.e., $\pi \in \mathcal{L}(V, \widetilde{V})$, $\pi^2 = \pi$, $\operatorname{im} \pi = \widetilde{V}$, and $\ker \pi = \Lambda(L)$. By virtue of (2.12), we have

$$\operatorname{lin} \pi(\Lambda(S)) = \pi(\operatorname{lin} \Lambda(S)) = \operatorname{im} \pi = \widetilde{V}.$$

Since $0 \in \Lambda(S)$, it follows that $0 \in \pi(\Lambda(S))$ and, therefore, $\operatorname{aff} \pi(\Lambda(S)) = \operatorname{lin} \pi(\Lambda(S)) = \widetilde{V}$. But if the affine hull of a convex set in a finite-dimensional space coincides with the entire space, then the interior of this set is nonempty (see, e.g., [14, Ch. 3, Theorem 1.12]). Thus,

$$\operatorname{int} \pi(\Lambda(S)) \neq \varnothing. \tag{2.13}$$

The equality

$$\Lambda(S) + \Lambda(L) = \Lambda(S) \tag{2.14}$$

immediately follows from the equality $S + L = S$; hence,

$$\pi^{-1}(\pi(\Lambda(S))) = \Lambda(S). \tag{2.15}$$

Indeed, if $v \in \pi^{-1}(\pi(\Lambda(S)))$, then $\pi v \in \pi(\Lambda(S))$, i.e., there exists $v^1 \in \Lambda(S)$ such that $\pi(v - v^1) = 0$. In other words, $v^2 = v - v^1 \in \ker \pi = \Lambda(L)$; this and (2.14) imply that $v = v^1 + v^2 \in \Lambda(S)$. Thus, $\pi^{-1}(\pi(\Lambda(S))) \subset \Lambda(S)$. The reverse inclusion is obvious.

Relations (2.13) and (2.15) imply (2.8), because the preimages of open sets under continuous mappings are open. This completes the proof of the lemma.

Lemma 6. *Suppose that U and V are Banach spaces, $A, \Lambda \in \mathcal{L}(U,V)$, the subspace $\Lambda(\ker A)$ is closed, $B \in \mathcal{L}(U, \mathbb{R}^s)$, $\bar{v} \in V$, and $\bar{w} \in \mathbb{R}^s$. Suppose also that* (2.6) *holds.*

Then the set $\Lambda(S)$ is closed.

Proof. Take an element $\bar{u} \in U$ such that $A\bar{u} + \bar{v} = 0$. We have

$$\Lambda(S) = \big\{v \in V \mid \exists\, u \in \ker A \text{ such that } v = \Lambda(\bar{u}+u) \text{ and } B(\bar{u}+u) + \bar{w} \le 0\big\}. \tag{2.16}$$

Set $U_0 = \ker A$ and $V_0 = \Lambda(\ker A)$. Consider the operator $\Lambda_0 \in \mathcal{L}(U_0, V_0 \times \mathbb{R}^s)$ defined by $\Lambda_0 u = (\Lambda u, Bu)$ and set

$$\Omega = \big\{(v,w) \in V \times \mathbb{R}^s \mid \exists\, u \in \ker A \text{ such that } v = \Lambda(\bar{u}+u) \text{ and } w \ge B(\bar{u}+u) + \bar{w}\big\}.$$

Obviously, the codimension of $\operatorname{im}\Lambda_0$ in $V_0 \times \mathbb{R}^s$ is finite; by Proposition 1, $\operatorname{im}\Lambda_0$ is closed. Moreover,

$$\Omega = \operatorname{im}\Lambda_0 + (\Lambda\bar{u}, B\bar{u} + \bar{w}) + \{0\} \times \mathbb{R}^s_+;$$

hence, Ω is closed by Lemma 3 since $\operatorname{im}\Lambda_0$ is closed. On the other hand, by virtue of (2.16), we have $\Lambda(S) = \{v \in V \mid (v,0) \in \Omega\}$; this and the closedness of Ω imply the closedness of $\Lambda(S)$. This completes the proof of the lemma.

Finally, we mention a simple corollary to the mean-value theorem (see [4]). Suppose that U and V are normed linear spaces and $\Phi\colon U \to V$ is a mapping that is twice Fréchet differentiable at a point $\bar{u} \in U$. Then, for $u^1, u^2 \in U$, we have

$$\|\Delta(u^1) - \Delta(u^2)\| = o(\|u^1\| + \|u^2\|)\|u^1 - u^2\|, \tag{2.17}$$

where

$$\Delta(u) = \Phi(\bar{u}+u) - \Phi'(\bar{u})u - \frac{1}{2}\Phi''(\bar{u})[u,u].$$

3. 2-REGULARITY CONDITIONS OF THE FIRST AND SECOND ORDER

Throughout the rest of the paper, we assume that the mappings F and g are twice Fréchet differentiable at a point $x_* \in D$. The linearization of the set D at x_* is the cone

$$H_1(x_*) = \big\{\xi \in X \mid F'(x_*)\xi = 0,\ \langle g_i'(x_*), \xi\rangle \le 0\ \forall\, i \in I(x_*)\big\}. \tag{3.1}$$

Definition 1. We say that the 2-*regularity condition* holds at the point x_* *in a direction* $h \in X$ if

$$\operatorname{im} F'(x_*) + F''(x_*)[h, H_1(x_*)] = Y \tag{3.2}$$

and there exist $\bar{\xi}^1 \in X$ and $\bar{\xi}^2 \in H_1(x_*)$ such that

$$F'(x_*)\bar{\xi}^1 + F''(x_*)[h, \bar{\xi}^2] = 0 \qquad \text{and} \qquad \langle g_i'(x_*), \bar{\xi}^1\rangle + g_i''(x_*)[h, \bar{\xi}^2] < 0 \quad \forall\, i \in I(x_*). \tag{3.3}$$

Note that the 2-regularity condition in the direction $h = 0$ coincides with the MFCQ (for $\bar{\xi} = \bar{\xi}^1$). Moreover, if the latter holds at the point x_*, then the 2-regularity condition holds at this point in any direction $h \in X$, including $h = 0$ (we can take $\bar{\xi}^1 = \bar{\xi}$ and $\bar{\xi}^2 = 0$).

In the absence of inequality constraints, the 2-regularity condition takes the form

$$\operatorname{im} F'(x_*) + F''(x_*)[h, \ker F'(x_*)] = Y \tag{3.4}$$

and is equivalent to the 2-regularity of the mapping F at the point x_* in the direction h, which was introduced in [4].

Remark 1. It is easy to verify that 2-regularity is stable with respect to small perturbations of h.

The above 2-regularity condition makes it possible to characterize the elements of the contingent cone (the first-order external tangent set) to the feasible set and obtain meaningful first-order necessary conditions for a local extremum (see Sections 4 and 5 in this paper). Thus, it is natural to call this condition a first-order 2-regularity condition. Along with it, we use a generally weaker second-order 2-regularity condition, which is needed to characterize the second-order tangent set and obtain second-order necessary conditions for a local extremum (see Sections 4 and 6 below).

For each $h \in X$, introduce a set of indices $I(x_*, h) = \{i \in I(x_*) \mid \langle g_i'(x_*), h\rangle = 0\}$ and set

$$S_2(x_*, h) = \Big\{x \in X \mid F'(x_*)x + F''(x_*)[h, h] = 0,\ \langle g_i'(x_*), x\rangle + g_i''(x_*)[h, h] \le 0\ \forall\, i \in I(x_*, h)\Big\}. \tag{3.5}$$

In the following definition, it is assumed that the mappings F and g are three times Fréchet differentiable at the point x_*.

Definition 2. We say that the *second-order* 2-*regularity condition* holds at the point x_* *in a direction* $h \in X$ if

$$0 \in \operatorname{int}\Big(\operatorname{im} F'(x_*) + 3F''(x_*)[h, S_2(x_*, h)] + F'''(x_*)[h, h, h]\Big) \tag{3.6}$$

and there exist $\bar{x}^1 \in X$ and $\bar{x}^2 \in S_2(x_*, h)$ such that

$$F'(x_*)\bar{x}^1 + 3F''(x_*)[h, \bar{x}^2] + F'''(x_*)[h, h, h] = 0 \tag{3.7}$$

and

$$\langle g_i'(x_*), \bar{x}^1\rangle + 3g_i''(x_*)[h, \bar{x}^2] + g_i'''(x_*)[h, h, h] < 0 \qquad \forall\, i \in I(x_*). \tag{3.8}$$

It is easy to show that if $h \in H_1(x_*)$, then the set $I(x_*)$ can be replaced by $I(x_*, h)$ in both (3.3) and (3.8): such a change does not affect the 2-regularity and second-order 2-regularity conditions in the direction h. Moreover, it is easy to see that if $S_2(x_*, h) \ne \varnothing$, then 2-regularity in the direction h implies second-order 2-regularity in this direction, but the converse is not generally true. Indeed, the inclusion

$$S_2(x_*, h) + H_1(x_*) \subset S_2(x_*, h) \tag{3.9}$$

follows immediately from (3.1) and (3.5). Since $S_2(x_*, h)$ is nonempty, (3.2) implies (3.6). Take elements $\widetilde{x}^1 \in X$ and $\widetilde{x}^2 \in S_2(x_*, h)$ such that $F'(x_*)\widetilde{x}^1 + 3F''(x_*)[h, \widetilde{x}^2] + F'''(x_*)[h, h, h] = 0$ (such elements exist because of the relation (3.6) proved above). We set $\bar{x}^1 = \widetilde{x}^1 + t\bar{\xi}^1$ and $\bar{x}^2 = \widetilde{x}^2 + \frac{1}{3}t\bar{\xi}^2$, where $\bar{\xi}^1 \in X$ and $\bar{\xi}^2 \in H_1(x_*)$ are the same as in (3.3). Then, for sufficiently large $t > 0$, we obtain (3.7) and (3.8), and (3.9) gives $\bar{x}^2 \in S_2(x_*, h)$.

Examples 3 and 4 below show that the converse implication does not hold. In the absence of inequality constraints, the 2-regularity and second-order 2-regularity conditions are equivalent (and, accordingly, both are equivalent to (3.4)).

4. CONTINGENT CONES AND SECOND-ORDER TANGENT SETS

The contingent cone to the set D at the point x_* is defined as follows:

$$T_D(x_*) = \Big\{h \in X \mid \exists\, \{t_k\} \subset \mathbb{R}_+ \setminus \{0\} \text{ such that } \{t_k\} \to 0 \text{ and } \operatorname{dist}(x_* + t_k h, D) = o(t_k)\Big\}. \tag{4.1}$$

Accordingly, the (external) second-order tangent set to D at x_* in a direction $h \in X$ is defined as

$$T_D^2(x_*, h) = \bigg\{x \in X \;\bigg|\; \exists\, \{t_k\} \subset \mathbb{R}_+ \setminus \{0\} \text{ such that } \{t_k\} \to 0 \text{ and } \operatorname{dist}\Big(x_* + t_k h + \frac{1}{2}t_k^2 x, D\Big) = o\big(t_k^2\big)\bigg\}. \tag{4.2}$$

As is known, both of these sets are always closed (see, e.g., [3]).

Consider the cone

$$H_2(x_*) = \Big\{h \in H_1(x_*) \;\Big|\; \exists\, x \in X \text{ such that } F'(x_*)x + F''(x_*)[h,h] = 0 \text{ and } \langle g_i'(x_*), x\rangle + g_i''(x_*)[h,h] \le 0 \ \forall\, i \in I(x_*,h)\Big\}$$
$$= \big\{h \in H_1(x_*) \mid S_2(x_*,h) \ne \varnothing\big\}. \tag{4.3}$$

Let $H_2^1(x_*)$ denote the cone consisting of all $h \in H_2(x_*)$ for which the 2-regularity condition holds at the point x_* in the direction h. Similarly, by $H_2^2(x_*)$ we denote the cone consisting of all $h \in H_2(x_*)$ for which the second-order 2-regularity condition holds at x_* in the direction h. It follows from what was said above that $H_2^1(x_*) \subset H_2^2(x_*)$.

The following assertion is a corollary to Theorem 6′ from [17]. We give its proof for completeness.

Theorem 1. *The following inclusions hold*:

$$H_2^1(x_*) \subset T_D(x_*) \subset H_2(x_*). \tag{4.4}$$

Proof. For simplicity, we assume that $g(x_*) = 0$ (otherwise, we replace g by a mapping with components g_i, $i \in I(x_*)$).

Consider the cone $K = \{0\} \times \mathbb{R}^m_-$ in the space $Y \times \mathbb{R}^m$. Let $\Phi\colon X \to Y \times \mathbb{R}^m$ be the mapping defined by $\Phi(x) = (F(x), g(x))$. It follows from (1.2) that

$$D = \{x \in X \mid \Phi(x) \in K\}. \tag{4.5}$$

Moreover, we have $\Phi(x_*) = 0$, and $\operatorname{im}\Phi'(x_*)$ is closed by Lemma 3. It is easy to derive from (3.1), (3.5), and (4.3) that

$$H_2(x_*) = \big\{h \in X \mid \Phi'(x_*)h \in K,\ \Phi''(x_*)[h,h] \in K + \operatorname{im}\Phi'(x_*)\big\}. \tag{4.6}$$

If $h \in T_D(x_*)$, then, according to (4.1) and (4.5), there exist sequences $\{t_k\} \subset \mathbb{R}_+ \setminus \{0\}$ and $\{r^k\} \subset X$ such that $\{t_k\} \to 0$, $r^k = o(t_k)$, and $\Phi(x_* + t_k h + r^k) \in K$ for any k. Hence,

$$K \ni \Phi(x_* + t_k h + r^k) = t_k\Phi'(x_*)h + \Phi'(x_*)r^k + \frac{1}{2}t_k^2\Phi''(x_*)[h,h] + o\big(t_k^2\big),$$

and so

$$K \ni t_k\Phi'(x_*)h + o(t_k) \qquad \text{and} \qquad K + \operatorname{im}\Phi'(x_*) \ni \frac{1}{2}t_k^2\Phi''(x_*)[h,h] + o\big(t_k^2\big);$$

by Lemma 3, the set on the left-hand side of the last inclusion is closed. This and (4.6) immediately imply that $h \in H_2(x_*)$.

Now, suppose that $h \in H_2^1(x_*)$. Take $\bar\xi^1 \in X$ and $\bar\xi^2 \in H_1(x_*)$ as in Definition 1. We set $U = X\times X$, $V = Y$, and $P = X\times H_1(x_*)$. Let Φ be a mapping defined by $\Phi(u) = F'(x_*)x^1 + F''(x_*)[h,x^2]$ for $u = (x^1,x^2) \in U$. We set $\bar u = (\bar\xi^1,\bar\xi^2) \in P$ and $\bar v = \Phi(\bar u)$. By virtue of the equality in (3.3), we have $\bar v = 0$. It follows from (3.2) and Robinson's stability theorem [18] (see also Theorem 2.87 and p. 71 in [3]) that the objects introduced above satisfy all assumptions of Lemma 1 for some $a > 0$, $\varepsilon_1 > 0$, and $\varepsilon_2 > 0$. Let $c > 0$, $\delta_1 > 0$, and $\delta_2 > 0$ be the same as in Lemma 1.

For $x \in X$, we set

$$\Delta_1(x) = F(x_* + x) - F'(x_*)x - \frac{1}{2}F''(x_*)[x,x], \tag{4.7}$$

$$\Delta_2(x) = g(x_* + x) - g'(x_*)x - \frac{1}{2}g''(x_*)[x,x]. \tag{4.8}$$

According to estimate (2.17), we have

$$\Delta_1(th) = o(t^2) \qquad \text{and} \qquad \Delta_2(th) = o(t^2). \tag{4.9}$$

Set

$$\tau(t) = 4\max\bigl\{\|\Delta_1(th)\|^{1/2}, \|\Delta_2(th)\|^{1/2}, t^{3/2}\bigr\}. \tag{4.10}$$

It follows from (4.9) and (4.10) that

$$\tau(t) = o(t) \qquad \text{and} \qquad \lim_{t\to 0+} \frac{t^2}{\tau(t)} = 0. \tag{4.11}$$

Take any $x \in S_2(x_*, h)$ (we have $S_2(x_*, h) \neq \varnothing$ by (4.3)); for any $x^1, x^2 \in X$ and $t > 0$, we set

$$x(x^1, x^2; t) = x_* + th + \tau(t)x^2 + \frac{1}{2}t^2x + t\tau(t)x^1 \tag{4.12}$$

and

$$\begin{aligned}\varphi_t(u) = \varphi_t(x^1, x^2) = {}& \frac{t^2}{2\tau(t)}F''(x_*)[h, x] + tF''(x_*)[h, x^1] \\ & + \frac{1}{2t\tau(t)}F''(x_*)\left[\tau(t)x^2 + \frac{1}{2}t^2x + t\tau(t)x^1,\ \tau(t)x^2 + \frac{1}{2}t^2x + t\tau(t)x^1\right] \\ & + \frac{1}{t\tau(t)}\Delta_1\bigl(x(x^1, x^2; t) - x_*\bigr).\end{aligned} \tag{4.13}$$

Relations (2.17) and (4.11) imply that, on any bounded set, the mapping φ_t satisfies the Lipschitz condition with a constant that tends to zero as $t \to 0+$. Moreover, according to (2.17) and (4.11),

$$\begin{aligned}\left\|\Delta_1\left(th + \tau(t)\bar{\xi}^2 + \frac{1}{2}t^2x + t\tau(t)\bar{\xi}^1\right)\right\| &\le \|\Delta_1(th)\| \\ &\quad + \left\|\Delta_1\left(th + \tau(t)\bar{\xi}^2 + \frac{1}{2}t^2x + t\tau(t)\bar{\xi}^1\right) - \Delta_1(th)\right\| \\ &= \|\Delta_1(th)\| + o(t\tau(t))\end{aligned} \tag{4.14}$$

and, similarly,

$$\left\|\Delta_2\left(th + \tau(t)\bar{\xi}^2 + \frac{1}{2}t^2x + t\tau(t)\bar{\xi}^1\right)\right\| = \|\Delta_2(th)\| + o(t\tau(t)). \tag{4.15}$$

It follows from (4.9)–(4.14) that $\varphi_t(\bar{u}) \to 0$ as $t \to 0+$.

Thus, for any sufficiently small $t > 0$, the mapping φ_t satisfies the Lipschitz condition with a constant $l < 1/a$ on $B_{\delta_1}(\bar{u})$, and $\|\bar{v} + \varphi_t(\bar{u})\| \le \delta_2$.

For $u = (x^1, x^2) \in U$, relations (4.7), (4.12), and (4.13), the definition of Φ, and the inclusions $h \in H_1(x_*)$ and $x \in S_2(x_*, h)$ (see also (3.1) and (3.5)) imply

$$\begin{aligned}F(x(x^1, x^2; t)) = {}& \tau(t)F'(x_*)x^2 + t\tau(t)F'(x_*)x^1 + t\tau(t)F''(x_*)[h, x^2] \\ & + \frac{1}{2}t^3F''(x_*)[h, x] + t^2\tau(t)F''(x_*)[h, x^1] \\ & + \frac{1}{2}F''(x_*)\left[\tau(t)x^2 + \frac{1}{2}t^2x + t\tau(t)x^1,\ \tau(t)x^2 + \frac{1}{2}t^2x + t\tau(t)x^1\right] + \Delta_1(x(t) - x_*) \\ = {}& \tau(t)F'(x_*)x^2 + t\tau(t)(\Phi(u) + \varphi_t(u)).\end{aligned}$$

Then, applying Lemma 1 to the set P and the mappings Φ and $\varphi = \varphi_t$, we conclude that for any sufficiently small $t > 0$, there exist $x^1(t) \in X$ and $x^2(t) \in H_1(x_*)$ (see (3.1)) such that

$$F(x(t)) = 0, \tag{4.16}$$

where $x(t) = x(x^1(t), x^2(t); t)$, and

$$\|x^1(t) - \bar{\xi}^1\| + \|x^2(t) - \bar{\xi}^2\| = \|u - \bar{u}\| \le c\|\bar{v} + \varphi_t(\bar{u})\| = c\|\varphi_t(\bar{u})\|. \tag{4.17}$$

In (4.17), a standard norm on $X \times X$ is assumed which is given by the sum of norms on X.

Since $\varphi_t(\bar{u}) \to 0$ as $t \to 0+$, it follows from (4.17) that

$$\|x^1(t) - \bar{\xi}^1\| \to 0 \qquad \text{and} \qquad \|x^2(t) - \bar{\xi}^2\| \to 0 \tag{4.18}$$

as $t \to 0+$; in particular, (4.11) and (4.12) imply $x(t) = x_* + th + o(t)$. Therefore, according to (4.1) and (4.16), it remains to be shown that $g(x(t)) \le 0$ for any sufficiently small $t > 0$.

The relations (4.8), (4.12), $x^2(t) \in H_1(x_*)$ (see also (3.1)), (4.10), (4.11), (4.15), and (4.18) imply

$$\begin{aligned} g(x(t)) &= tg'(x_*)h + \tau(t)g'(x_*)x^2(t) + \frac{1}{2}t^2\big(g'(x_*)x + g''(x_*)[h,h]\big) \\ &\quad + t\tau(t)\big(g'(x_*)x^1(t) + g''(x_*)[h, x^2(t)]\big) + \frac{1}{2}t^3 g''(x_*)[h,x] + t^2\tau(t)g''(x_*)[h, x^1(t)] \\ &\quad + \frac{1}{2}g''(x_*)\left[\tau(t)x^2(t) + \frac{1}{2}t^2x + t\tau(t)x^1(t),\, \tau(t)x^2(t) + \frac{1}{2}t^2x + t\tau(t)x^1(t)\right] + \Delta_2(x(t) - x_*) \\ &\le tg'(x_*)h + \frac{1}{2}t^2\big(g'(x_*)x + g''(x_*)[h,h]\big) + t\tau(t)\big(g'(x_*)\bar{\xi}^1 + g''(x_*)[h, \bar{\xi}^2]\big) + o(t\tau(t)). \end{aligned} \tag{4.19}$$

If $i \in \{1, \dots, m\} \setminus I(x_*, h)$, then $\langle g'(x_*), h\rangle < 0$ because $h \in H_1(x_*)$ (see (3.1)), and (4.19) implies that $g_i(x(t)) < 0$ for any sufficiently small $t > 0$. If $i \in I(x_*, h)$, then $\langle g'(x_*), h\rangle = 0$ and, since $x \in S_2(x_*, h)$ (see (3.5)), relation (4.19) and the inequalities in (3.3) give

$$g_i(x(t)) \le t\tau(t)\big(g'(x_*)\bar{\xi}^1 + g''(x_*)[h, \bar{\xi}^2]\big) + o(t\tau(t)) < 0$$

for any sufficiently small $t > 0$. This completes the proof of the theorem.

Thus, $H_2(x_*)$ is an outer approximation of the contingent cone $T_D(x_*)$, and $H_2^1(x_*)$ is its inner approximation. In the absence of inequality constraints, Theorem 1 was first obtained in [4]. An earlier version of this result under stronger smoothness requirements can be found in [19].

It is easy to see from the proof of Theorem 1 suggested above that both inclusions in (4.4) remain valid when the contingent cone $T_D(x_*)$ is replaced by the so-called interior tangent cone

$$T_D^{\mathrm{i}}(x_*) = \big\{h \in X \mid \operatorname{dist}(x_* + th, D) = o(t),\ t \ge 0\big\}$$

to the set D at the point x_*.

Definition 3. We say that the 2-*regularity condition* holds at the point x_* if the 2-regularity condition holds at this point in any direction $h \in H_2(x_*) \setminus \{0\}$, i.e., if $H_2(x_*) \setminus \{0\} = H_2^1(x_*) \setminus \{0\}$.

Theorem 1 implies the following corollary.

Corollary 1. *If the* 2*-regularity condition holds at* x_**, then* $T_D(x_*) = H_2(x_*)$.

Moreover, since $T_D(x_*)$ is closed, the equality $T_D(x_*) = H_2(x_*)$ holds even if $H_2^1(x_*)$ is just dense in $H_2(x_*)$.

Recall that if the MFCQ holds at the point x_*, then the 2-regularity condition holds at this point in any direction $h \in X$, i.e., $H_2^1(x_*) = H_2(x_*)$. Moreover, in this case, we have $H_2(x_*) = H_1(x_*)$. Indeed, for any $h \in H_1(x_*)$, we can take an element $\widetilde{x} \in X$ such that $F'(x_*)\widetilde{x} = -\frac{1}{2}F''(x_*)[h, h]$ (such an element exists because $\operatorname{im} F'(x_*) = Y$). We set $x = \widetilde{x} + t\bar{\xi}$, where $\bar{\xi} \in X$ is the same as in (1.3). For any sufficiently large $t > 0$, we then have $x \in S_2(x_*, h)$; according to (4.3), it follows that $h \in H_2(x_*)$.

Thus, if the MFCQ holds at the point x_*, Corollary 1 reduces to the traditional description of the contingent cone by the equality $T_D(x_*) = H_1(x_*)$. However, the conditions of Theorem 1 and Corollary 1 are much less restrictive than the MFCQ.

In the absence of equality constraints, analogs of Theorem 1 and Corollary 1 were obtained in [11, 12].

In the following theorem, it is assumed that the mappings F and g are three times Fréchet differentiable at the point x_*. For any $h, x \in X$, we define a set of indices

$$I(x_*, h, x) = \bigl\{i \in I(x_*, h) \mid \langle g_i'(x_*), x\rangle + g_i''(x_*)[h,\, h] = 0\bigr\}$$

and put

$$S_3(x_*, h) = \Bigl\{x^2 \in S_2(x_*, h) \bigm| \exists\, x^1 \in X \text{ such that } F'(x_*)x^1 + 3F''(x_*)[h, x^2] + F'''(x_*)[h, h, h] = 0,$$
$$\langle g_i'(x_*), x^1\rangle + 3g_i''(x_*)[h, x^2] + g_i'''(x_*)[h, h, h] \le 0\ \forall\, i \in I(x_*, h, x^2)\Bigr\}. \tag{4.20}$$

Theorem 2. *For any $h \in X$, the following inclusion holds*:

$$T_D^2(x_*, h) \subset S_3(x_*, h). \tag{4.21}$$

If $h \in H_2^2(x_*)$, then the reverse inclusion is also valid, i.e.,

$$T_D^2(x_*, h) = S_3(x_*, h). \tag{4.22}$$

Proof. Again, without loss of generality, we assume that $g(x_*) = 0$.

If $T_D^2(x_*, h) \ne \varnothing$, then (4.1) and (4.2) imply $h \in T_D^2(x_*)$; therefore, by virtue of the second inclusion in (4.4), we have $h \in H_2(x_*)$.

Let $x \in T_D^2(x_*, h)$. It follows from (1.2) and (4.2) that there exist sequences $\{t_k\} \subset \mathbb{R}_+ \setminus \{0\}$ and $\{r^k\} \subset X$ such that $\{t_k\} \to 0$, $r^k = o(t_k^2)$, and $\Phi\bigl(x_* + t_k h + \frac{1}{2}t_k^2 x + r^k\bigr) \in K$ for all k.

Let $\widetilde{m}$ denote the number of elements in the set $I(x_*, h)$. Consider the cone $K = \{0\} \times \mathbb{R}_-^{\widetilde{m}}$ in $Y \times \mathbb{R}^{\widetilde{m}}$, the mapping $\widetilde{g}\colon X \to \mathbb{R}^{\widetilde{m}}$ with components $g_i(x)$, $i \in I(x_*, h)$, and the mapping $\Phi\colon X \to Y \times \mathbb{R}^{\widetilde{m}}$ defined by $\Phi(x) = (F(x), \widetilde{g}(x))$. We have $\Phi(x_*) = 0$; (3.1), (4.3), and the inclusion $h \in H_2(x_*)$ imply $h \in \ker \Phi'(x_*)$. Moreover, it follows from (3.5) that

$$S_2(x_*, h) = \bigl\{x \in X \mid \Phi'(x_*)x + \Phi''(x_*)[h, h] \in K\bigr\}. \tag{4.23}$$

For any k, the inclusion $h \in \ker \Phi'(x_*)$ yields

$$K \ni \Phi\left(x_* + t_k h + \frac{1}{2}t_k^2 x + r^k\right) = \frac{1}{2}t_k^2\bigl(\Phi'(x_*)x + \Phi''(x_*)[h, h]\bigr) + o\bigl(t_k^2\bigr).$$

By virtue of (4.23), this implies $x \in S_2(x_*, h)$.

Now, let $\widetilde{m}$ denote the cardinality of the set $I(x_*, h, x)$ and $\widetilde{g}\colon X \to \mathbb{R}^{\widetilde{m}}$ be the mapping with components $g_i(x)$, $i \in I(x_*, h, x)$. Consider the cone $K \subset Y \times \mathbb{R}^{\widetilde{m}}$ and the mapping $\Phi\colon X \to Y \times \mathbb{R}^{\widetilde{m}}$

defined in the same way as above but for the new $\widetilde{m}$ and $\widetilde{g}$. Again, we have $\Phi(x_*) = 0$, $\operatorname{im}\Phi'(x_*)$ is closed by Lemma 3, and $h \in \ker \Phi'(x_*)$. Definition (4.20) implies

$$S_3(x_*, h) = \bigl\{x \in S_2(x_*, h) \mid 3\Phi''(x_*)[h, x] + \Phi'''(x_*)[h, h, h] \in K + \operatorname{im}\Phi'(x_*)\bigr\}. \tag{4.24}$$

Moreover, by the definition of the set $I(x_*, h, x)$, it follows from (4.3) and the inclusion $h \in H_2(x_*)$ that $\Phi'(x_*)x + \Phi''(x_*)[h, h] = 0$. Hence, for any k, we have

$$K \ni \Phi\biggl(x_* + t_k h + \frac{1}{2}t_k^2 x + r^k\biggr) = \Phi'(x_*)r^k + \frac{1}{2}t_k^3\left(\Phi''(x_*)[h, x] + \frac{1}{3}\Phi'''(x_*)[h, h, h]\right) + o\bigl(t_k^3\bigr)$$

and, therefore,

$$K + \operatorname{im}\Phi'(x_*) \ni \frac{1}{2}t_k^3\left(\Phi''(x_*)[h, x] + \frac{1}{3}\Phi'''(x_*)[h, h, h]\right) + o\bigl(t_k^3\bigr);$$

according to Lemma 3, the set on the left-hand side of this inclusion is closed. This and relation (4.24) immediately imply $x \in S_3(x_*, h)$.

Now, suppose that $h \in H_2^2(x_*)$ and $x = x^2 \in S_3(x_*, h)$. Take any element $x^1 \in X$ such that

$$F'(x_*)x^1 + 3F''(x_*)[h, x^2] + F'''(x_*)[h, h, h] = 0 \tag{4.25}$$

and

$$\langle g_i'(x_*), x^1\rangle + 3g_i''(x_*)[h, x^2] + g_i'''(x_*)[h, h, h] \le 0 \qquad \forall\, i \in I(x_*, h, x^2) \tag{4.26}$$

(such an element exists by virtue of (4.20)). Take $\bar{x}^1 \in X$ and $\bar{x}^2 \in S_2(x_*, h)$ as in Definition 2. For every $\theta \in (0, 1]$, we set $\widehat{x}^1(\theta) = (1-\theta)x^1 + \bar{x}^1$ and $\widehat{x}^2(\theta) = (1-\theta)x^2 + \bar{x}^2$. Let us show that $\widehat{x}^2(\theta) \in T_D^2(x_*, h)$; this inclusion implies that $x^2 \in T_D^2(x_*, h)$ since $T_D^2(x_*, h)$ is closed and $\widehat{x}^2(\theta) \to x^2$ as $\theta \to 0+$.

It follows from (3.5), (4.20), (3.7), (3.8), (4.25), and (4.26) that $\widehat{x}^2(\theta) \in S_2(x_*, h)$,

$$F'(x_*)\widehat{x}^1(\theta) + 3F''(x_*)[h, \widehat{x}^2(\theta)] + F'''(x_*)[h, h, h] = 0, \tag{4.27}$$

and

$$\langle g_i'(x_*), \widehat{x}^1(\theta)\rangle + 3g_i''(x_*)[h, \widehat{x}^1(\theta)] + g_i'''(x_*)[h, h, h] < 0 \qquad \forall\, i \in I(x_*, h, x^2). \tag{4.28}$$

We set $U = X \times X$, $V = Y$, and $P = X \times S_2(x_*, h)$ and define a mapping Φ as $\Phi(u) = F'(x_*)x^1 + 3F''(x_*)[h, x^2] + F'''(x_*)[h, h, h]$ for $u = (x^1, x^2) \in U$. Let $\bar{u} = (\widehat{x}^1(\theta), \widehat{x}^2(\theta)) \in P$ and $\bar{v} = \Phi(\bar{u})$. By virtue of (4.27), we have $\bar{v} = 0$. It follows from (3.6) and Robinson's stability theorem that the objects introduced above satisfy all the conditions of Lemma 1 for some $a > 0$, $\varepsilon_1 > 0$, and $\varepsilon_2 > 0$. Let the numbers $c > 0$, $\delta_1 > 0$, and $\delta_2 > 0$ be chosen as in Lemma 1.

For any $x \in X$, we define $\Delta_1(x)$ and $\Delta_2(x)$ according to (4.7) and (4.8). For $x^1, x^2 \in X$ and $t > 0$, we set

$$x(x^1, x^2; t) = x_* + th + \frac{1}{2}t^2x^2 + \frac{1}{3!}t^3x^1 \tag{4.29}$$

and

$$\varphi_t(u) = \varphi_t(x^1, x^2) = tF''(x_*)[h, x^1] + \frac{3}{t^3}F''(x_*)\left[\frac{1}{2}t^2x^2 + \frac{1}{3!}t^3x^1,\ \frac{1}{2}t^2x^2 + \frac{1}{3!}t^3x^1\right]$$
$$+ \frac{3!}{t^3}\Delta_1(x(x^1, x^2; t) - x_*) - F'''(x_*)[h, h, h]. \tag{4.30}$$

Estimate (2.17) implies that, on any bounded set, the mapping $\varphi_t\colon U \to V$ satisfies the Lipschitz condition with constant tending to zero as $t \to 0+$. Moreover, according to (4.7), we have

$$\Delta_1\biggl(th + \frac{1}{2}t^2x^2 + \frac{1}{3!}t^3x^1\biggr) = \frac{1}{3!}t^3F'''(x_*)[h,h,h] + o(t^3); \tag{4.31}$$

similarly, according to (4.8), we have

$$\Delta_2\biggl(th + \frac{1}{2}t^2x^2 + \frac{1}{3!}t^3x^1\biggr) = \frac{1}{3!}t^3F'''(x_*)[h,h,h] + o(t^3). \tag{4.32}$$

It follows from (4.30) and (4.31) that $\varphi_t(\bar{u}) \to 0$ as $t \to 0+$.

Thus, for any sufficiently small $t > 0$, the mapping φ_t satisfies the Lipschitz condition on $B_{\delta_1}(\bar{u})$ with a constant $l < 1/a$, and $\|\bar{v} + \varphi_t(\bar{u})\| \le \delta_2$.

Now, for $u = (x^1, x^2) \in U$, it follows from (4.7), (4.29)–(4.31), the definition of Φ, and the inclusion $h \in H_1(x_*)$ (see also (3.1)) that

$$\begin{aligned}F(x(x^1,x^2;t)) &= \frac{1}{2}t^2F'(x_*)x^2 + \frac{1}{3!}t^3F'(x_*)x^1 + \frac{1}{2}t^2F''(x_*)[h,h] + \frac{1}{2}t^3F''(x_*)[h,x^2] \\ &\quad + \frac{1}{3!}t^4F''(x_*)[h,x^1] + \frac{1}{2}F''(x_*)\biggl[\frac{1}{2}t^2x^2 + \frac{1}{3!}t^3x^1,\ \frac{1}{2}t^2x^2 + \frac{1}{3!}t^3x^1\biggr] + \Delta_1(x(t) - x_*) \\ &= \frac{1}{2}t^2\bigl(F'(x_*)x^2 + F''(x_*)[h,h]\bigr) + \frac{1}{3!}t^3(\Phi(u) + \varphi_t(u)) = 0.\end{aligned}$$

Applying Lemma 1 to the set P and the mappings Φ and $\varphi = \varphi_t$, we see that for any sufficiently small $t > 0$, there exist $x^1(t) \in X$ and $x^2(t) \in S_2(x_*,h)$ (see (3.5)) such that, for $x(t) = x(x^1(t), x^2(t); t)$, we have

$$F(x(t)) = 0 \tag{4.33}$$

and

$$\|x^1(t) - \widehat{x}^1(\theta)\| + \|x^2(t) - \widehat{x}^2(\theta)\| = \|u - \bar{u}\| \le c\|\bar{v} + \varphi_t(\bar{u})\| = c\|\varphi_t(\bar{u})\|. \tag{4.34}$$

In (4.34), it is assumed that the norm on $X \times X$ is the sum of the norms on the copies of X.

Since $\varphi_t(\bar{u}) \to 0$ as $t \to 0+$, it follows from (4.34) that

$$\|x^1(t) - \widehat{x}^1(\theta)\| \to 0 \qquad \text{and} \qquad \|x^2(t) - \widehat{x}^2(\theta)\| \to 0 \tag{4.35}$$

as $t \to 0+$; in particular, by virtue of (4.29), we have

$$x(t) = x_* + th + \frac{1}{2}t^2\widehat{x}^2(\theta) + \frac{1}{2}t^2(x^2(t) - \widehat{x}^2(\theta)) + \frac{1}{3!}t^3x^1(t) = x_* + th + \frac{1}{2}t^2\widehat{x}^2(\theta) + o(t^2).$$

Thus, according to (4.2) and (4.33), to prove the inclusion $\widehat{x}^2(\theta) \in T_D^2(x_*,h)$, it remains to show that $g(x(t)) \le 0$ for any sufficiently small $t > 0$.

Relations (4.8), (4.29), (4.32), and (4.35) give

$$\begin{aligned}g(x(t)) &= tg'(x_*)h + \frac{1}{2}t^2\bigl(g'(x_*)x^2(t) + g''(x_*)[h,h]\bigr) + \frac{1}{3!}t^3g'(x_*)x^1(t) + \frac{1}{2}t^3g''(x_*)[h,x^2(t)] \\ &\quad + \frac{1}{3!}t^4g''(x_*)[h,x^1(t)] + \frac{1}{2}g''(x_*)\biggl[\frac{1}{2}t^2x^2 + \frac{1}{3!}t^3x^1,\ \frac{1}{2}t^2x^2 + \frac{1}{3!}t^3x^1\biggr] + \Delta_2(x(t) - x_*) \\ &= tg'(x_*)h + \frac{1}{2}t^2\bigl(g'(x_*)x^2(t) + g''(x_*)[h,h]\bigr) \\ &\quad + \frac{1}{3!}t^3\Bigl(g'(x_*)x^1(t) + 3g''(x_*)[h,x^2(t)] + g'''(x_*)[h,h,h]\Bigr) + o(t^3).\end{aligned} \tag{4.36}$$

If $i \in \{1,\dots,m\} \setminus I(x_*,h)$, then $\langle g'(x_*),h\rangle < 0$ because $h \in H_1(x_*)$ (see (3.1)), and (4.36) implies that $g_i(x(t)) < 0$ for any sufficiently small $t > 0$. If $i \in I(x_*,h) \setminus I(x_*,h,\widehat{x}^2(\theta))$, then $\langle g'(x_*),h\rangle = 0$, the inclusion $\widehat{x}^2(\theta) \in S_2(x_*,h)$ implies $\langle g_i'(x_*),\widehat{x}^2(\theta)\rangle + g_i''(x_*)[h,h] < 0$ (see (3.5)), and according to (4.35) and (4.36) we have

$$g(x(t)) = \frac{1}{2}t^2\bigl(g'(x_*)x^2(t) + g''(x_*)[h,h]\bigr) + o(t^2) = \frac{1}{2}t^2\bigl(g'(x_*)\widehat{x}^2(\theta) + g''(x_*)[h,h]\bigr) + o(t^2) < 0$$

for any sufficiently small $t > 0$. Finally, if $i \in I(x_*,h,\widehat{x}^2(\theta))$, then $\langle g'(x_*),h\rangle = 0$, and since $\widehat{x}^2(t) \in S_2(x_*,h)$, it follows from (4.35), (4.36), and (4.28) that

$$\begin{aligned} g_i(x(t)) &\le \frac{1}{3!}t^3\Bigl(g'(x_*)x^1(t) + 3g''(x_*)[h,x^2(t)] + g'''(x_*)[h,h,h]\Bigr) + o(t^3) \\ &= \frac{1}{3!}t^3\Bigl(g'(x_*)\widehat{x}^1(\theta) + 3g''(x_*)[h,\widehat{x}^1(\theta)] + g'''(x_*)[h,h,h]\Bigr) + o(t^3) < 0 \end{aligned}$$

for any sufficiently small $t > 0$. This completes the proof of the theorem.

It is easy to see from the proof of Theorem 2 that both assertions of this theorem remain valid when the second-order tangent set $T_D^2(x_*,h)$ is replaced by the so-called inner second-order tangent set

$$T_D^{\mathrm{i},2}(x_*,h) = \left\{x \in X \;\middle|\; \operatorname{dist}\left(x_* + th + \frac{1}{2}t^2x, D\right) = o(t^2),\ t \ge 0\right\}$$

to D at the point x_* in the direction $h \in X$.

5. FIRST-ORDER NECESSARY CONDITIONS

This section is devoted to first-order necessary conditions for a local minimum in problem (1.1), (1.2). These conditions are of first order in the sense that they involve only the first derivative of the objective function of the problem. Moreover, these conditions are a natural development of the usual first-order necessary conditions for a local extremum. Thus, we assume that the function f is Fréchet differentiable at the point x_* under consideration.

Theorem 1 immediately implies the following first-order necessary condition.

Proposition 2. *If x_* is a local solution to problem* (1.1), (1.2), *then*

$$\langle f'(x_*),h\rangle \ge 0 \qquad \forall\, h \in H_2^1(x_*).$$

The rest of this section is devoted to the derivation of a first-order necessary condition in the Lagrangian form.

For an arbitrary $h \in X$, let $G(x_*,h)\colon X \times X \to Y$ be a linear operator defined by

$$G(x_*,h)(x^1,x^2) = F'(x_*)x^1 + F''(x_*)[h,x^2]. \tag{5.1}$$

Note that this operator appears on the left-hand side of (3.2). We also define a cone

$$C_2(x_*) = \bigl\{h \in H_2(x_*) \mid \langle f'(x_*),h\rangle \le 0\bigr\}, \tag{5.2}$$

which is a restriction of the critical cone of problem (1.1), (1.2) at the point x_* (see (5.14) below). Finally, we define a generalized Lagrangian for problem (1.1), (1.2) by relation (1.7).

Theorem 3. *Suppose that x_* is a local solution to problem* (1.1), (1.2) *and the subspaces* $\operatorname{im} F'(x_*)$ *and* $G(x_*,h)(X \times \ker F'(x_*))$ *are closed.*

Then, for any $h \in C_2(x_*)$, *there exist* $\lambda_0 = \lambda_0(h) \geq 0$, $\lambda^1 = \lambda^1(h) \in Y^*$, $\lambda^2 = \lambda^2(h) \in Y^*$, $\mu^1 = \mu^1(h) \in \mathbb{R}^m$, *and* $\mu^2 = \mu^2(h) \in \mathbb{R}^m$ *such that* λ_0, λ^2, *and* μ^2 *do not vanish simultaneously and*

$$\frac{\partial L_2}{\partial x}(x_*, h, \lambda_0, \lambda^1, \lambda^2, \mu^1, \mu^2) = 0, \qquad (F'(x_*))^*\lambda^2 + (g'(x_*))^*\mu^2 = 0, \tag{5.3}$$

$$\mu^1 \geq 0, \qquad \langle \mu^1, g(x_*)\rangle = 0, \qquad \mu^2 \geq 0, \qquad \langle \mu^2, g(x_*)\rangle = 0. \tag{5.4}$$

Proof. First, suppose that $h \in H_2^1(x_*)$. Consider the minimization problem

$$\langle f'(x_*), \xi\rangle \to \min, \qquad \xi \in H_2(x_*). \tag{5.5}$$

Proposition 2 and (5.2) imply

$$\langle f'(x_*), h\rangle = 0. \tag{5.6}$$

Moreover, according to Remark 1, there exists a $\delta > 0$ such that $H_2(x_*) \cap B_\delta(h) \subset H_2^1(x_*)$; hence, by Proposition 2, we have

$$\langle f'(x_*), \xi\rangle \geq 0 \qquad \forall\, \xi \in H_2(x_*) \cap B_\delta(h).$$

Together with (5.6), this means that h is a local solution to problem (5.5).

Now, consider the problem

$$\langle f'(x_*), \xi\rangle \to \min, \qquad (x, \xi) \in D_2(x_*), \tag{5.7}$$

$$D_2(x_*) = \Big\{(x, \xi) \in X \times H_1(x_*) \;\Big|\; F'(x_*)x + F''(x_*)[\xi, \xi] = 0,$$
$$\langle g_i'(x_*), x\rangle + g_i''(x_*)[\xi, \xi] \leq 0 \;\forall\, i \in I(x_*, h)\Big\}. \tag{5.8}$$

By the definition of the set $H_2(x_*)$ (see (3.5) and (4.3)), there exists an element $x(h) \in X$ such that $(x(h), h) \in D_2(x_*)$. It follows from the above considerations that such a point $(x(h), h)$ is a local solution to problem (5.7), (5.8); it is easy to see that the condition of 2-regularity in the direction h implies that the constraints of problem (5.7), (5.8) satisfy the Robinson regularity condition at this solution (see, e.g., [3, (3.13)]). Using the dual cone lemma [2, Section 3.3.4] and (1.7), it is easy to derive the required relations (5.3) and (5.4) for $\lambda_0 = 1$ and some $\lambda^1 \in Y^*$, $\lambda^2 \in Y^*$, $\mu^1 \in \mathbb{R}^m$, and $\mu^2 \in \mathbb{R}^m$.

Now, suppose that $h \notin H_2^1(x_*)$. Let s denote the number of elements in the set $I(x_*)$. Consider the mapping $\widetilde{g}\colon X \to \mathbb{R}^s$ with components $g_i(x)$, $i \in I(x_*)$.

First, suppose that condition (3.2) is violated. Set $U = X \times X$, $V = Y$, and $S = X \times H_1(x_*)$ and define linear operators A, Λ, and B as follows: $Au = F'(x_*)x$, $\Lambda u = G(x_*, h)u$, and $Bu = \widetilde{g}'(x_*)x$ for $u = (x, \xi) \in U$. We set $\bar{v} = 0$ and $\bar{w} = 0$. Lemma 6 implies that the convex cone $G(x_*, h)(X \times H_1(x_*))$ is closed. The violation of condition (3.2) means that this cone does not coincide with the entire Y; hence, by the second separation theorem (see [15, p. 210]), there exist $y \in Y$ and $\lambda^2 \in Y^*$ such that

$$\langle (F'(x_*))^*\lambda^2, x\rangle + \langle (F''(x_*)[h])^*\lambda^2, \xi\rangle = \langle \lambda^2, G(x_*, h)(x, \xi)\rangle > \langle \lambda^2, y\rangle \qquad \forall\, x \in X, \quad \forall\, \xi \in H_1(x_*)$$

(see (5.1); the inequality $\lambda^2 \neq 0$ holds automatically). This relation implies that

$$(F'(x_*))^*\lambda^2 = 0 \qquad \text{and} \qquad \langle (F''(x_*)[h])^*\lambda^2, \xi\rangle \geq 0 \quad \forall\, \xi \in H_1(x_*).$$

Using the dual cone lemma [2, Section 3.3.4] and relations (3.1) and (1.7), we obtain the required relations (5.3) and (5.4) for $\lambda_0 = 0$, some $\lambda^1 \in Y^*$, the $\lambda^2 \in Y^* \setminus \{0\}$ specified above, $\mu^1 = 0$, and $\mu^2 = 0$.

Finally, suppose that condition (3.2) holds but condition (3.3) is violated. The latter means that the set

$$S = \Big\{(y, z) \in Y \times \mathbb{R}^{\widetilde{m}} \mid \exists\, \xi^1 \in X, \xi^2 \in H_1(x_*) \text{ such that } y = G(x_*, h)(\xi^1, \xi^2) \\ \text{and } z > \widetilde{g}'(x_*)\xi^1 + \widetilde{g}''(x_*)[h, \xi^2]\Big\} \tag{5.9}$$

does not contain 0; moreover, the set S is obviously convex, and $\operatorname{int} S \neq \varnothing$ by (3.2). By the first separation theorem (see [15, p. 209]), there exist $\lambda^2 \in Y^*$ and $\widetilde{\mu}^2 \in \mathbb{R}^s$ such that they do not vanish simultaneously and

$$\langle \lambda^2, y\rangle + \langle \widetilde{\mu}^2, z\rangle \geq 0 \qquad \forall\, (y, z) \in S. \tag{5.10}$$

According to (5.9), we have $S + \{0\} \times \mathbb{R}^s_+ \subset S$, and (5.10) implies $\widetilde{\mu}^2 \geq 0$.

Moreover, according to (5.9) and (5.10), we have

$$\big\langle (F'(x_*))^*\lambda^2 + (\widetilde{g}'(x_*))^*\widetilde{\mu}^2, \xi^1\big\rangle + \big\langle (F''(x_*)[h])^*\lambda^2 + (\widetilde{g}''(x_*)[h])^*\widetilde{\mu}^2, \xi^2\big\rangle \\ = \big\langle \lambda^2, G(x_*, h)(\xi^1, \xi^2)\big\rangle + \big\langle \widetilde{\mu}^2, \widetilde{g}'(x_*)\xi^1 + \widetilde{g}''(x_*)[h, \xi^2]\big\rangle \geq 0 \\ \forall\, \xi^1 \in X, \qquad \forall\, \xi^2 \in H_1(x_*).$$

It follows that

$$(F'(x_*))^*\lambda^2 + (\widetilde{g}'(x_*))^*\widetilde{\mu}^2 = 0 \tag{5.11}$$

and

$$\big\langle (F''(x_*)[h])^*\lambda^2 + (\widetilde{g}''(x_*)[h])^*\widetilde{\mu}^2, \xi\big\rangle \geq 0 \qquad \forall\, \xi \in H_1(x_*).$$

The last relation, the dual cone lemma [2, Section 3.3.4], and (3.1) imply the existence of $\lambda^1 \in Y^*$ and $\widetilde{\mu}^1 \in \mathbb{R}^s$ such that $\widetilde{\mu}^1 \geq 0$ and

$$(F'(x_*))^*\lambda^1 + (\widetilde{g}'(x_*))^*\widetilde{\mu}^1 + (F''(x_*)[h])^*\lambda^2 + (\widetilde{g}''(x_*)[h])^*\widetilde{\mu}^2 = 0. \tag{5.12}$$

Let $\mu^1 \in \mathbb{R}^m$ and $\mu^2 \in \mathbb{R}^m$ be defined by the equalities $\mu^1_i = \widetilde{\mu}^1_i$ and $\mu^2_i = \widetilde{\mu}^2_i$ for $i \in I(x_*)$ and $\mu^1_i = \mu^2_i = 0$ for $i \in \{1, 2, \dots, m\} \setminus I(x_*)$. The required relations (5.3) and (5.4) for $\lambda_0 = 0$, these $\mu^1 \in \mathbb{R}^m$, $\mu^2 \in \mathbb{R}^m$, and $\lambda^1 \in Y^*$, $\lambda^2 \in Y^*$ specified above follow from (1.7), (5.11), and (5.12); moreover, λ^2 and μ^2 do not vanish simultaneously. This completes the proof of the theorem.

Let

$$C^1_2(x_*) = C_2(x_*) \cap H^1_2(x_*). \tag{5.13}$$

It is easy to see that if $h \in C^1_2(x_*)$, then relations (5.3) and (5.4) hold only for $\lambda_0 > 0$. This observation and the proof of Theorem 3 imply that for any $h \in H_2(x_*)$, relations (5.3) and (5.4) hold for $\lambda_0 = 0$ and some $\lambda^1, \lambda^2 \in Y^*$ and $\mu^1, \mu^2 \in \mathbb{R}^m$ such that λ^2 and μ^2 do not vanish simultaneously if and only if the 2-regularity condition in the direction h is violated at x_*.

If the MFCQ holds at the point x_*, then Theorem 3 becomes the usual first-order necessary condition in the form (1.6). Indeed, as mentioned above, in this case we have $H^1_2(x_*) = H_2(x_*) = H_1(x_*)$, and it follows from (5.2) and (5.13) that $C^1_2(x_*)$ coincides with the usual critical cone

$$C(x_*) = C_1(x_*) = \big\{h \in H_1(x_*) \mid \langle f'(x_*), h\rangle \leq 0\big\} \tag{5.14}$$

for problem (1.1), (1.2) at the point x_*. In particular, $C_2^1(x_*)$ contains $h = 0$. Applying Theorem 3 with $h = 0$ and using the equality

$$L_2(x, 0, \lambda_0, \lambda^1, \lambda^2, \mu^1, \mu^2) = L(x, \lambda_0, \lambda, \mu)$$
$$\forall\, x \in X, \quad \forall \lambda^1 = \lambda \in Y^*, \quad \forall \lambda^2 \in Y^*, \quad \forall \mu^1 = \mu \in \mathbb{R}^m, \quad \forall \mu^2 \in \mathbb{R}^m,$$

which follows from (1.4) and (1.7), we see that relations (5.3) and (5.4) imply the required relations (1.6) (recall that λ_0 cannot equal zero in this case).

Moreover, it is easy to show that if the MFCQ holds at x_*, then the second equality in (5.3) and the condition on μ^2 in (5.4) may hold simultaneously only for $\lambda^2 = 0$ and $\mu^2 = 0$. Thus, using the equality

$$L_2(x, h, \lambda_0, \lambda^1, 0, \mu^1, 0) = L(x, \lambda_0, \lambda, \mu) \qquad \forall\, x, h \in X, \quad \forall \lambda^1 = \lambda \in Y^*, \quad \forall \mu^1 = \mu \in \mathbb{R}^m,$$

which follows from (1.4) and (1.7), we reduce relations (5.3) and (5.4) to (1.6) for any $h \in C(x_*)$ (not only for $h = 0$).

At the same time, in the irregular case, when the MFCQ does not hold, relations (5.3) and (5.4) (possibly, with $\lambda^2 \neq 0$ and/or $\mu^2 \neq 0$) give meaningful information about the point x_* under consideration.

Example 1. Let $X = \mathbb{R}^3$, $Y = \mathbb{R}$, $m = 1$, $f(x) = \langle l, x\rangle$ for $l \in \mathbb{R}^3$, $F(x) = x_1x_3$, and $g(x) = x_1^2 + x_2^2 - x_3^2$. The point $x_* = 0$ is feasible for problem (1.1), (1.2); moreover, $F(x_*) = g(x_*) = 0$ and $F'(x_*) = g'(x_*) = 0$. Thus, the MFCQ is violated and the standard first-order necessary conditions hold trivially at the point x_* for any l.

However, it follows from (3.1), (3.5), (4.3), and (5.2) that

$$C_2(x_*) = \bigl\{h \in \mathbb{R}^3 \mid h_1 = 0,\ h_2^2 \le h_3^2,\ l_2h_2 + l_3h_3 \le 0\bigr\}$$

and, as is easy to verify, $C_2^1(x_*) = C_2(x_*) \setminus \{0\}$.

For any $h \in C_2(x_*)$, relations (5.3) and (5.4) hold for any numbers λ^1, λ^2, $\mu^1 \ge 0$, and $\mu^2 \ge 0$ satisfying the equalities

$$l_1 + \lambda^2 h_3 = 0, \qquad l_2 + 2\mu^2 h_2 = 0, \qquad \text{and} \qquad l_3 - 2\mu^2 h_3 = 0. \tag{5.15}$$

It is easy to show that if $l_2 \neq 0$ or $l_3 \neq 0$, then there exists an $h \in C_2(x_*) \setminus \{0\}$ such that the last two equalities in (5.15) hold for no μ^2. Therefore, according to Theorem 3, x_* cannot be a local solution to problem (1.1), (1.2) in this case.

At the same time, if $l_2 = l_3 = 0$, then (5.15) holds for any $h \in C_2(x_*) \setminus \{0\}$ with $\lambda^2 = -l_1/h_3$ and $\mu^2 = 0$. It is easy to see that in this case x_* is indeed a solution to problem (1.1), (1.2). Note that (1.5) holds only for $\lambda_0 = 0$ at this solution.

Thus, in this example, Theorem 3 completely characterizes the presence (or absence) of a local extremum at the point x_*. However, the situation certainly changes when f, F, or g contain terms of order higher than 2 (see Example 2 below).

6. SECOND-ORDER NECESSARY CONDITIONS

This section is devoted to second-order necessary conditions for a local minimum in problem (1.1), (1.2). These conditions are of second order in the sense that they include the first two derivatives of the objective function of the problem and are a natural development of the usual second-order necessary optimality conditions. Thus, we assume that the function f is twice Fréchet differentiable at the point x_* under consideration. We also need to assume that the mappings F and g are three times Fréchet differentiable at x_*.

We set

$$C_2^2(x_*) = C_2(x_*) \cap H_2^2(x_*). \tag{6.1}$$

The following proposition gives a second-order necessary condition.

Proposition 3. *Let x_* be a local solution to problem* (1.1), (1.2).

Then, for any $h \in C_2^2(x_*)$,

$$\langle f'(x_*), x\rangle + f''(x_*)[h, h] \geq 0 \qquad \forall\, x \in S_3(x_*, h). \tag{6.2}$$

Proof. By virtue of (6.1) and Theorem 2, for any $h \in C_2^2(x_*)$ and $x \in S_3(x_*, h)$, there exists a mapping $r\colon \mathbb{R}_+ \to X$ such that $r(t) = o(t^2)$ and $x_* + th + \frac{1}{2}t^2x + r(t) \in D$ for all $t \geq 0$. Since x_* is a local solution to problem (1.1), (1.2), it follows from (5.2) and (6.1) that

$$\begin{aligned} 0 &\leq f\left(x_* + th + \frac{1}{2}t^2x + r(t)\right) - f(x_*) \\ &= \langle f'(x_*), h\rangle t + \frac{1}{2}\big(\langle f'(x_*), x\rangle + f''(x_*)[h, h]\big)t^2 + o\big(t_k^2\big) \\ &\leq \frac{1}{2}\big(\langle f'(x_*), x\rangle + f''(x_*)[h, h]\big)t^2 + o(t^2) \end{aligned}$$

for any sufficiently small $t \geq 0$. This immediately implies (6.2), which proves the proposition.

If the MFCQ holds at x_*, then $C_2^2(x_*) = C_2^1(x_*) = C_2(x_*) = C(x_*)$ according to the above considerations. Moreover, in this case, $S_3(x_*, h) = S_2(x_*, h)$ for any $h \in X$. Indeed, given $x = x^2 \in S_2(x_*, h)$, take an element $\widetilde{x} \in X$ such that $F'(x_*)\widetilde{x} = -3F''(x_*)[h, x] - F'''(x_*)[h, h, h]$ (such an element exists because $\operatorname{im} F'(x_*) = Y$). Let $x^1 = \widetilde{x} + t\bar{\xi}$, where $\bar{\xi} \in X$ is the same as in (1.3). All the conditions on the right-hand side of (4.20) are satisfied for sufficiently large $t > 0$, which means that $x \in S_3(x_*, h)$. The validity of Proposition 3 in this case is well known (see, e.g., [3, Lemma 3.44]).

Let us proceed to a second-order necessary condition in the Lagrangian form.

Theorem 4. *Suppose that x_* is a local solution to problem* (1.1), (1.2) *and the subspaces* $\operatorname{im} F'(x_*)$ *and* $G(x_*, h)(X \times \ker F'(x_*))$ *are closed.*

Then, for any $h \in C_2(x_*)$, there exist $\lambda_0 = \lambda_0(h) \geq 0$, $\lambda^1 = \lambda^1(h) \in Y^*$, $\lambda^2 = \lambda^2(h) \in Y^*$, $\mu^1 = \mu^1(h) \in \mathbb{R}^m$, and $\mu^2 = \mu^2(h) \in \mathbb{R}^m$ such that λ_0, λ^2, and μ^2 do not vanish simultaneously, relations (5.3) *and* (5.4) *hold, and*

$$\frac{\partial^2 L_2}{\partial x^2}\left(x_*, h, \lambda_0, \lambda^1, \frac{1}{3}\lambda^2, \mu^1, \frac{1}{3}\mu^2\right)[h, h] \geq 0. \tag{6.3}$$

Proof. Let $\widehat{m}$ and $\widetilde{m}$ denote the cardinalities of the sets $I(x_*, h)$ and $I(x_*)$, respectively. Let $\widehat{g}\colon X \to \mathbb{R}^{\widehat{m}}$ be the mapping with components $g_i(x)$, $i \in I(x_*, h)$, and $\widetilde{g}\colon X \to \mathbb{R}^{\widetilde{m}}$ be the mapping with components $g_i(x)$, $i \in I(x_*)$.

Consider two cases. First, suppose that $h \in C_2^2(x_*)$. Let us prove that in this case the required assertion is true for $\lambda_0 = 1$.

Set $U = X \times X$, $V = Y \times Y$, and $s = \widehat{m} + \widetilde{m}$. Define linear operators A, B and a linear functional l by

$$Au = \big(F'(x_*)x^2, F'(x_*)x^1 + 3F''(x_*)[h, x^2]\big),$$

$$Bu = \big(\widehat{g}'(x_*)x^2, \widetilde{g}'(x_*)x^1 + 3\widetilde{g}''(x_*)[h, x^2]\big),$$

and $\langle l, u\rangle = \langle f'(x_*), x^2\rangle$ for $u = (x^1, x^2) \in U$. We set $a = f''(x_*)[h, h]$,

$$v = \big(F''(x_*)[h, h], F'''(x_*)[h, h, h]\big), \qquad \text{and} \qquad w = \big(\widehat{g}''(x_*)[h, h], \widetilde{g}'''(x_*)[h, h, h]\big).$$

It follows from (3.5) and (4.3) that $S_2(x_*, h) \neq \varnothing$. For an arbitrary $x \in S_2(x_*, h)$, (3.5) implies

$$F''(x_*)[h, S_2(x_*, h)] \subset F''(x_*)[h, x] + F''(x_*)[h, \ker F'(x_*)]. \tag{6.4}$$

By (3.6),

$$\operatorname{int}\big(\operatorname{im} F'(x_*) + 3F''(x_*)[h, S_2(x_*, h)]\big) \neq \varnothing.$$

Therefore, taking into account (6.4), we obtain

$$\operatorname{int}\big(\operatorname{im} F'(x_*) + 3F''(x_*)[h, \ker F'(x_*)]\big) \neq \varnothing;$$

since the set on the left-hand side of this inequality is a linear subspace, we have

$$\operatorname{im} F'(x_*) + 3F''(x_*)[h, \ker F'(x_*)] = Y.$$

It follows easily from this equality and the definition of the operator A that $\operatorname{im} A = \operatorname{im} F'(x_*) \times Y$, and so $\operatorname{im} A$ is a closed subspace (because $\operatorname{im} F'(x_*)$ is closed). Moreover, definition (3.5) and the second-order 2-regularity condition in the direction h imply that the set S defined by (2.1) is nonempty; indeed, it contains the pair $(\bar{x}^1, \bar{x}^2) \in X \times S_2(x_*, h)$, which satisfies (3.7) and (3.8). Finally, (3.5), (4.20), and Proposition 3 imply (2.2). By Lemma 2, the required relations (5.3), (5.4), and (6.3) hold for $\lambda_0 = 1$ and some $\lambda^1, \lambda^2 \in Y^*$ and $\mu^1, \mu^2 \in \mathbb{R}^m$.

Consider the second case. Suppose that $h \notin C_2^2(x_*)$, i.e., $h \in C_2(x_*) \setminus H_2^2(x_*)$ (see (6.1)). Let us prove that in this case the required assertion is true for $\lambda_0 = 0$.

First, suppose that

$$\operatorname{int}\big(\operatorname{im} F'(x_*) + 3F''(x_*)[h, S_2(x_*, h)]\big) = \varnothing. \tag{6.5}$$

We set $U = X \times X$, $V = Y$, and $s = \widehat{m}$. Define linear operators A, Λ, and B as $Au = F'(x_*)x^2$, $\Lambda u = F'(x_*)x^1 + 3F''(x_*)[h, x^2]$, and $Bu = \widehat{g}'(x_*)x^2$ for $u = (x^1, x^2) \in U$. Set $\bar{v} = F''(x_*)[h, h]$ and $\bar{w} = \widehat{g}''(x_*)[h, h]$. Note that $G(x_*, h)(X \times \ker F'(x_*))$ is closed if and only if $\Lambda(\ker A)$ is closed. The inclusion $h \in C_2(x_*)$, together with (4.3) and (5.2), implies that the set $S = X \times S_2(x_*, h)$ defined by (2.6) is nonempty. Applying Lemma 5, we see that (6.5) may hold only when (2.7) is violated, i.e.,

$$\operatorname{cl}\big(\operatorname{lin}\big(\operatorname{im} F'(x_*) + 3F''(x_*)[h, S_2(x_*, h)]\big)\big) \neq Y. \tag{6.6}$$

If $-F'''(x_*)[h, h, h] \in \operatorname{im} F'(x_*) + 3F''(x_*)[h, S_2(x_*, h)]$, then (6.6) is equivalent to

$$\operatorname{cl}\big(\operatorname{lin}\big(\operatorname{im} F'(x_*) + 3F''(x_*)[h, S_2(x_*, h)] + F'''(x_*)[h, h, h]\big)\big) \neq Y. \tag{6.7}$$

Thus, (6.6) means that either (6.7) holds or

$$-F'''(x_*)[h, h, h] \notin \operatorname{im} F'(x_*) + 3F''(x_*)[h, S_2(x_*, h)]. \tag{6.8}$$

Moreover, by Lemma 6, $\operatorname{im} F'(x_*) + 3F''(x_*)[h, S_2(x_*, h)]$ is closed.

It follows from the second separation theorem (see [15, p. 210]) that in each of these cases, there exists a $\lambda^2 \in Y^* \setminus \{0\}$ such that

$$\big\langle \lambda^2, F'(x_*)x^1 + 3F''(x_*)[h, x^2] + F'''(x_*)[h, h, h]\big\rangle \geq 0 \qquad \forall\, x^1 \in X, \quad \forall\, x^2 \in S_2(x_*, h).$$

Set $U = X \times X$, $V = Y$, and $s = \widehat{m}$ and define linear operators A and B and a linear functional l by the equalities $Au = F'(x_*)x^2$, $Bu = \widehat{g}'(x_*)x^2$, and $\langle l, u\rangle = \langle \lambda^2, F'(x_*)x^1 + 3F''(x_*)[h, x^2]\rangle$ for $u = (x^1, x^2) \in U$. Set $a = \langle \lambda^2, F'''(x_*)[h, h, h]\rangle$, $v = F''(x_*)[h, h]$, and $w = \widehat{g}''(x_*)[h, h]$. Applying

Lemma 2, we obtain the required relations (5.3) and (5.4) for $\lambda_0 = 0$, some $\lambda^1 \in Y^*$ and $\mu^1 \in \mathbb{R}^m$, the $\lambda^2 \in Y^* \setminus \{0\}$ specified above, and $\mu^2 = 0$.

Now, suppose that

$$\operatorname{int}\bigl(\operatorname{im} F'(x_*) + 3F''(x_*)[h, S_2(x_*, h)]\bigr) \neq \varnothing \tag{6.9}$$

but either (3.6) is violated or there exist no $\bar{x}^1 \in X$ and $\bar{x}^2 \in S_2(x_*, h)$ satisfying (3.7) and (3.8). Obviously, in this case $0 \notin \operatorname{int} S$, where

$$S = \Bigl\{(y, z) \in Y \times \mathbb{R}^{\widetilde{m}} \bigm| \exists\, x^1 \in X,\ x^2 \in S_2(x_*, h) \text{ such that}$$
$$y = F'(x_*)x^1 + 3F''(x_*)[h, x^2] + F'''(x_*)[h, h, h] \text{ and}$$
$$z > \widetilde{g}'(x_*)x^1 + 3\widetilde{g}''(x_*)[h, x^2] + \widetilde{g}'''(x_*)[h, h, h]\Bigr\}. \tag{6.10}$$

Clearly, the set S is convex.

Let us show that $\operatorname{int} S \neq \varnothing$. We set $U = X \times X$, $V = Y$, and $P = X \times S_2(x_*, h)$ and define a mapping Φ by $\Phi(u) = F'(x_*)x^1 + 3F''(x_*)[h, x^2] + F'''(x_*)[h, h, h]$ for $u = (x^1, x^2) \in U$. It follows from (6.9) that $\operatorname{int} \Phi(P) \neq \varnothing$. This means that there exists a $\bar{u} = (\widehat{x}^1, \widehat{x}^2) \in P$ such that $\bar{v} = \Phi(\bar{u}) \in \operatorname{int} \Phi(P)$. Moreover, by Robinson's stability theorem, the objects introduced above satisfy the conditions of Lemma 1 for some $a > 0$, $\varepsilon_1 > 0$, and $\varepsilon_2 > 0$. Let $c > 0$, $\delta_1 > 0$, and $\delta_2 > 0$ be the numbers defined according to Lemma 1. Take any $y \in B_{\delta_2}(\bar{v})$ and consider the mapping $\varphi\colon U \to V$ defined by $\varphi(u) \equiv -y$. This mapping satisfies the Lipschitz condition with constant $l = 0$ and the condition $\|\bar{v} + \varphi(\bar{u})\| = \|y - \bar{v}\| \le \delta_2$ on U. By Lemma 1, there exists a $u \in P$ such that

$$\Phi(u) = y \qquad \text{and} \qquad \|u - \bar{u}\| \le c\|y - \bar{v}\| \le c\delta_2,$$

i.e.,

$$B_{\delta_2}(\bar{v}) \subset \Phi(P \cap B_{c\delta_2}(\bar{u})). \tag{6.11}$$

Consider the set

$$\widetilde{S} = B_{\delta_2}(\bar{v}) \times \Bigl\{z \in \mathbb{R}^{\widetilde{m}} \Bigm|$$
$$z_i > \max_{j \in I(x_*)} \Bigl(\|g_j'(x_*)\|(\|\widehat{x}^1\| + c\delta_2) + \|g_j''(x_*)[h]\|(\|\widehat{x}^2\| + c\delta_2) + \|g_j'''(x_*)[h, h, h]\|\Bigr),\ i \in I(x_*)\Bigr\}.$$

Obviously, $\operatorname{int} \widetilde{S} \neq \varnothing$; moreover, (6.10), (6.11), and the definition of P and Φ imply $\widetilde{S} \subset S$. Thus, $\operatorname{int} S \neq \varnothing$.

By the first separation theorem (see [15, p. 209]), there exist $\widetilde{\lambda}^2 \in Y^*$ and $\widetilde{\mu}^2 \in \mathbb{R}^{\widetilde{m}}$ such that they do not vanish simultaneously and

$$\langle \widetilde{\lambda}^2, y \rangle + \langle \widetilde{\mu}^2, z \rangle \ge 0 \qquad \forall\, (y, z) \in S. \tag{6.12}$$

According to (6.10), we have $S + \{0\} \times \mathbb{R}_+^{\widetilde{m}} \subset S$; this inclusion and the inequality in (6.12) imply $\widetilde{\mu}^2 \ge 0$.

Relations (6.10) and (6.12) yield

$$\bigl\langle (F'(x_*))^* \widetilde{\lambda}^2 + (\widetilde{g}'(x_*))^* \widetilde{\mu}^2, x^1 \bigr\rangle + 3\bigl\langle (F''(x_*)[h])^* \widetilde{\lambda}^2 + (\widetilde{g}''(x_*)[h])^* \widetilde{\mu}^2, x^2 \bigr\rangle$$
$$+ \bigl\langle \widetilde{\lambda}^2, F'''(x_*)[h, h, h] \bigr\rangle + \bigl\langle \widetilde{\mu}^2, \widetilde{g}'''(x_*)[h, h, h] \bigr\rangle \ge 0$$
$$\forall\, x^1 \in X, \quad \forall\, x^2 \in S_2(x_*, h).$$

Therefore,

$$(F'(x_*))^*\widetilde{\lambda}^2 + (\widetilde{g}'(x_*))^*\widetilde{\mu}^2 = 0 \tag{6.13}$$

and

$$3\big\langle (F''(x_*)[h])^*\widetilde{\lambda}^2 + (\widetilde{g}''(x_*)[h])^*\widetilde{\mu}^2, x\big\rangle + \big\langle \widetilde{\lambda}^2, F'''(x_*)[h,h,h]\big\rangle + \big\langle \widetilde{\mu}^2, \widetilde{g}'''(x_*)[h,h,h]\big\rangle \geq 0$$
$$\forall\, x \in S_2(x_*, h).$$

Using the last relation and (3.5) and applying Lemma 2 with $U = X$, $V = Y$, $s = \widehat{m}$, $A = F'(x_*)$, $B = \widehat{g}'(x_*)$, $l = (F''(x_*)[h])^*\widetilde{\lambda}^2 + (\widetilde{g}''(x_*)[h])^*\widetilde{\mu}^2$, $a = \langle \widetilde{\lambda}^2, F'''(x_*)[h,h,h]\rangle + \langle \widetilde{\mu}^2, \widetilde{g}'''(x_*)[h,h,h]\rangle$, $v = F''(x_*)[h,h]$, and $w = \widehat{g}''(x_*)[h,h]$, we see that there exist $\lambda^1 \in Y^*$ and $\widehat{\mu}^1 \in \mathbb{R}^{\widehat{m}}$ such that $\widehat{\mu}^1 \geq 0$,

$$(F'(x_*))^*\lambda^1 + (\widehat{g}'(x_*))^*\widehat{\mu}^1 + 3(F''(x_*)[h])^*\widetilde{\lambda}^2 + 3(\widetilde{g}''(x_*)[h])^*\widetilde{\mu}^2 = 0, \tag{6.14}$$

and

$$\langle \lambda, F''(x_*)[h,h]\rangle + \langle \widehat{\mu}^1, \widehat{g}''(x_*)[h,h]\rangle + \langle \widetilde{\lambda}^2, F'''(x_*)[h,h,h]\rangle + \langle \widetilde{\mu}^2, \widetilde{g}'''(x_*)[h,h,h]\rangle \geq 0. \tag{6.15}$$

Set $\lambda^2 = 3\widetilde{\lambda}^2$ and define vectors $\mu^1 \in \mathbb{R}^m$ and $\mu^2 \in \mathbb{R}^m$ by the equalities $\mu_i^1 = \widehat{\mu}_i^1$ for $i \in I(x_*, h)$, $\mu_i^1 = 0$ for $i \in \{1, 2, \ldots, m\} \backslash I(x_*, h)$, $\mu_i^2 = 3\widetilde{\mu}_i^2$ for $i \in I(x_*)$, and $\mu_i^2 = 0$ for $i \in \{1, 2, \ldots, m\} \backslash I(x_*)$. It follows from (1.7) and (6.13)–(6.15) that the required relations (5.3), (5.4), and (6.3) hold for $\lambda_0 = 0$ and the $\lambda^1 \in Y^*$, $\mu^1 \in \mathbb{R}^m$, $\lambda^2 \in Y^*$, and $\mu^2 \in \mathbb{R}^m$ specified above; moreover, λ^2 and μ^2 do not vanish simultaneously. This completes the proof of the theorem.

It is easy to show that if $h \in C_2^2(x_*)$, then relations (5.3), (5.4), and (6.3) hold only for $\lambda_0 > 0$. It follows from this observation and the proof of Theorem 4 that, for any $h \in H_2(x_*)$, the validity of relations (5.3), (5.4), and (6.3) for $\lambda_0 = 0$ and some $\lambda^1, \lambda^2 \in Y^*$ and $\mu^1, \mu^2 \in \mathbb{R}^m$ such that λ^2 and μ^2 do not vanish simultaneously is equivalent to the violation of the second-order 2-regularity condition at x_* in the direction h.

The following example describes a situation in which Theorem 3 does not detect the absence of an extremum at a feasible point under consideration, while Theorem 4 does.

Example 2. Let $X = \mathbb{R}^4$, $Y = \mathbb{R}^2$, $m = 1$, $f(x) = x_1$, $F(x) = (x_1x_3 + x_3^3, x_1^2 + x_2^2 - x_3^2)$, and $g(x) = x_1^2 - x_4^2$. The point $x_* = 0$ is feasible for problem (1.1), (1.2); moreover, $F(x_*) = 0$, $g(x_*) = 0$, $F'(x_*) = 0$, $g'(x_*) = 0$, and the MFCQ is violated at x_*.

As in Example 1, we obtain

$$C_2(x_*) = \big\{h \in \mathbb{R}^4 \mid h_1 = 0,\ h_2^2 = h_3^2\big\}$$

and $C_2^1(x_*) = \{h \in C_2(x_*) \mid h_2 \neq 0,\ h_4 \neq 0\}$.

For any $h \in C_2^1(x_*) \setminus \{0\}$, relations (5.3) and (5.4) hold for $\lambda_0 = 1$ and any $\lambda^1, \lambda^2 \in \mathbb{R}^2$ and $\mu^1 \geq 0$, $\mu^2 \geq 0$ satisfying the equalities

$$1 + \lambda_1^2 h_3 = 0, \qquad \lambda_2^2 h_2 = 0, \qquad \lambda_2^2 h_3 = 0, \qquad \text{and} \qquad \mu^2 h_4 = 0. \tag{6.16}$$

Thus, the first-order necessary conditions provided by Theorem 3 hold at the point x_*. At the same time, for $h = (0,1,1,1) \in C_2^1(x_*)$, (6.16) implies the equalities $\lambda^2 = (-1, 0)$ and $\mu^2 = 0$; therefore,

$$\frac{\partial^2 L_2}{\partial x^2}\left(x_*, h, 1, \lambda^1, \frac{1}{3}\lambda^2, \mu^1, \frac{1}{3}\mu^2\right)[h,h] = -2\mu^1 + 2\lambda_1^2 < 0$$

for all λ^1, λ^2, $\mu^1 \geq 0$, and μ^2 satisfying (6.16). Thus, according to Theorem 4, x_* is not a local solution to problem (1.1), (1.2).

Note that replacing the inequality constraint by the equality in this example yields

$$C_2(x_*) = \{h \in \mathbb{R}^4 \mid h_1 = h_4 = 0,\ h_2^2 = h_3^2\}$$

and $C_2^1(\bar{x}) = \varnothing$. Therefore, previous results concerning problems with 2-regular equality constraints do not apply to this example.

In [20], a second-order necessary condition was obtained that may be meaningful even when the MFCQ is violated. Namely, if x_* is a local solution to problem (1.1), (1.2), then, for any $h \in C(x_*)$, there exist $\lambda_0 = \lambda_0(h) \geq 0$, $\lambda = \lambda(h) \in Y^*$, and $\mu = \mu(h) \in \mathbb{R}^m$ such that they do not vanish simultaneously, conditions (1.5) hold, and

$$\frac{\partial^2 L}{\partial x^2}(x_*, \lambda_0, \lambda, \mu)[h, h] \geq 0. \tag{6.17}$$

Let us show that this assertion is a special case of Theorem 4.

Indeed, suppose that $h \in C_2(x_*)$. Choose λ_0, λ^1, λ^2, μ^1, and μ^2 according to Theorem 4. If $\lambda^2 = 0$ and $\mu^2 = 0$, then $\lambda_0 > 0$ and the required assertion for $\lambda = \lambda^1$ and $\mu = \mu^1$ follows from (1.4), (5.2)–(5.4), and (6.3). Now, suppose that λ^2 and μ^2 do not vanish simultaneously. Relations (3.1), (4.3), and (5.2)–(5.4) imply that

$$\langle \lambda^2, F''(x_*)[h, h]\rangle + \langle \mu^2, g''(x_*)[h, h]\rangle = -\lambda_0\langle f'(x_*), h\rangle - \langle \lambda^1, F'(x_*)h\rangle - \langle \mu^1, g'(x_*)h\rangle \geq 0. \tag{6.18}$$

Then the required inequality for $\lambda_0 = 0$, $\lambda = \lambda^2$, and $\mu = \mu^2$ follows from the second relation in (5.3) and relations (5.4) and (6.18).

Now, suppose that $h \in C(x_*) \setminus C_2(x_*) = C(x_*) \setminus H_2(x_*)$, i.e., $h \in H_1(x_*)$ and $S_2(x_*, h) \neq \varnothing$ (see (4.3), (5.2), and (5.14)).

Again, let $\widehat{m}$ denote the number of elements in $I(x_*, h)$. Consider the cone $K = \{0\} \times \mathbb{R}^{\widehat{m}}_-$ in $Y \times \mathbb{R}^{\widehat{m}}$, the mapping $\widehat{g}\colon X \to \mathbb{R}^{\widehat{m}}$ with components $g_i(x)$, $i \in I(x_*, h)$, and the mapping $\Phi\colon X \to Y \times \mathbb{R}^{\widehat{m}}$ defined by $\Phi(x) = (F(x), \widehat{g}(x))$. The set $S_2(x_*, h)$ is empty if and only if

$$\Phi''(x_*)[h, h] \notin K + \operatorname{im}\Phi'(x_*),$$

and by Lemmas 3 and 4, the convex cone $K + \operatorname{im}\Phi'(x_*)$ is closed. By the second separation theorem (see [15, p. 210]), there exists a $\nu = (\lambda, \widehat{\mu}) \in Y^* \times \mathbb{R}^{\widehat{m}}$ such that

$$\langle \nu, \eta + \Phi'(x_*)\xi\rangle \leq 0 < \langle \nu, \Phi''(x_*)[h, h]\rangle \qquad \forall\, \xi \in X, \quad \forall\, \eta \in K.$$

It follows from these inequalities and the definition of K and Φ that $\widehat{\mu} \geq 0$,

$$(F'(x_*))^*\lambda + (\widehat{g}'(x_*))^*\widehat{\mu} = 0, \qquad \text{and} \qquad \langle \lambda, F''(x_*)[h, h]\rangle + \langle \widehat{\mu}, \widehat{g}''(x_*)[h, h]\rangle > 0,$$

which implies the required assertion for $\lambda_0 = 0$, the λ specified above, and the $\mu \in \mathbb{R}^m$ defined by the equalities $\mu_i = \widehat{\mu}_i$ for $i \in I(x_*, h)$ and $\mu_i = 0$ for $i \in \{1, 2, \dots, m\} \setminus I(x_*, h)$.

In relation to the above-mentioned necessary extremum condition from [20], we suggest the following definition.

Definition 4. We say that the *second-order regularity condition* holds at x_* *in a direction* $h \in X$ if $\operatorname{im} F'(x_*) = Y$ and there exists an $\bar{x} \in X$ such that

$$F'(x_*)\bar{x} + F''(x_*)[h, h] = 0 \tag{6.19}$$

and

$$\langle g_i'(x_*), \bar{x}\rangle + g_i''(x_*)[h, h] < 0 \qquad \forall\, i \in I(x_*). \tag{6.20}$$

Take an arbitrary $h \in C(x_*)$. It is easy to see that if the second-order regularity condition in this direction h is satisfied, then (1.5) and (6.17) hold only for $\lambda_0 > 0$, although the MFCQ may be violated. This fact follows directly from Theorem 4. Indeed, the second-order regularity in the direction $h \in X$ at x_* readily implies the second-order 2-regularity in this direction. Moreover, it follows from (3.5), (4.3), (5.2), (6.19), and (6.20) that $h \in C_2(x_*)$. However, according to (3.1), (4.3), and (5.2)–(5.4), the second-order 2-regularity in the direction $h \in C_2(x_*)$ implies (6.18) and

$$\langle \lambda^2, F'(x_*)\bar{x}\rangle + \langle \mu^2, g'(x_*)\bar{x}\rangle = 0,$$

where $\bar{x} \in X$ is chosen as in Definition 4. Summing the relations obtained yields

$$\big\langle \lambda^2, F'(x_*)\bar{x} + F''(x_*)[h,h]\big\rangle + \big\langle \mu^2, g'(x_*)\bar{x} + g''(x_*)[h,h]\big\rangle \geq 0.$$

By virtue of (5.4), (6.19), and (6.20), this inequality may hold only for $\mu^2 = 0$, and the second relation in (5.3), together with $\operatorname{im} F'(x_*) = Y$, implies $\lambda^2 = 0$. We have thus shown that if the second-order regularity condition in a direction $h \in C_2(x_*)$ holds at the point x_*, then relations (5.3) and (5.4) may hold only for $\lambda^2 = 0$ and $\mu^2 = 0$. In other words, in this case Theorem 4 completely reduces to the necessary condition from [20] with $\lambda_0 > 0$.

On the other hand, if the second-order regularity condition does not hold at x_*, then, as is easy to see, the necessary condition from [20] holds automatically for $\lambda_0 = 0$. At the same time, Theorem 4 gives meaningful information about the point x_* under consideration.

Example 3. Let $X = \mathbb{R}$, $m = 1$, $f(x) = x$, and $g(x) = x^3$. It is easy to show that at the feasible point $x_* = 0$, problem (1.1), (1.2) satisfies the necessary condition from [20], $C_2(x_*) = \mathbb{R}_-$, and $C_2^2(x_*) = \mathbb{R} \setminus \{0\}$, but the second-order necessary condition provided by Theorem 4 is violated and, therefore, x_* is not a local solution to problem (1.1), (1.2). Note also that $C_2^1(x_*) = \varnothing$ and the first-order necessary condition provided by Theorem 3 holds for $\lambda_0 = 0$.

Finally, the following example shows that the second-order 2-regularity may be weaker than the 2-regularity even when the point under consideration is a local solution to problem (1.1), (1.2).

Example 4. Let $X = \mathbb{R}^2$, $m = 2$, $f(x) = x_1x_2$, and $g(x) = (x_1^3, x_2^3)$. Then the point $x_* = 0$ is a local solution to problem (1.1), (1.2); moreover, $H_2^1(x_*) = \varnothing$ and $C_2^2(x_*) = H_2^2(x_*) = \{h \in \mathbb{R}^2 \mid h_1 < 0,\ h_2 < 0\}$. It is easy to show that the first-order necessary condition from Theorem 3 may hold at the point x_* for $h \in C_2^2(x_*)$ and $\lambda_0 = 0$, while the second-order necessary condition from Theorem 4 holds only for $\lambda_0 > 0$.

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