

Vasily E. Tarasov (Ed.)

# **Handbook of Fractional Calculus with Applications**

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Volume 4: Applications in Physics, Part A

Series edited by Jose Antonio Tenreiro Machado

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## Preface

Fractional calculus (FC) was initially developed in 1695, nearly at the same time as the conventional calculus. However, FC attracted limited attention and remained a pure mathematical exercise, in spite of the contributions of important mathematicians, physicists, and engineers. FC had rapid further development during the last few decades, both in mathematics and applied sciences, being nowadays recognized as an excellent tool for describing complex systems, phenomena involving long range memory effects, and non-locality. A huge number of research papers and books devoted to this subject have been published, and presently several specialized conferences and workshops are organized each year. The FC popularity in all fields of science is due to its successful application in mathematical models, namely in the form of FC operators and fractional integral and differential equations. Presently, we are witnessing considerable progress both in regard to theoretical aspects and applications of FC in areas such as physics, engineering, biology, medicine, economy, or finance.

The popularity of FC has attracted many researchers from all over the world, and there is a demand for works covering all areas of science in a systematic and rigorous form. In fact, the literature devoted to FC and its applications is huge, but readers are confronted with a high heterogeneity and, in some cases, with misleading and inaccurate information. The Handbook of Fractional Calculus with Applications (HFCA) intends to fill that gap and provides the readers with a solid and systematic treatment of the main aspects and applications of FC. Motivated by these ideas, the editors of the volumes involved a team of internationally recognized experts for a joint publishing project, offering a survey of their own and other important results in their fields of research. As a result of these joint efforts, a modern encyclopedia of FC and its applications, reflecting present day scientific knowledge, is now available with the HFCA. This work is distributed by several distinct volumes, each one developed under the supervision of its editors.

The fourth and fifth volumes of HFCA are devoted to the application of fractional calculus (FC) and fractional differential equations in different areas of physics. These volumes describe the fundamental physical effects and, first of all, those that belong to fractional relaxation-oscillation or diffusion-wave phenomena. FC allows describing spatial non-locality and fading memory of power-law type, the openness of physical systems and dissipation, long-range interactions, and other physical phenomena. The most well-known physical phenomena and processes, which are described by fractional differential equations, include fractional viscoelasticity, spatial and frequency dispersion of power type, nonexponential relaxation, anomalous diffusion, and many others. The fourth volume of HFCA focuses on the application of FC in various aspects of classical mechanics and continuum mechanics. The most important basic models and phenomena include the fractional oscillator and Van der Pol oscillator, fractional relaxation and fractional oscillation phenomena, discrete long-range

interactions, and nonlinear systems with power-law memory. In the 13 chapters, the non-locality and memory of the power-law type are described in phenomena, such as viscoelasticity, thermodynamics, thermodiffusion in deformable solid, in gradient mechanics, and mechanics of fractal media.

My special thanks go to the authors of individual chapters that are excellent surveys of selected classical and new results in several important fields of FC. The editors believe that the HFCA will represent a valuable and reliable reference work for all scholars and professionals willing to develop research in the challenging, relevant, and timely scientific area.

Vasily E. Tarasov

# Contents

**Preface — V**

J. A. Tenreiro Machado and António M. Lopes

**Fractional Van der Pol oscillator — 1**

Dumitru Baleanu and Sami I. Muslih

**Fractional Lagrangian and Hamiltonian mechanics with memory — 23**

Rudolf Gorenflo and Francesco Mainardi

**Fractional relaxation-oscillation phenomena — 45**

Vasily E. Tarasov

**Fractional calculus and long-range interactions — 75**

Mark Edelman

**Dynamics of nonlinear systems with power-law memory — 103**

Aleksander Stanislavsky

**Fractional oscillator basics — 133**

Francesco Mainardi

**Fractional viscoelasticity — 153**

Raoul R. Nigmatullin and Dumitru Baleanu

**Relationships between 1D and space fractals and fractional integrals and their applications in physics — 183**

Yuriy Povstenko

**Thermodiffusion in a deformable solid: fractional calculus approach — 221**

Elias C. Aifantis

**Fractional generalizations of gradient mechanics — 241**

Jun Li and Martin Ostoja-Starzewski

**Application of fractional calculus to fractal media — 263**

Francesco Paolo Pinnola and Massimiliano Zingales

**Fractional-order constitutive equations in mechanics and thermodynamics — 277**

**Index — 305**