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Directional Stability Theorem and Directional Metric Regularity

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We develop a new regularity concept, unifying metric regularity, Robinson's constraint qualification, and directional regularity. We present the directional stability theorem and the related concept of directional metric regularity. On one hand, our directional stability theorem immediately implies Robinson's stability theorem [Arutyunov, A. V. 2005. Covering of nonlinear maps on cone in neighborhood of abnormal point. *Math. Notes* **77** 447–460.] as a particular case, while on the other hand, our theorem easily implies various stability results under the directional regularity condition, widely used in sensitivity analysis. Some applications of this kind are also presented.

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1. Introduction. Throughout this paper, let X be a normed linear space, let $Y = \mathbf{R}^l$, and let Q be a fixed closed convex set in Y. Let $F\colon X \to Y$ be a smooth mapping (our smoothness hypotheses will be specified below). Recall that the mapping F is called metrically regular with respect to Q at $\bar{x} \in F^{-1}(Q)$ if the estimate

$$\operatorname{dist}(x, F^{-1}(Q-y)) = O(\operatorname{dist}(F(x)+y, Q))$$

holds for $(x, y) \in X \times Y$ close to $(\bar{x}, 0)$ (Bonnans and Shapiro [6, p. 65]). This notion dates from Robinson [16] (or even from the classical works (Lyusternik [12], Graves [9]); see also Dmitruk et al. [7]). For more recent developments and extensions of the metric regularity theory, see Mordukhovich [13], Mordukhovich and Shao [14], Mordukhovich and Wang [15], Ioffe [10], and references therein.

As is well known (see, e.g., Bonnans and Shapiro [6, Proposition 2.89]), metric regularity is equivalent to the so-called Robinson's constraint qualification (CQ) at $\bar{x}$, which consists of saying that

$$0 \in \operatorname{int}(F(\bar{x}) + \operatorname{im} F'(\bar{x}) - Q).$$

The fact that Robinson's CQ implies metric regularity is a consequence of the so-called Robinson's stability theorem [17] (see also Bonnans and Shapiro [6, Theorem 2.87]). To state the latter, let Σ be a topological space (the space of parameters), let $F\colon \Sigma \times X \to Y$ be a mapping satisfying the appropriate continuity and smoothness requirements, and for each $\sigma \in \Sigma$, set

$$D(\sigma) = \{x \in X \mid F(\sigma, x) \in Q\}. \tag{1}$$

For a given (base) parameter value $\bar{\sigma} \in \Sigma$, Robinson's CQ at $\bar{x} \in D(\bar{\sigma})$ takes the form

$$0 \in \operatorname{int}\left(F(\bar{\sigma}, \bar{x}) + \operatorname{im} \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x}) - Q\right), \tag{2}$$

and this condition implies that the estimate

$$\operatorname{dist}(x, D(\sigma)) = O(\operatorname{dist}(F(\sigma, x), Q)) \tag{3}$$

holds for $(\sigma, x) \in \Sigma \times X$ close to $(\bar{\sigma}, \bar{x})$.

Another very useful regularity concept is the so-called directional regularity at $\bar{x}$ with respect to a given direction $d \in \Sigma$, which becomes relevant when Σ is a normed linear space. This condition has the form

$$0 \in \operatorname{int}\left(F(\bar{\sigma}, \bar{x}) + \operatorname{im} \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x}) + \operatorname{cone}\left\{\frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d\right\} - Q\right), \tag{4}$$

and it finds numerous applications, especially in sensitivity analysis for optimization problems (Bonnans and Shapiro [6]). In the context of mathematical programming problems, directional regularity is known as Gollan's condition [8], and it was extended to the general setting in Bonnans and Cominetti [4].

In this paper, we present the directional stability theorem (with a quite simple and self-contained proof), of which Robinson's stability theorem [17] is a particular case. The significance of this result is in that it enables unification of the diverse regularity concepts playing a crucial role in modern optimization theory and variational analysis. Specifically, the results employing Robinson's CQ and those employing the directional regularity condition were previously derived separately (see, e.g., Bonnans and Shapiro [6]). For example, each new result about the existence of a feasible arc of a needed given form under the directional regularity condition required a new (and usually highly nontrivial) proof. Now, the results of this kind can be derived directly from Theorem 4.1 presented below, and without any auxiliary technical tools, e.g., from the multifunctions theory.

In addition, our directional stability theorem suggests a new form of weakened (restricted) metric regularity, which may be a meaningful concept for the cases when the usual metric regularity does not hold. We believe that this concept may find multiple applications in the future, but of course, these applications could be mainly expected in the field of nonsmooth problems.

It is important to stress that in our setting, Y is a finite-dimensional space. This is strongly related to the method of proof being used, because it relies on (completely finite-dimensional) Brouwer's fixed point theorem. On the other hand, this setting (with finite-dimensional Y but possibly infinite-dimensional X) is rich enough to cover many applications (e.g., in optimal control, not to mention mathematical programming, semidefinite programming, semi-infinite programming, etc.).

This paper is organized as follows. In §2, we present some auxiliary lemmas. In §3, we prove our basic stability theorem for constraint systems comprised by equality constraints and set constraints. Section 4 contains the directional stability theorem, and in §5, we develop the related directional metric regularity concept. Section 6 deals with the case of directional regularity. Finally, in §7, we present some applications to sensitivity analysis of optimization problems, in the cases of directionally regular constraints, Hölder stable solutions, and empty sets of Lagrange multipliers.

Some comments on our (fairly standard) notation are in order. For a given normed linear space X, X^* is its (topologically) dual space, and $B_\delta(x) = \{\xi \in X \mid \|\xi - x\| \le \delta\}$ is a ball centered at $x \in X$ and of radius $\delta > 0$. If $K \subset X$ is a cone, $K^\circ = \{l \in X^* \mid \langle l, \xi\rangle \le 0 \ \forall\, \xi \in K\}$ stands for its polar cone. For a given set $S \subset X$, $\operatorname{int} S$ stands for its interior, $\operatorname{cl} S$ stands for its closure, $\operatorname{span} S$ stands for the linear space spanned by S ($\operatorname{span} S$ is not necessarily closed), $\operatorname{conv} S$ ($\operatorname{cone} S$, $\operatorname{aff} S$) stands for its convex (conic, affine) hull, i.e., the smallest convex set (cone, affine set) containing S, and $S^\perp = \{l \in X^* \mid \langle l, x\rangle = 0 \ \forall\, x \in S\}$ is the annihilator of S. Furthermore, if $0 \in S$, then $\operatorname{ri} S$ is the relative interior of S, i.e., its interior with respect to $\operatorname{span} S$. The convex hull of a finite set will be referred to as a finitely generated set. Furthermore, $\operatorname{dist}(x, S) = \inf_{\xi \in S} \|\xi - x\|$ is the distance from $x \in X$ to S, and if S is a closed convex set in a finite-dimensional X, $\pi_S(x)$ stands for the projection of x onto S, i.e., the (uniquely defined) point $\xi \in S$ such that $\|x - \xi\| = \operatorname{dist}(x, S)$. For a given point $x \in S$, $R_S(x) = \operatorname{cone}(S - x)$, is the so-called radial cone to S at x,

$$T_S(x) = \{h \in X \mid \exists \{t_k\} \subset \mathbf{R}_+ \backslash \{0\} \text{ such that } \{t_k\} \to 0,\ \operatorname{dist}(x + t_k h, S) = o(t_k)\}$$

is the contingent cone to S at x, and $N_S(x) = (T_S(x))^\circ$ is the normal cone to S at x (if $x \notin S$, then $N_S(x) = \varnothing$ by definition). Recall that for a convex S, $T_S(x) = \operatorname{cl} R_S(x)$.

If Y is another normed linear space, $\mathscr{L}(X, Y)$ stands for the space of continuous linear operators from X to Y. For a given linear operator $A\colon X \to Y$, $\operatorname{im} A$ stands for its range (image space).

2. Auxiliary lemmas. The proof of our basic stability theorem employs the following lemmas. Let Z be a normed linear space.

The first lemma can be regarded as a far-reaching extension of some well-known results on the right inverse mapping, to the case when a linear operator is considered only on a given convex cone in Z rather than on the entire Z. In the case of infinite-dimensional Y, such results are closely related to the Banach open mapping theorem and its generalizations (see Arutyunov [1]). Note also that here, we establish not only the "restricted" covering property but the existence of a continuous inverse function as well.

LEMMA 2.1. *Let $K \subset Z$ be a closed convex cone, and let $\bar{A} \in \mathscr{L}(Z, Y)$.*

If $\bar{y} \in Y$ satisfies the inclusion $\bar{y} \in \operatorname{int} \bar{A}(K)$, then there exist $\delta > 0$ and $c > 0$ such that for each $A \in \mathscr{L}(X, Y)$ close enough to $\bar{A}$, there exists a continuous mapping $\varphi_A\colon \operatorname{cone} B_\delta(\bar{y}) \to K$ satisfying the following requirements:

$$A\varphi_A(y) = y, \quad \|\varphi_A(y)\| \le c\|y\| \qquad \forall\, y \in \operatorname{cone} B_\delta(\bar{y}). \tag{5}$$

Proof. Let $\eta^i = \bar{y} + \varepsilon e^i$, $i = 1, \ldots, l$, $\eta^{l+1} = \bar{y} - \varepsilon \sum_{i=1}^{l} e^i$, where $e^1, \ldots, e^l$ is a standard basis in $Y = \mathbf{R}^l$, with $\varepsilon > 0$ being fixed. It is easy to see that $\bar{y} \in \operatorname{int} \operatorname{conv}\{\eta^1, \ldots, \eta^{l+1}\}$; i.e., there exists $\delta_1 > 0$ such that

$$B_{2\delta_1}(\bar{y}) \subset \operatorname{conv}\{\eta^1, \ldots, \eta^{l+1}\}. \tag{6}$$

On the other hand, by the condition $\bar{y} \in \operatorname{int} \bar{A}(K)$, one can choose $\varepsilon > 0$ small enough, so that $\eta^1, \ldots, \eta^{l+1} \in \bar{A}(K)$; i.e., there exist $\xi^i \in K$ such that $\bar{A}\xi^i = \eta^i$, $i = 1, \ldots, l+1$. For each $A \in \mathscr{L}(X, Y)$ close enough to $\bar{A}$, the points $\eta^i(A) = A\xi^i$ are close to η^i, $i = 1, \ldots, l+1$, and hence, by (6), it can be easily derived that

$$B_{\delta_1}(\bar{y}) \subset S(A), \tag{7}$$

where $S(A) = \operatorname{conv}\{\eta^1(A), \ldots, \eta^{l+1}(A)\}$.

Furthermore, each point $y \in S(A)$ can be uniquely expanded as $y = \sum_{i=1}^{l+1} \theta_i \eta^i(A)$, where $\theta_i \geq 0$, $i = 1, \ldots, l+1$, $\sum_{i=1}^{l+1} \theta_i = 1$. The numbers $\theta_i = \theta_i(A; y)$ are the so-called barycentric coordinates of y in the l-dimensional simplex $S(A)$ (Ioffe and Tikhomirov [11, §3.5.2]). Fix some $\delta \in (0, \delta_1]$ and define the mapping φ_A: $\operatorname{cone} B_\delta(\bar{y}) \to K$ as follows:

$$\varphi_A(0) = 0, \qquad \varphi_A(y) = \frac{\|y\|}{\gamma} \sum_{i=1}^{l+1} \theta_i\left(A; \frac{\gamma}{\|y\|} y\right) \xi^i, \quad y \in (\operatorname{cone} B_\delta(\bar{y})) \setminus \{0\},$$

where

$$\gamma = \begin{cases} \|\bar{y}\|, & \text{if } \bar{y} \neq 0, \\ \delta, & \text{if } \bar{y} = 0. \end{cases}$$

It can be easily seen that if δ is taken small enough, then in any case, $\gamma y/\|y\| \in B_{\delta_1}(\bar{y})$ $\forall\, y \in (\operatorname{cone} B_\delta(\bar{y})) \setminus \{0\}$, and according to (7), this mapping is correctly defined. Moreover,

$$\begin{aligned} A\varphi_A(y) &= \frac{\|y\|}{\gamma} \sum_{i=1}^{l+1} \theta_i\left(A; \frac{\gamma}{\|y\|} y\right) A\xi^i \\ &= \frac{\|y\|}{\gamma} \sum_{i=1}^{l+1} \theta_i\left(A; \frac{\gamma}{\|y\|} y\right) \eta^i(A) \\ &= \frac{\|y\|}{\gamma} \frac{\gamma}{\|y\|} y \\ &= y, \\ \|\varphi_A(y)\| &\leq \frac{\|y\|}{\gamma} \sum_{i=1}^{l+1} \theta_i\left(A; \frac{\gamma}{\|y\|} y\right) \|\xi^i\| \\ &\leq \frac{\|y\|}{\gamma} \Bigl(\max_{i=1,\ldots,l+1} \|\xi^i\|\Bigr) \sum_{i=1}^{l+1} \theta_i\left(A; \frac{\gamma}{\|y\|} y\right) \\ &= \frac{1}{\gamma} \Bigl(\max_{i=1,\ldots,l+1} \|\xi^i\|\Bigr) \|y\|; \end{aligned}$$

i.e., (5) holds with $c = \max_{i=1,\ldots,l+1} \|\xi^i\|/\gamma$. Continuity of φ_A can be easily verified. □

Lemma 2.2. *Let $A \in \mathscr{L}(Z, Y)$, and let $P \subset Z$ be a closed convex set.*
If for some $\bar{y} \in Y$ it holds that

$$\bar{y} \in \operatorname{int} A(P), \tag{8}$$

then there exist $z^i \in P$, $i = 1, \ldots, l+1$ such that

$$\bar{y} \in \operatorname{int} A(\operatorname{conv}\{z^1, \ldots, z^{l+1}\}). \tag{9}$$

Proof. According to (8), there exist $\eta^i \in Y$, $i = 1, \ldots, l+1$ such that, on one hand,

$$\bar{y} \in \operatorname{int} \operatorname{conv}\{\eta^1, \ldots, \eta^{l+1}\}, \tag{10}$$

and on the other hand, $\eta^i \in A(P)$ $\forall\, i = 1, \ldots, l+1$ (for instance, one can take $\eta^i = \bar{y} + \varepsilon e^i$, $i = 1, \ldots, l$, $\eta^{l+1} = \bar{y} - \varepsilon \sum_{i=1}^{l} e^i$, where $e^1, \ldots, e^l$ is a standard basis in $Y = \mathbf{R}^l$, with $\varepsilon > 0$ being small enough). Then, there exist $z^i \in P$ such that $\eta^i = Az^i$ $\forall\, i = 1, \ldots, l+1$. It can be easily seen that

$$\operatorname{conv}\{\eta^1, \ldots, \eta^{l+1}\} \subset A(\operatorname{conv}\{z^1, \ldots, z^{l+1}\}).$$

The latter relation combined with (10) implies (9). □

The role of Lemma 2.2 in the proof of Theorem 3.1 below can be characterized as follows: sometimes it is possible to replace, without affecting the related regularity conditions, an arbitrary convex closed set in the constraints by its finitely generated (and hence finite-dimensional) subset, which is more tractable.

LEMMA 2.3. *Let $P \subset Z$ be a closed convex set, and let $\widetilde{P} \subset Z$ be a finitely generated set. Assume that for a given $\bar{z} \in \widetilde{P}$, it holds that $\widetilde{P} \setminus \{\bar{z}\} \subset \operatorname{int} P$.*

Then, there exists $\varepsilon > 0$ such that the inclusion

$$z + (\operatorname{cone}(\widetilde{P} - \bar{z})) \cap B_\varepsilon(0) \subset P \tag{11}$$

holds for each $z \in P$ close enough to $\bar{z}$.

PROOF. It can be easily seen that there exists $\varepsilon > 0$ such that $\bar{z} + (\operatorname{cone}(\widetilde{P} - \bar{z})) \cap (B_\varepsilon(0) \setminus \{0\}) \subset \operatorname{int} P$. We next prove by a contradiction argument that this ε is the one we need. Indeed, suppose that there exist sequences $\{z^k\} \subset P$ and $\{\zeta^k\} \subset \operatorname{cone}(\widetilde{P} - \bar{z})$ such that $\{z^k\} \to \bar{z}$, $\|\zeta^k\| \le \varepsilon$, and $z^k + \zeta^k \notin P$ $\forall\, k$. Since P is convex, one may suppose that $\|\zeta^k\| = \varepsilon$ $\forall\, k$. Then, without loss of generality, one may suppose that the sequence $\{\zeta^k\}$ converges to some $\zeta \in \operatorname{cone}(\widetilde{P} - \bar{z})$ (recall that $\operatorname{cone}(\widetilde{P} - \bar{z})$ is finitely generated, and hence finite dimensional and closed), and moreover, $\|\zeta\| = \varepsilon$. Then, $\{z^k + \zeta^k\} \to \bar{z} + \zeta$; hence $\bar{z} + \zeta \notin \operatorname{int} P$, which contradicts the choice of ε. □

3. Basic stability theorem: Equality-type constraints and set constraints. In this section, we present our basic stability result in the following setting. Let Z be a normed linear space, and let P be a fixed closed convex set in Z. Let $\Phi\colon \Sigma \times Z \to Y$ be a given mapping, and assume that $\bar{z} \in \Delta(\bar{\sigma})$, where for each $\sigma \in \Sigma$

$$\Delta(\sigma) = \{z \in P \mid \Phi(\sigma, z) = 0\}. \tag{12}$$

We shall employ the following hypotheses (H):

(H1) The restriction of Φ to $\Sigma \times P$ is continuous at $(\bar{\sigma}, \bar{z})$.

(H2) For each $(\sigma, z) \in \Sigma \times P$ close enough to $(\bar{\sigma}, \bar{z})$, there exists $A_{\sigma, z} \in \mathscr{L}(Z, Y)$ such that for $\tilde{z} \in P$ the estimate

$$\Phi(\sigma, \tilde{z}) - \Phi(\sigma, z) - A_{\sigma, z}(\tilde{z} - z) = o(\|\tilde{z} - z\|)$$

holds uniformly in (σ, z), and the mapping $(\sigma, z) \to A_{\sigma, z}$ is continuous at $(\bar{\sigma}, \bar{z})$. Any selection of $A_{\sigma, z}$ satisfying these assumptions will be referred to as a *derivative* of Φ with respect to z at (σ, z), and will be denoted by $(\partial\Phi/\partial z)(\sigma, z)$.

For (H1) and (H2) to be satisfied, it suffices to assume that Φ is continuous at $(\bar{\sigma}, \bar{z})$ and Fréchet differentiable with respect to z near $(\bar{\sigma}, \bar{z})$, and its derivative with respect to z is continuous at $(\bar{\sigma}, \bar{z})$. In this case, the derivative of Φ with respect to z is uniquely defined near (σ, z). On the other hand, (H1) and (H2) may hold even when Φ is defined only on $\Sigma \times P$.

THEOREM 3.1. *Let $\bar{z} \in \Delta(\bar{\sigma})$, and let Φ satisfy hypotheses* (H1) *and* (H2). *Assume that* $\operatorname{ri} P \neq \varnothing$.

If $\bar{y} \in Y$ satisfies the inclusion

$$\bar{y} \in \operatorname{int} \frac{\partial\Phi}{\partial z}(\bar{\sigma}, \bar{z})(P - \bar{z}), \tag{13}$$

then there exists $\delta > 0$ such that the estimate

$$\operatorname{dist}(z, \Delta(\sigma)) = O(\|\Phi(\sigma, z)\|) \tag{14}$$

holds for $(\sigma, z) \in \Sigma \times P$ close to $(\bar{\sigma}, \bar{z})$ and satisfying the inclusion

$$-\Phi(\sigma, z) \in \operatorname{cone} B_\delta(\bar{y}). \tag{15}$$

Proof. To begin with, without loss of generality, we may assume that $\operatorname{int} P \neq \varnothing$. Indeed, if $\operatorname{int} P = \varnothing$, then replace Z by $\widetilde{Z} = \operatorname{span}(P - \bar{z})$, replace Φ by the mapping $\widetilde{\Phi}\colon \Sigma \times \widetilde{Z} \to Y$ defined by

$$\widetilde{\Phi}(\sigma, z) = \Phi(\sigma, \bar{z} + z),$$

and for each $\sigma \in \Sigma$ replace $\Delta(\sigma)$ by the set

$$\tilde{\Delta}(\sigma) = \{z \in P - \bar{z} \mid \widetilde{\Phi}(\sigma, \bar{z} + z) = 0\}.$$

Then $\Delta(\sigma) = \tilde{\Delta}(\sigma) + \bar{z}$, and it is evident that estimate (14) holds for $(\sigma, z) \in \Sigma \times P$ close to $(\bar{\sigma}, \bar{z})$ and satisfying the inclusion (15) if and only if the estimate

$$\operatorname{dist}(z, \tilde{\Delta}(\sigma)) = O(\|\widetilde{\Phi}(\sigma, z)\|)$$

holds for $(\sigma, z) \in \Sigma \times (P - \bar{z})$ close to $(\bar{\sigma}, 0)$ and satisfying the inclusion

$$-\widetilde{\Phi}(\sigma, z) \in \operatorname{cone} B_\delta(\bar{y}).$$

Furthermore, (13) is evidently equivalent to the inclusion

$$\bar{y} \in \operatorname{int} \frac{\partial \widetilde{\Phi}}{\partial z}(\bar{\sigma}, 0)(P - \bar{z}).$$

On the other hand, the interior of the set $P - \bar{z}$ with respect to $\operatorname{span} Z = \operatorname{aff}(P - \bar{z})$ coincides with $\operatorname{ri}(P - \bar{z})$, which is assumed to be nonempty. Thus, throughout the rest of the proof, we suppose that $\operatorname{int} P \neq \varnothing$.

According to (13) and Lemma 2.2, there exist $z^i \in P$, $i = 1, \ldots, l+1$ such that

$$\bar{y} \in \operatorname{int} \frac{\partial \Phi}{\partial z}(\bar{\sigma}, \bar{z})(\widetilde{P} - \bar{z}), \tag{16}$$

where

$$\widetilde{P} = \operatorname{conv}\{z^1, \ldots, z^{l+1}, \bar{z}\}.$$

Since $\operatorname{int} P \neq \varnothing$, by a small perturbation of the points $z^i \in P$, it can be achieved that $z^i \in \operatorname{int} P \; \forall\, i = 1, \ldots, l+1$ (and hence $\widetilde{P} \backslash \{\bar{z}\} \subset \operatorname{int} P$) and the inclusion (16) will still be valid.

From (16), it evidently follows that

$$\bar{y} \in \operatorname{int} \frac{\partial \Phi}{\partial z}(\bar{\sigma}, \bar{z})(\operatorname{cone}(\widetilde{P} - \bar{z})).$$

Then, by (H2) and Lemma 2.1, there exist $\delta_1 > 0$ and $c > 0$ such that for (σ, z) close enough to $(\bar{\sigma}, \bar{z})$, there exists a continuous mapping $\varphi_{\sigma, z}\colon \operatorname{cone} B_{\delta_1}(\bar{y}) \to \operatorname{cone}(\widetilde{P} - \bar{z})$ satisfying the following requirements:

$$\frac{\partial \Phi}{\partial z}(\sigma, z)\varphi_{\sigma, z}(y) = y, \quad \|\varphi_{\sigma, z}(y)\| \leq c\|y\| \qquad \forall\, y \in \operatorname{cone} B_{\delta_1}(\bar{y}). \tag{17}$$

Furthermore, by (17), by Lemma 2.3, and by (H2), one can choose $\delta_2 > 0$ such that for any (σ, z) close enough to $(\bar{\sigma}, \bar{z})$ and any $y \in (\operatorname{cone} B_{\delta_1}(\bar{y})) \cap B_{\delta_2}(0)$ it holds that $z + \varphi_{\sigma, z}(y) \in P$ and

$$\left\| \Phi(\sigma, z + \varphi_{\sigma, z}(y)) - \Phi(\sigma, z) - \frac{\partial \Phi}{\partial z}(\sigma, z)\varphi_{\sigma, z}(y) \right\| \leq \frac{\gamma}{2c}\|\varphi_{\sigma, z}(y)\|, \tag{18}$$

where

$$\gamma = \begin{cases} \min\left\{\dfrac{\delta_1}{2\|\bar{y}\|}, 1\right\}, & \text{if } \bar{y} \neq 0, \\[2ex] 1, & \text{if } \bar{y} = 0. \end{cases} \tag{19}$$

For such (σ, z), define the mapping $G_{\sigma, z}\colon (\operatorname{cone} B_{\delta_1}(\bar{y})) \cap B_{\delta_2}(0) \to Y$,

$$G_{\sigma, z}(y) = y - \Phi(\sigma, z + \varphi_{\sigma, z}(y)),$$

and for each $y \in (\operatorname{cone} B_{\delta_1}(\bar{y})) \cap B_{\delta_2}(0)$, set

$$\Omega_{\sigma,z}(y) = -\Phi(\sigma, z) - G_{\sigma,z}(y). \tag{20}$$

By (H2), $G_{\sigma,z}$ is evidently continuous on its domain, and according to (17) and (18),

$$\begin{aligned}\|\Omega_{\sigma,z}(y)\| &= \left\|\Phi(\sigma, z+\varphi_{\sigma,z}(y)) - \Phi(\sigma, z) - \frac{\partial \Phi}{\partial z}(\sigma, z)\varphi_{\sigma,z}(y)\right\| \\ &\le \frac{\gamma}{2}\|y\| \quad \forall\, y \in (\operatorname{cone} B_{\delta_1}(\bar{y})) \cap B_{\delta_2}(0).\end{aligned} \tag{21}$$

It can be easily seen that there exists $\delta \in (0, \delta_1/2]$ possessing the following property: if $\eta \in \operatorname{cone} B_{\delta}(\bar{y})$, then $\|\|\bar{y}\|\eta/\|\eta\| - \bar{y}\| \le \delta_1/2$. If $\Phi(\sigma, z) \ne 0$ (the opposite case is trivial), set $\delta_2(\sigma, z) = \|\Phi(\sigma, z)\|$. According to (H1), for (σ, z) close enough to $(\bar{\sigma}, \bar{z})$ and satisfying (15), and for any $y \in (\operatorname{cone} B_{\delta_1}(\bar{y})) \cap B_{2\delta_2(\sigma,z)}(0)$, we then obtain: if $\bar{y} \ne 0$, then according to (19), (21), it holds that

$$\begin{aligned}\left\|-\frac{\|\bar{y}\|}{\|\Phi(\sigma,z)\|}\Phi(\sigma, z) - \frac{\|\bar{y}\|}{\|\Phi(\sigma,z)\|}\Omega_{\sigma,z}(y) - \bar{y}\right\| &\le \left\|-\frac{\|\bar{y}\|}{\|\Phi(\sigma,z)\|}\Phi(\sigma, z) - \bar{y}\right\| + \frac{\|\bar{y}\|}{\|\Phi(\sigma,z)\|}\frac{\delta_1}{4\|\bar{y}\|}\|y\|, \\ &\le \frac{\delta_1}{2} + \frac{\delta_1}{2} \\ &= \delta_1,\end{aligned}$$

and hence taking into account (20),

$$G_{\sigma,z}(y) = \frac{\|\Phi(\sigma,z)\|}{\|\bar{y}\|}\left(-\frac{\|\bar{y}\|}{\|\Phi(\sigma,z)\|}\Phi(\sigma, z) - \frac{\|\bar{y}\|}{\|\Phi(\sigma,z)\|}\Omega_{\sigma,z}(y)\right) \in \operatorname{cone} B_{\delta_1}(\bar{y}).$$

On the other hand, if $\bar{y} = 0$, then $\operatorname{cone} B_{\delta_1}(\bar{y}) = Y$, and the inclusion $G_{\sigma,z}(y) \in \operatorname{cone} B_{\delta_1}(\bar{y})$ holds trivially. Finally, according to (20),

$$\|G_{\sigma,z}(y)\| \le \|\Phi(\sigma, z)\| + \|\Omega_{\sigma,z}(y)\| \le \|\Phi(\sigma, z)\| + \tfrac{1}{2}\|y\| \le 2\delta_2(\sigma, z).$$

Therefore, for (σ, z) close enough to $(\bar{\sigma}, \bar{z})$, $G_{\sigma,z}$ continuously maps the convex compact set $(\operatorname{cone} B_{\delta_1}(\bar{y})) \cap B_{2\delta_2(\sigma,z)}(0)$ into itself. Thus, according to Brouwer's theorem, this mapping has a fixed point in this set; that is, there exists $y = y(\sigma, z) \in (\operatorname{cone} B_{\delta_1}(\bar{y})) \cap B_{2\delta_2(\sigma,z)}(0)$ such that $y = G_{\sigma,z}(y)$; i.e.,

$$\Phi(\sigma, z+\varphi_{\sigma,z}(y)) = 0.$$

Moreover, according to (17),

$$\|\varphi_{\sigma,z}(y)\| \le c\|y\| \le 2c\delta_2(\sigma, z) = O(\|\Phi(\sigma, z)\|), \tag{22}$$

and it remains to recall (12) and the inclusion $z + \varphi_{\sigma,z}(y) \in P$. □

Applying Theorem 3.1 with $\bar{y} = 0$, we obtain

Corollary 3.1. *Under the assumptions of Theorem 3.1, if*

$$0 \in \operatorname{int} \frac{\partial \Phi}{\partial z}(\bar{\sigma}, \bar{z})(P - \bar{z}),$$

then the estimate (14) holds for $(\sigma, z) \in \Sigma \times P$ close to $(\bar{\sigma}, \bar{z})$.

Remark 3.1. Theorem 3.1 has an evident directional flavor, since (13) can be replaced by the assumption that there exists $\theta \ge 0$ such that

$$\theta\bar{y} \in \operatorname{int} \frac{\partial \Phi}{\partial z}(\bar{\sigma}, \bar{z})(P - \bar{z}).$$

It can be easily seen (e.g., using the separation argument) that in its turn, the latter condition can be presented in the following *homogeneous* form:

$$\bar{y} \in \operatorname{int} \frac{\partial \Phi}{\partial z}(\bar{\sigma}, \bar{z})(R_P(\bar{z})).$$

4. Directional stability theorem. We now get back to the setting discussed in §1: for each $\sigma \in \Sigma$, let $D(\sigma)$ be defined according to (1).

THEOREM 4.1. *Let $\bar{x} \in D(\bar{\sigma})$, let F be continuous at $(\bar{\sigma}, \bar{x})$ and Fréchet differentiable with respect to x near $(\bar{\sigma}, \bar{x})$, and let its derivative with respect to x be continuous at $(\bar{\sigma}, \bar{x})$.*

If $\bar{y} \in Y$ satisfies the inclusion

$$\bar{y} \in \operatorname{int}\left(F(\bar{\sigma}, \bar{x}) + \operatorname{im} \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x}) - Q\right), \tag{23}$$

then there exists $\delta > 0$ such that the estimate

$$\operatorname{dist}(x, D(\sigma)) = O(\|F(\sigma, x) - q\|) \tag{24}$$

holds for $(\sigma, x, q) \in \Sigma \times X \times Q$ close to $(\bar{\sigma}, \bar{x}, F(\bar{\sigma}, \bar{x}))$ and satisfying the inclusion

$$-(F(\sigma, x) - q) \in \operatorname{cone} B_{\delta}(\bar{y}). \tag{25}$$

PROOF. Set $Z = X \times Y$, $P = X \times Q$, and define the mapping $\Phi\colon \Sigma \times Z \to Y$,

$$\Phi(\sigma, z) = F(\sigma, x) - y, \quad z = (x, y). \tag{26}$$

Set $\bar{z} = (\bar{x}, F(\bar{\sigma}, \bar{x})) \in P$. Then, for an arbitrary triple $(\sigma, x, q) \in \Sigma \times X \times Q$ close to $(\bar{\sigma}, \bar{x}, F(\bar{\sigma}, \bar{x}))$ and satisfying (25), we have: $z = (x, q) \in P$, z is close to $\bar{z}$, and (15) holds. Moreover, $\operatorname{ri} P = X \times \operatorname{ri} Q \neq \varnothing$ (since Q is a convex set in a finite-dimensional space $Y = \mathbf{R}^l$), and hence, according to Theorem 3.1 and (12), there exists $r = r(\sigma, z) = (\xi, \eta) \in Z$ such that

$$z + r \in P, \qquad \Phi(\sigma, z + r) = 0, \qquad \|r\| = O(\|\Phi(\sigma, z)\|).$$

Then, from (26), we obtain

$$q + \eta \in Q, \qquad F(\sigma, x + \xi) - (q + \eta) = 0, \qquad \|\xi\| + \|\eta\| = O(\|F(\sigma, x) - q\|),$$

and hence

$$F(\sigma, x + \xi) \in Q, \qquad \|\xi\| = O(\|F(\sigma, x) - q\|).$$

This implies (24). □

Note that with $q = \pi_Q(F(\sigma, x))$, the assertion of Theorem 4.1 takes the following form: if (23) holds, then there exists $\delta > 0$ such that the estimate (3) holds for $(\sigma, x) \in \Sigma \times X$ close to $(\bar{\sigma}, \bar{x})$ and satisfying the inclusion

$$-(F(\sigma, x) - \pi_Q(F(\sigma, x))) \in \operatorname{cone} B_{\delta}(\bar{y}).$$

Among the immediate consequences of this fact is Robinson's stability theorem (see Robinson [17], or Bonnans and Shapiro [6, Theorem 2.87]). To obtain the latter, it suffices to apply the result above with $\bar{y} = 0$. Note that our argument does not rely on any set-valued analysis, and in particular, the Robinson-Ursescu stability theorem [18, 16] (see also Bonnans and Shapiro [6, Theorem 2.83]) is not employed here. However, we stress it again that in our setting, Y is a finite-dimensional space. (Note, however, that X is not supposed to be complete!)

COROLLARY 4.1. *Under the assumptions of Theorem 4.1, if Robinson's CQ (2) is satisfied, then the estimate (3) holds for $(\sigma, x) \in \Sigma \times X$ close to $(\bar{\sigma}, \bar{x})$.*

Of course, Corollary 4.1 can also be derived from Corollary 3.1.

REMARK 4.1. Clearly, assumption (23) in Theorem 4.1 can be replaced by the assumption that there exists $\theta \geq 0$ such that

$$\theta \bar{y} \in \operatorname{int}\left(F(\bar{\sigma}, \bar{x}) + \operatorname{im} \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x}) - Q\right),$$

and it can be easily seen that the latter condition can be presented in the following *homogeneous* form:

$$\bar{y} \in \operatorname{int}\left(\operatorname{im} \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x}) - R_Q(F(\bar{\sigma}, \bar{x}))\right). \tag{27}$$

5. Directional metric regularity. In this section, we temporarily get back to the nonparametric case to introduce and study the following concept.

DEFINITION 5.1. We say that the mapping $F: X \to Y$ is metrically regular at $\bar{x} \in F^{-1}(Q)$ with respect to Q in a direction $\bar{y} \in Y$ if there exists $\delta > 0$ such that the estimate

$$\operatorname{dist}(x, F^{-1}(Q - y)) = O(\|F(x) + y - q\|)$$

holds for $(x, y, q) \in X \times Y \times Q$ close to $(\bar{x}, 0, F(\bar{x}))$ and satisfying the inclusion

$$-(F(x) + y - q) \in \operatorname{cone} B_\delta(\bar{y}).$$

Clearly, metric regularity in a direction $\bar{y} = 0$ is equivalent to the usual metric regularity (to prove that the former implies the latter, it suffices to take $q = \pi_Q(F(x) + y)$).

By Theorem 4.1 and Remark 4.1, under the appropriate smoothness assumptions, condition

$$\bar{y} \in \operatorname{int}\big(\operatorname{im} F'(\bar{x}) - R_Q(F(\bar{x}))\big) \tag{28}$$

(compare with (27)) implies metric regularity in a direction $\bar{y}$. The converse implication is at issue in our next result.

PROPOSITION 5.1. *Let $\bar{x} \in F^{-1}(Q)$, let F be Fréchet differentiable near $\bar{x}$, and let its derivative be continuous at $\bar{x}$.*

Then, F is metrically regular at $\bar{x}$ with respect to Q in a direction $\bar{y} \in Y$ if and only if condition (28) holds.

PROOF. Suppose that F is metrically regular at $\bar{x}$ with respect to Q in a direction $\bar{y} \in Y$, but (28) does not hold. Then, by the separation argument, there exists $\mu \in Y \setminus \{0\}$ such that

$$\langle \mu, \bar{y} \rangle \le 0 \le \langle \mu, \eta \rangle \quad \forall\, \eta \in \operatorname{im} F'(\bar{x}) - R_Q(F(\bar{x})),$$

which is equivalent to

$$\langle \mu, \bar{y} \rangle \le 0, \quad \mu \in (\operatorname{im} F'(\bar{x}))^\perp \cap (R_Q(F(\bar{x})))^\circ. \tag{29}$$

Set $\theta = \delta/\|\mu\|$, where $\delta > 0$ is taken from Definition 5.1. Then, for $x = \bar{x}$, $q = F(\bar{x})$, and $y(t) = -t(\bar{y} - \theta\mu)$, $t > 0$, it holds that

$$-(F(x) + y(t) - q) = t(\bar{y} - \theta\mu) \in \operatorname{cone} B_\delta(\bar{y}),$$

and hence, by Definition 5.1, the estimate

$$\begin{aligned} \operatorname{dist}(x, F^{-1}(Q - y(t))) &= O(\|F(x) + y(t) - q\|) \\ &= O(t\|\bar{y} - \theta\mu\|) \\ &= O(t) \end{aligned}$$

holds for $t > 0$. This means that for each $t > 0$ small enough, there exists $x(t) \in X$ such that

$$F(x(t)) \in Q - y(t) = Q + t(\bar{y} - \theta\mu), \qquad \|x(t) - \bar{x}\| = O(t).$$

Then, by (29), we obtain

$$\langle \mu, F(x(t)) - F(\bar{x}) \rangle \le -\theta\|\mu\|^2 t,$$

while on the other hand,

$$\begin{aligned} \langle \mu, F(x(t)) - F(\bar{x}) \rangle &= \langle \mu, F'(\bar{x})(x(t) - \bar{x}) \rangle + o(\|x(t) - \bar{x}\|) \\ &= o(t), \end{aligned}$$

which is a contradiction. □

Theorem 4.1 and Proposition 5.1 imply the following important property: under the assumptions of Proposition 5.1, metric regularity in a given direction $\bar{y} \in Y$ is stable subject to perturbations of F, such that the corresponding perturbation of $F(\bar{x})$ and $F'(\bar{x})$ is small enough. Specifically, let F be metrically regular at $\bar{x}$ with respect to Q in a direction $\bar{y}$. Then, by Proposition 5.1, condition (28) holds. Let the space Σ be comprised by mappings σ: $X \to Y$, which are Fréchet differentiable near $\bar{x}$, let $\bar{\sigma} = F$, and let a topology in Σ be defined in such a way that $\sigma(x) \to F(\bar{x})$, $\sigma'(x) \to F'(\bar{x})$ as $\sigma \to \bar{\sigma}$ and $x \to \bar{x}$. Applying Theorem 4.1, we obtain the following: For any mapping $\tilde{F}$: $X \to Y$ Fréchet differentiable near $\bar{x}$ and such that for $x \in X$ close enough to $\bar{x}$, $\tilde{F}(x)$ and $\tilde{F}'(x)$ are close enough to $F(\bar{x})$ and $F'(\bar{x})$, respectively, and for any $\tilde{x} \in \tilde{F}^{-1}(Q)$ close enough to $\bar{x}$, $\tilde{F}$ is metrically regular at $\tilde{x}$ with respect to Q in a direction $\bar{y}$. Moreover, the following *uniform* version of this statement is valid: there exist neighborhood $\mathcal{O}$ of $(\bar{x}, 0, F(\bar{x}))$ in $X \times Y \times Y$, $\delta > 0$ and $c > 0$ (all independent of $\tilde{F}$) such that the estimate

$$\operatorname{dist}(x, \tilde{F}^{-1}(Q - y)) \leq c\|\tilde{F}(x) + y - q\|$$

holds for $(x, y, q) \in \mathcal{O}$ such that $q \in Q$ and

$$-(\tilde{F}(x) + y - q) \in \operatorname{cone} B_\delta(\bar{y}).$$

6. Applications: The case of directional regularity. Throughout the rest of this paper, let Σ be a normed linear space. It can be easily seen that for any $d \in \Sigma$, the directional regularity condition (4) can be expressed in the following equivalent form:

$$\frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d \in \operatorname{int}\left(R_Q(F(\bar{\sigma}, \bar{x})) - \operatorname{im} \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})\right). \tag{30}$$

Indeed, suppose first that (4) holds while (30) is violated. Then, by the separation argument, there exists $\mu \in Y^* \backslash \{0\}$ such that

$$\left\langle \mu, \frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d \right\rangle \leq 0 \leq \langle \mu, \eta \rangle \quad \forall \eta \in R_Q(F(\bar{\sigma}, \bar{x})) - \operatorname{im} \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x}), \tag{31}$$

and hence

$$\left\langle \mu, F(\bar{\sigma}, \bar{x}) - q + y + \theta \frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d \right\rangle \leq 0 \quad \forall q \in Q, \ \forall y \in \operatorname{im} \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x}), \ \forall \theta \geq 0, \tag{32}$$

where the inclusion $Q - F(\bar{\sigma}, \bar{x}) \subset R_Q(F(\bar{\sigma}, \bar{x}))$ was taken into account. Thus

$$\langle \mu, \eta \rangle \leq 0 \quad \forall \eta \in F(\bar{\sigma}, \bar{x}) + \operatorname{im} \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x}) + \operatorname{cone}\left\{\frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d\right\} - Q, \tag{33}$$

which contradicts (4).

Now, suppose that (30) holds while (4) is violated. Then, again by the separation argument, there exists $\mu \in Y^* \backslash \{0\}$ such that (33) holds, while the latter can be written in the form (32). From (32), it easily follows that

$$\left\langle \mu, \frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d \right\rangle \leq 0 \leq \left\langle \mu, \frac{1}{\theta}(q - F(\bar{\sigma}, \bar{x})) - y \right\rangle \quad \forall q \in Q, \ \forall y \in \operatorname{im} \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x}), \ \forall \theta > 0,$$

while the latter evidently gives (31), which contradicts (30).

We next demonstrate how Theorem 4.1 can be used to easily prove the two lemmas playing the crucial role in sensitivity analysis under the directional regularity condition. The first lemma essentially corresponds to Bonnans and Shapiro [6, Lemma 4.10].

LEMMA 6.1. *Under the assumptions of Theorem 4.1, let F be Fréchet differentiable at $(\bar{\sigma}, \bar{x})$.*

If the directional regularity condition (4) holds at $\bar{x}$ with respect to a direction $d \in \Sigma$, then for any sequences $\{t_k\} \subset \mathbf{R}_+ \backslash \{0\}$, $\{\rho^k\} \subset \Sigma$ and $\{x^k\} \subset X$ such that $\{t_k\} \to 0$, $\rho^k = o(t_k)$, and

$$\|x^k - \bar{x}\| = O(t_k), \qquad \operatorname{dist}(F(\bar{\sigma} + t_k d + \rho^k, x^k), Q) = o(t_k), \tag{34}$$

the estimate

$$\operatorname{dist}(x^k, D(\bar{\sigma} + t_k d + \rho^k)) = O(\operatorname{dist}(F(\bar{\sigma} + t_k d + \rho^k, x^k), Q)) \tag{35}$$

holds.

PROOF. Since directional regularity condition (4) is equivalent to (30), and by Remark 4.1, Theorem 4.1 is applicable with $\bar{y} = -(\partial F/\partial \sigma)(\bar{\sigma}, \bar{x})d$. Hence, there exists $\delta > 0$ such that the estimate (24) holds for

$(\sigma, x, q) \in \Sigma \times X \times Q$ close to $(\bar{\sigma}, \bar{x}, F(\bar{\sigma}, \bar{x}))$ and satisfying the inclusion

$$F(\sigma, x) - q \in \operatorname{cone} B_\delta\left(\frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d\right). \tag{36}$$

For each k set $\tau_k = \operatorname{dist}(F(\bar{\sigma} + t_k d + \rho^k, x^k), Q)$, and suppose that $\tau_k \neq 0$ (since the opposite case is trivial). By taking $\gamma > 0$ large enough, we can ensure the inclusion

$$\gamma \frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d + B_1(0) \subset \operatorname{cone} B_\delta\left(\frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d\right). \tag{37}$$

For each k set

$$\theta_k = (\gamma + 1)\frac{\tau_k}{t_k}. \tag{38}$$

Note that according to the second relation in (34), $\theta_k \to 0$ as $k \to \infty$.

According to the first relation in (34), for each k, we have

$$F(\bar{\sigma} + t_k d + \rho^k, x^k) = F(\bar{\sigma}, \bar{x}) + t_k \frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d + \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})(x^k - \bar{x}) + o(t_k),$$

and thus, by setting $y^k = \pi_Q(F(\bar{\sigma} + t_k d + \rho^k, x^k))$ and taking into account the second relation in (34), we derive the equality

$$\frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})(x^k - \bar{x}) = y^k - F(\bar{\sigma}, \bar{x}) - t_k \frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d + o(t_k).$$

By this equality and the first relation in (34), for each k large enough,

$$\begin{aligned}
F(\bar{\sigma} + t_k d + \rho^k, x^k - \theta_k(x^k - \bar{x})) &= F(\bar{\sigma} + t_k d + \rho^k, x^k) - \theta_k \frac{\partial F}{\partial x}(\bar{\sigma} + t_k d + \rho^k, x^k)(x^k - \bar{x}) + o(\theta_k \|x^k - \bar{x}\|) \\
&= F(\bar{\sigma} + t_k d + \rho^k, x^k) - \theta_k \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})(x^k - \bar{x}) + o(\theta_k t_k) \\
&= F(\bar{\sigma} + t_k d + \rho^k, x^k) - \theta_k(y^k - F(\bar{\sigma}, \bar{x})) + \theta_k t_k \frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d + o(\theta_k t_k),
\end{aligned}$$

and hence, taking into account (38), we obtain that

$$\begin{aligned}
&F(\bar{\sigma} + t_k d + \rho^k, x^k - \theta_k(x^k - \bar{x})) - (y^k - \theta_k(y^k - F(\bar{\sigma}, \bar{x}))) \\
&\quad = F(\bar{\sigma} + t_k d + \rho^k, x^k) - y^k + (\gamma + 1)\tau_k \frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d + o(\tau_k) \\
&\quad = \tau_k\left(\frac{F(\bar{\sigma} + t_k d + \rho^k, x^k) - y^k}{\|F(\bar{\sigma} + t_k d + \rho^k, x^k) - y^k\|} + \gamma \frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d\right) + \left(\tau_k \frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d + o(\tau_k)\right).
\end{aligned}$$

According to (37), the first term in the right-hand side belongs to $\operatorname{cone} B_\delta((\partial F/\partial \sigma)(\bar{\sigma}, \bar{x})d)$. The second term also belongs to $\operatorname{cone} B_\delta((\partial F/\partial \sigma)(\bar{\sigma}, \bar{x})d)$ for all k large enough, and hence, for such k inclusion, (36) holds with $\sigma = \bar{\sigma} + t_k d + \rho^k$, $x = x^k - \theta_k(x^k - \bar{x})$, $q = y^k - \theta_k(y^k - F(\bar{\sigma}, \bar{x}))$.

Furthermore, employing the well-known fact that the projection operator π_Q is Lipschitz continuous with modulus 1, by the mean value theorem and the first relation in (34), we obtain

$$\begin{aligned}
\|y^k - F(\bar{\sigma}, \bar{x})\| &= \|\pi_Q(F(\bar{\sigma} + t_k d + \rho^k, x^k)) - \pi_Q(F(\bar{\sigma}, \bar{x}))\| \\
&\leq \|F(\bar{\sigma} + t_k d + \rho^k, x^k) - F(\bar{\sigma}, \bar{x})\| \\
&= O(t_k) + O(\|x^k - \bar{x}\|) \\
&= O(t_k).
\end{aligned}$$

Thus, from (24), the mean value theorem, the first relation in (34), and (38), it follows that

$$\begin{aligned}
\operatorname{dist}(x^k, D(\bar{\sigma} + t_k d + \rho^k)) &\leq \operatorname{dist}(x^k - \theta_k(x^k - \bar{x}), D(\bar{\sigma} + t_k d + \rho^k)) + \theta_k \|x^k - \bar{x}\| \\
&= O(\|F(\bar{\sigma} + t_k d + \rho^k, x^k - \theta_k(x^k - \bar{x})) - (y^k - \theta_k(y^k - F(\bar{\sigma}, \bar{x})))\|) + \theta_k \|x^k - \bar{x}\| \\
&= O(\|F(\bar{\sigma} + t_k d + \rho^k, x^k) - y^k\|) + O(\theta_k \|x^k - \bar{x}\|) + O(\theta_k \|y^k - F(\bar{\sigma}, \bar{x})\|) \\
&= O(\operatorname{dist}(F(\bar{\sigma} + t_k d + \rho^k, x^k), Q)) + O(\theta_k t_k) \\
&= O(\operatorname{dist}(F(\bar{\sigma} + t_k d + \rho^k, x^k), Q)).
\end{aligned}$$

This gives (35). □

The assertion in Bonnans and Shapiro [6, Lemma 4.10] is slightly stronger than in our Lemma 6.1, but the assumptions are stronger too: F is supposed to have Lipschitz-continuous derivative near $(\bar{\sigma}, \bar{x})$. However, it seems that what is actually needed in sensitivity analysis is precisely our Lemma 6.1. Specifically, this lemma can be immediately used to derive sufficient conditions for the existence (for given $\xi \in X$ or $\xi^1, \xi^2 \in X$) of feasible arcs of the form $\bar{x} + t\xi + o(t)$, $\bar{x} + t\xi + O(t^2)$ (see Bonnans and Shapiro [6, Lemma 4.57]), or $\bar{x} + t\xi^1 + t^2\xi^2 + o(t^2)$, corresponding to the given arc $\bar{\sigma} + td + o(t)$ (or $\bar{\sigma} + td + O(t^2)$) in the space of parameters, $t \geq 0$. Analysis *along* such arcs leads to the most sharp sensitivity results in the case when the solution can be expected to possess Lipschitzian stability (see Bonnans and Shapiro [6, §§4.5 and 4.7]).

On the other hand, assuming that F has a Lipschitz-continuous derivative near $(\bar{\sigma}, \bar{x})$, one can easily modify the above proof of Lemma 6.1 to establish Bonnans and Shapiro [6, Lemma 4.10] in full generality.

Our next lemma essentially corresponds to Bonnans and Shapiro [6, Lemma 4.109].

Lemma 6.2. *Under the assumptions of Theorem 4.1, let F be Fréchet differentiable at $(\bar{\sigma}, \bar{x})$.*

If the directional regularity condition (4) holds at $\bar{x}$ with respect to a direction $d \in \Sigma$, then there exists $c > 0$ possessing the following property: for any sequences $\{t_k\} \subset \mathbf{R}_+ \backslash \{0\}$, $\{\rho^k\} \subset \Sigma$, and $\{x^k\} \subset X$ such that $\{t_k\} \to 0$, $\rho^k = o(t_k)$, $\{x^k\} \to \bar{x}$, and

$$\operatorname{dist}(F(\bar{\sigma} + t_k d + \rho^k, x^k), Q) = o(t_k), \tag{39}$$

and for any $\theta > 0$, the inequality

$$\operatorname{dist}(x^k, D(\bar{\sigma} + (1+\theta)t_k d + \rho^k)) \leq c\theta t_k \tag{40}$$

holds for each k large enough.

Proof. By the same argument as in the proof of Lemma 6.1, we obtain the existence of $\delta > 0$ such that the estimate (24) holds for $(\sigma, x, q) \in \Sigma \times X \times Q$ close to $(\bar{\sigma}, \bar{x}, F(\bar{\sigma}, \bar{x}))$ and satisfying the inclusion (36).

For each k, we have

$$F(\bar{\sigma} + (1+\theta)t_k d + \rho^k, x^k) = F(\bar{\sigma} + t_k d + \rho^k, x^k) + \theta t_k \frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d + o(t_k),$$

and thus, by setting $y^k = \pi_Q(F(\bar{\sigma} + t_k d + \rho^k, x^k))$ and taking into account (39), for each k large enough, we obtain

$$\begin{aligned} F(\bar{\sigma} + (1+\theta)t_k d + \rho^k, x^k) - y^k &= \theta t_k \frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d + o(t_k) \\ &\in \operatorname{cone} B_\delta\left(\frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d\right), \end{aligned} \tag{41}$$

i.e., inclusion (36) holds with $\sigma = \bar{\sigma} + (1+\theta)t_k d + \rho^k$, $x = x^k$, $q = y^k$.

From (24), it follows that

$$\operatorname{dist}(x^k, D(\bar{\sigma} + (1+\theta)t_k d + \rho^k)) = O(\|F(\bar{\sigma} + (1+\theta)t_k d + \rho^k, x^k) - y^k\|),$$

and by the equality in (41), the needed estimate (40) with some $c > 0$ follows. □

Lemma 6.2 can be immediately used to derive sufficient conditions for the existence (for given $\xi^1, \xi^2 \in X$) of feasible arcs of the form $\bar{x} + t^{1/2}\xi^1 + t\xi^2 + o(t)$, corresponding to the arc $\bar{\sigma} + td + o(t)$ in the space of parameters, $t \geq 0$. Known sensitivity results in the case when the solution can be expected to possess Hölder stability only are based on the analysis *along* such arcs (see Bonnans and Shapiro [6, §§4.5 and 4.8]).

7. Applications: Sensitivity analysis in the cases of Hölder stability and empty sets of Lagrange multipliers. The cases appearing in the title of this section were previously studied by means of square root-linear feasible arcs mentioned at the very end of the previous section (see Bonnans and Shapiro [6, §4.8.3] or the original works (Bonnans [3], Bonnans and Cominetti [5])). However, in these cases, it seems quite natural to consider pure square root arcs; that is, $\bar{x} + t^{1/2}\xi + o(t^{1/2})$, $t \geq 0$, for a given $\xi \in X$. This line of analysis is more direct than those used earlier, and in particular, it does not appeal to any duality argument, and the resulting theory (presented below) is quite complete and self-contained. Morever, we believe that *short* arcs do the job in the case when there are no Lagrange multipliers, while *longer* ones should come into play when the set of Lagrange multipliers is nonempty.

Throughout the rest of this paper, let us assume that the set Q possesses the so-called *conicity* property at $F(\bar{\sigma}, \bar{x})$; that is, within some neighborhood of 0, the set $Q - F(\bar{\sigma}, \bar{x})$ coincides with $R_Q(F(\bar{\sigma}, \bar{x}))$ (the same property can also be expressed by saying that $R_Q(F(\bar{\sigma}, \bar{x}))$ is closed, or by the equality $T_Q(F(\bar{\sigma}, \bar{x})) = R_Q(F(\bar{\sigma}, \bar{x}))$). Note that the conicity property is automatic when Q is a polyhedral set (and hence, in the case of mathematical programming problems). Also, as was pointed out by the referee, conicity can actually be replaced (with the appropriate changes in the analysis and statements) by a much weaker property of cone reducibility, as defined in Bonnans and Shapiro [6, Definition 3.135]. This gives rise to a possibility of applications to semidefinite programming problems (Bonnans and Shapiro [6, Example 3.140]).

Proposition 7.1. *Let $\bar{x} \in D(\bar{\sigma})$, and let F be Fréchet differentiable at $(\bar{\sigma}, \bar{x})$ and twice Fréchet differentiable with respect to x at $(\bar{\sigma}, \bar{x})$.*

Then, for any $d \in \Sigma$, the following assertions are valid:

(i) *If for a given $\xi \in X$ there exist sequences $\{t_k\} \subset \mathbf{R}_+ \backslash \{0\}$ and $\{\rho^k\} \subset \Sigma$ such that $\{t_k\} \to 0$, $\rho^k = o(t_k)$, and*

$$\operatorname{dist}\big(\bar{x} + t_k^{1/2}\xi, D(\bar{\sigma} + t_k d + \rho^k)\big) = o\big(t_k^{1/2}\big),$$

then $\xi \in ((\partial F/\partial x)(\bar{\sigma}, \bar{x}))^{-1}(T_Q(F(\bar{\sigma}, \bar{x})))$ and

$$\frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi, \xi] \in \operatorname{cl}\left(R_Q(F(\bar{\sigma}, \bar{x})) - \operatorname{im}\frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})\right). \tag{42}$$

(ii) *Let F be Fréchet differentiable with respect to x near $(\bar{\sigma}, \bar{x})$, and let its derivative with respect to x be continuous at $(\bar{\sigma}, \bar{x})$. Suppose that the set Q possesses the conicity property at $F(\bar{\sigma}, \bar{x})$, and the directional regularity condition (4) holds at $\bar{x}$ with respect to a direction d. Then, for any $\xi \in ((\partial F/\partial x)(\bar{\sigma}, \bar{x}))^{-1}(T_Q(F(\bar{\sigma}, \bar{x})))$ satisfying (42), and for any mapping ρ: $\mathbf{R}_+ \to \Sigma$ such that $\rho(t) = o(t)$, the estimate*

$$\operatorname{dist}(\bar{x} + t^{1/2}\xi, D(\bar{\sigma} + td + \rho(t))) = o(t^{1/2}) \tag{43}$$

holds for $t \geq 0$.

Proof. We first prove (i). Fix a sequence $\{x^k\} \subset X$ such that $x^k = \bar{x} + t_k^{1/2}\xi + o(t_k^{1/2})$ (note that by necessity $\{x^k\} \to \bar{x}$) and $x^k \in D(\bar{\sigma} + t_k d + \rho^k)$ $\forall\, k$. Then,

$$\begin{aligned} Q \ni\ & F(\bar{\sigma} + t_k d + \rho^k, x^k) \\ &= F\big(\bar{\sigma} + t_k d + \rho^k, \bar{x} + t_k^{1/2}\xi + o\big(t_k^{1/2}\big)\big) \\ &= F(\bar{\sigma}, \bar{x}) + t_k^{1/2}\frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})\xi + o\big(t_k^{1/2}\big), \end{aligned} \tag{44}$$

$$\begin{aligned} Q - \operatorname{im}\frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x}) \ni\ & F(\bar{\sigma} + t_k d + \rho^k, x^k) - \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})(x^k - \bar{x}) \\ &= F(\bar{\sigma}, \bar{x}) + t_k\left(\frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi, \xi]\right) + o(t_k). \end{aligned} \tag{45}$$

From (44), it follows that

$$t_k^{1/2}\frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})\xi + o\big(t_k^{1/2}\big) \in Q - F(\bar{\sigma}, \bar{x}) \subset R_Q(F(\bar{\sigma}, \bar{x})).$$

Dividing the left- and right-hand sides of the latter relation by $t_k^{1/2}$, and passing onto the limit as $k \to \infty$, we come to the inclusion $(\partial F/\partial x)(\bar{\sigma}, \bar{x})\xi \in \operatorname{cl} R_Q(F(\bar{\sigma}, \bar{x})) = T_Q(F(\bar{\sigma}, \bar{x}))$, i.e., $\xi \in ((\partial F/\partial x)(\bar{\sigma}, \bar{x}))^{-1}(T_Q(F(\bar{\sigma}, \bar{x})))$.

Moreover, from (45), it follows that

$$\begin{aligned} t_k\left(\frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi, \xi]\right) + o(t_k) &\in Q - F(\bar{\sigma}, \bar{x}) - \operatorname{im}\frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x}) \\ &\subset R_Q(F(\bar{\sigma}, \bar{x})) - \operatorname{im}\frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x}). \end{aligned}$$

Dividing the left- and right-hand sides of the latter relation by t_k, and passing onto the limit as $k \to \infty$, we come to the inclusion (42).

We now prove (ii). For that purpose, we first show that the set of $\xi \in X$ satisfying (43) is closed. We argue by a contradiction. Suppose that there exists a sequence $\{\xi^k\} \subset X$ such that it converges to some $\bar{\xi} \in X$, and for each k, (43) holds with $\xi = \xi^k$, but at the same time, there exist $\gamma > 0$ and a sequence $\{t_j\} \subset \mathbf{R}_+ \backslash \{0\}$ such that $\{t_j\} \to 0$, and for each j,

$$\operatorname{dist}\big(\bar{x} + t_j^{1/2}\bar{\xi}, D(\bar{\sigma} + t_j d + \rho(t_j))\big) \geq \gamma t_j^{1/2}. \tag{46}$$

The latter means that $\forall\, x \in D(\bar{\sigma} + t_j d + \rho(t_j))$,

$$\big\|\bar{x} + t_j^{1/2}\bar{\xi} - x\big\| \geq \gamma t_j^{1/2}. \tag{47}$$

On the other hand, for each k large enough it holds that $\|\xi^k - \bar{\xi}\| \leq \gamma/2$. Hence, according to (46) and (47),

$$\begin{aligned}\big\|\bar{x} + t_j^{1/2}\xi^k - x\big\| &\geq \big\|\bar{x} + t_j^{1/2}\bar{\xi} - x\big\| - t_j^{1/2}\|\xi^k - \bar{\xi}\| \\ &\geq \gamma t_j^{1/2} - \tfrac{1}{2}\gamma t_j^{1/2} \\ &= \tfrac{1}{2}\gamma t_j^{1/2};\end{aligned}$$

that is,

$$\operatorname{dist}\big(\bar{x} + t_j^{1/2}\xi^k, D(\bar{\sigma} + t_j d + \rho(t_j))\big) \geq \tfrac{1}{2}\gamma t_j^{1/2}.$$

But this contradicts (43) with $\xi = \xi^k$.

From (30) and (42), it follows that for each fixed $\theta \in [0, 1)$, the following inclusion holds:

$$\frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d + \frac{1}{2}\theta^2 \frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi, \xi] \in \operatorname{int}\bigg(R_Q(F(\bar{\sigma}, \bar{x})) - \operatorname{im}\frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})\bigg).$$

By the same argument as in the proof of Lemma 6.1, it can now be shown that Theorem 4.1 is applicable with $\bar{y} = -(\partial F/\partial \sigma)(\bar{\sigma}, \bar{x})d - \frac{1}{2}\theta^2(\partial^2 F/\partial x^2)(\bar{\sigma}, \bar{x})[\xi, \xi]$, and hence there exists $\delta > 0$ such that the estimate (24) holds for $(\sigma, x, q) \in \Sigma \times X \times Q$ close to $(\bar{\sigma}, \bar{x}, F(\bar{\sigma}, \bar{x}))$ and satisfying the inclusion

$$F(\sigma, x) - q \in \operatorname{cone} B_\delta\bigg(\frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[\theta\xi, \theta\xi]\bigg). \tag{48}$$

For each $t > 0$ small enough, by setting $\sigma = \bar{\sigma} + td + \rho(t)$, $x = \bar{x} + t^{1/2}\theta\xi$, $q = F(\bar{\sigma}, \bar{x}) + t^{1/2}(\partial F/\partial x)(\bar{\sigma}, \bar{x})\theta\xi$, and taking into account the conicity property of Q at $F(\bar{\sigma}, \bar{x})$ and the inclusion $\xi \in ((\partial F/\partial x)(\bar{\sigma}, \bar{x}))^{-1}(T_Q(F(\bar{\sigma}, \bar{x})))$, we obtain $q \in Q$ and

$$\begin{aligned}F(\sigma, x) - q &= t\bigg(\frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[\theta\xi, \theta\xi]\bigg) + o(t) \\ &\in \operatorname{cone} B_\delta\bigg(\frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[\theta\xi, \theta\xi]\bigg);\end{aligned} \tag{49}$$

i.e., (48) holds. Hence, by (24) and the equality in (49), we obtain the estimate

$$\operatorname{dist}(\bar{x} + t^{1/2}\theta\xi, D(\bar{\sigma} + td + \rho(t))) = O(\|F(\sigma, x) - q\|) = O(t).$$

It remains to note that $\theta\xi \to \xi$ as $\theta \to 1$, and to employ the above-proved fact that the set of $\xi \in X$ satisfying (43) is closed. $\square$

Let $f\colon \Sigma \times X \to \mathbf{R}$ be a smooth function, and for each $\sigma \in \Sigma$, consider the optimization problem

$$\begin{aligned}&\text{minimize} \quad f(\sigma, x) \\ &\text{subject to} \quad x \in D(\sigma).\end{aligned} \tag{50}$$

Let $\bar{x}$ be a local solution of problem (50) with $\sigma = \bar{\sigma}$ and define the critical cone

$$C(\bar{\sigma}, \bar{x}) = \bigg\{\xi \in \bigg(\frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})\bigg)^{-1}(T_Q(F(\bar{\sigma}, \bar{x}))) \,\bigg|\, \bigg\langle \frac{\partial f}{\partial x}(\bar{\sigma}, \bar{x}), \xi\bigg\rangle \leq 0\bigg\} \tag{51}$$

and the following *second-order tightened* critical cone

$$C_2(\bar{\sigma}, \bar{x}) = \bigg\{\xi \in C(\bar{\sigma}, \bar{x}) \,\bigg|\, \frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi, \xi] \in \operatorname{cl}\bigg(R_Q(F(\bar{\sigma}, \bar{x})) - \operatorname{im}\frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})\bigg)\bigg\} \tag{52}$$

of this problem at $\bar{x}$. Furthermore, define the set

$$\Lambda(\bar{\sigma}, \bar{x}) = \left\{ \lambda \in N_Q(F(\bar{\sigma}, \bar{x})) \;\middle|\; \frac{\partial L}{\partial x}(\bar{\sigma}, \bar{x}, \lambda) = 0 \right\} \tag{53}$$

of Lagrange multipliers associated with $\bar{x}$, where L: $\Sigma \times X \times Y^* \to \mathbf{R}$,

$$L(\sigma, x, \lambda) = f(\sigma, x) + \langle \lambda, F(\sigma, x) \rangle$$

is the Lagrangian of problem (50).

Let $\delta > 0$ be fixed small enough, so that $\bar{x}$ is a global solution of problem (50) with $\sigma = \bar{\sigma}$ and with the additional constraint $x \in B_\delta(\bar{x})$. Define the local optimal value function of problem (50) as follows: v: $\Sigma \to \mathbf{R}$,

$$v(\sigma) = \inf_{x \in D(\sigma) \cap B_\delta(\bar{x})} f(\sigma, x).$$

With this definition, $v(\bar{\sigma}) = f(\bar{\sigma}, \bar{x})$. Of course, v depends on the choice of δ, but such optimal value function is a completely relevant object for local (asymptotic) analysis.

For each $d \in \Sigma$, consider the following auxiliary optimization problem:

$$\begin{aligned} &\text{minimize} \quad \left\langle \frac{\partial f}{\partial x}(\bar{\sigma}, \bar{x}), \xi \right\rangle \\ &\text{subject to} \quad \xi \in \Xi_{1/2}(\bar{\sigma}, \bar{x}; d), \end{aligned} \tag{54}$$

where

$$\Xi_{1/2}(\bar{\sigma}, \bar{x}; d) = \left\{ \xi \in C(\bar{\sigma}, \bar{x}) \;\middle|\; \frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi, \xi] \in \operatorname{cl}\left(R_Q(F(\bar{\sigma}, \bar{x})) - \operatorname{im} \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x}) \right) \right\}, \tag{55}$$

and let $v_{1/2}(\bar{\sigma}, \bar{x}; d)$ be the optimal value of problem (54):

$$v_{1/2}(\bar{\sigma}, \bar{x}; d) = \inf_{\xi \in \Xi_{1/2}(\bar{\sigma}, \bar{x}; d)} \left\langle \frac{\partial f}{\partial x}(\bar{\sigma}, \bar{x}), \xi \right\rangle. \tag{56}$$

Recall that the directional regularity condition (4) with respect to a direction $d \in \Sigma$ is equivalent to (30), and hence, an arbitrary $\xi \in C(\bar{\sigma}, \bar{x})$ close enough to 0 satisfies (42), and from (55) it follows that such ξ belongs to $\Xi_{1/2}(\bar{\sigma}, \bar{x}; d)$. In particular,

$$0 \in \Xi_{1/2}(\bar{\sigma}, \bar{x}; d), \tag{57}$$

and from (56), it immediately follows that

$$v_{1/2}(\bar{\sigma}, \bar{x}; d) \le 0. \tag{58}$$

THEOREM 7.1. *Under the assumptions of Theorem 4.1, let F be Fréchet differentiable at $(\bar{\sigma}, \bar{x})$ and twice Fréchet differentiable with respect to x at $(\bar{\sigma}, \bar{x})$. Let f be Fréchet differentiable with respect to x at $(\bar{\sigma}, \bar{x})$. Furthermore, let $\bar{x}$ be a local solution of problem (50) with $\sigma = \bar{\sigma}$, let the set Q possess the conicity property at $F(\bar{\sigma}, \bar{x})$, and let the directional regularity condition (4) hold at $\bar{x}$ with respect to a direction $d \in \Sigma$.*

Then, for any mapping ρ: $\mathbf{R}_+ \to \Sigma$ such that $\rho(t) = o(t)$, the estimate

$$\limsup_{t \to 0+} \frac{v(\bar{\sigma} + td + \rho(t)) - v(\bar{\sigma})}{t^{1/2}} \le v_{1/2}(\bar{\sigma}, \bar{x}; d) \tag{59}$$

holds. Moreover, the $\xi = 0$ is a feasible point of problem (54), and in particular, (58) holds.

If $v_{1/2}(\bar{\sigma}, \bar{x}; d)$ is finite, the estimate (59) can be written in the form of inequality

$$v(\bar{\sigma} + td + \rho(t)) \le v(\bar{\sigma}) + v_{1/2}(\bar{\sigma}, \bar{x}; d)t^{1/2} + o(t^{1/2}) \tag{60}$$

for $t \ge 0$.

Proof. According to (51) and (55), from Assertion (ii) of Proposition 7.1, it follows that for any element $\xi \in \Xi_{1/2}(\bar{\sigma}, \bar{x}; d)$ (recall that $\Xi_{1/2}(\bar{\sigma}, \bar{x}; d) \neq \varnothing$; see (57)), there exists a mapping $r: \mathbf{R}_+ \to X$ such that $r(t) = o(t^{1/2})$ and

$$\bar{x} + t^{1/2}\xi + r(t) \in D(\bar{\sigma} + td + \rho(t))$$

for each $t \geq 0$ small enough. Then, for such t, it holds that

$$\begin{aligned} v(\bar{\sigma} + td + \rho(t)) - v(\bar{\sigma}) &\leq f(\bar{\sigma} + td + \rho(t), \bar{x} + t^{1/2}\xi + r(t)) - f(\bar{\sigma}, \bar{x}) \\ &= \left\langle \frac{\partial f}{\partial x}(\bar{\sigma}, \bar{x}), \xi \right\rangle t^{1/2} + o(t^{1/2}), \end{aligned}$$

and since ξ is an arbitrary element in $\Xi_{1/2}(\bar{\sigma}, \bar{x}; d)$, (56) implies (59). $\square$

Theorem 7.1 is closely related to Bonnans and Shapiro [6, Proposition 4.117], where, however, the upper bound was expressed in terms of the optimal value of the auxiliary problem

$$\begin{aligned} &\text{minimize} \quad \left\langle \frac{\partial f}{\partial x}(\bar{\sigma}, \bar{x}), \xi \right\rangle \\ &\text{subject to} \quad \xi \in \widetilde{\Xi}_{1/2}(\bar{\sigma}, \bar{x}; d), \end{aligned} \tag{61}$$

where, under the conicity assumption,

$$\begin{aligned} \widetilde{\Xi}_{1/2}(\bar{\sigma}, \bar{x}; d) = \Bigg\{ \xi \in C(\bar{\sigma}, \bar{x}) \;\Bigg|\; & \frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi, \xi] \\ & \in \mathrm{cl}\left(R_Q(F(\bar{\sigma}, \bar{x})) + \mathrm{span}\left\{ \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})\xi \right\} \right) - \mathrm{im}\, \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x}) \Bigg\} \end{aligned} \tag{62}$$

(see problem $(\mathscr{P}\mathscr{Q}^3)$ in Bonnans and Shapiro [6, p. 359]). It can be easily seen from (55), (62) that the set $\widetilde{\Xi}_{1/2}(\bar{\sigma}, \bar{x}; d)$ is, in general, smaller than $\Xi_{1/2}(\bar{\sigma}, \bar{x}; d)$, and in particular, the set appearing in the constraints defining $\widetilde{\Xi}_{1/2}(\bar{\sigma}, \bar{x}; d)$ is not necessarily closed. Note, however, that our auxiliary problem (54) corresponds to problem $(\mathscr{D}\mathscr{Q}_*^{3,R})$ in Bonnans and Shapiro [6, p. 361], and according to Bonnans and Shapiro [6, Lemma 4.118], the optimal values of problems (54) and (61) are actually the same. (These can be shown via the chain of auxiliary subproblems, by employing the duality argument.)

If $\Lambda(\bar{\sigma}, \bar{x}) \neq \varnothing$, then under the assumptions of Theorem 7.1, $v_{1/2}(\bar{\sigma}, \bar{x}; d) = 0$. This follows from (51), (55), and the following observation: according to (53), for any $\lambda \in \Lambda(\bar{\sigma}, \bar{x})$ and any $\xi \in ((\partial F/\partial x)(\bar{\sigma}, \bar{x}))^{-1}(T_Q(F(\bar{\sigma}, \bar{x})))$, it holds that

$$\begin{aligned} \left\langle \frac{\partial f}{\partial x}(\bar{\sigma}, \bar{x}), \xi \right\rangle &= -\left\langle \left(\frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x}) \right)^* \lambda, \xi \right\rangle \\ &= -\left\langle \lambda, \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})\xi \right\rangle \\ &\geq 0. \end{aligned}$$

In this case, estimate (59) does not make much sense, since the stronger estimates are available (see, e.g., Bonnans and Shapiro [6, Propositions 4.21, 4.22]).

On the other hand, if $\Lambda(\bar{\sigma}, \bar{x}) = \varnothing$, estimate (59) may be meaningful. For instance, if Q is a polyhedral set, then under the assumptions of Theorem 7.1, $v_{1/2}(\bar{\sigma}, \bar{x}; d) < 0$. Indeed, if we suppose that

$$-\frac{\partial f}{\partial x}(\bar{\sigma}, \bar{x}) \in \left(\left(\frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x}) \right)^{-1} (T_Q(F(\bar{\sigma}, \bar{x}))) \right)^{\circ},$$

then from the Farkas lemma (Bonnans and Shapiro [6, Proposition 2.42]) and (53), it follows that $\Lambda(\bar{\sigma}, \bar{x})$ cannot be empty. Thus, there exists $\xi \in ((\partial F/\partial x)(\bar{\sigma}, \bar{x}))^{-1}(T_Q(F(\bar{\sigma}, \bar{x})))$ such that $\langle (\partial f/\partial x)(\bar{\sigma}, \bar{x}), \xi \rangle < 0$, and according to the discussion above, $\theta\xi \in \Xi_{1/2}(\bar{\sigma}, \bar{x}; d)$, for all $\theta > 0$ small enough.

The next example demonstrates that under the assumptions of Theorem 7.1, $v_{1/2}(\bar{\sigma}, \bar{x}; d)$ can be equal to $-\infty$. In such cases, estimate (59) means that the rate of decrease of $v(\bar{\sigma} + td + \rho(t))$ as t grows from 0 is higher than that of $-t^{1/2}$.

EXAMPLE 7.1. Let $s=1$, $n=3$, $l=2$, $f(\sigma,x)=x_1$, $F(\sigma,x)=(F_1(\sigma,x),F_2(\sigma,x))=(x_1-x_3-x_3^3, x_1^2+x_2^2-x_3^2-\sigma)$, $Q=\{(0,y_2)\in\mathbf{R}^2 \mid y_2\le 0\}$. It is easy to see that $D(\bar\sigma)=\{0\}$, and in particular, $\bar x=0$ is a solution of problem (50) with $\sigma=\bar\sigma$. Moreover, $(\partial F_1/\partial\sigma)(\bar\sigma,\bar x)\ne 0$, $F_2(\bar\sigma,\bar x)=0$, and for any $d>0$ and any $\bar\xi\in\mathbf{R}^3$ such that $\bar\xi_1=\bar\xi_3$ it holds that

$$\begin{aligned}\frac{\partial F_1}{\partial\sigma}(\bar\sigma,\bar x)d+\frac{\partial F_1}{\partial x}(\bar\sigma,\bar x)\bar\xi &= \bar\xi_1-\bar\xi_3\\ &=0,\\ \frac{\partial F_2}{\partial\sigma}(\bar\sigma,\bar x)d+\frac{\partial F_2}{\partial x}(\bar\sigma,\bar x)\bar\xi &= -d\\ &<0.\end{aligned}$$

This means that Gollan's condition (see Bonnans and Shapiro [6, (4.21)]) holds, and, as mentioned in §1, the latter is equivalent to the directional regularity condition (4).

Take $d=1$. It is easy to see that $\Xi_{1/2}(\bar\sigma,\bar x;d)=\{\xi\in\mathbf{R}^3 \mid \xi_1-\xi_3=0,\ \xi_1^2+\xi_2^2-\xi_3^2\le 1\}$, and, e.g., the $\xi=(\theta,\pm d^{1/2},\theta)$ are feasible points of problem (54), (55) for each $\theta\in\mathbf{R}$. At the same time, for such ξ,

$$\begin{aligned}\left\langle\frac{\partial f}{\partial x}(\bar\sigma,\bar x),\xi\right\rangle &= \xi_1\\ &=\theta\to-\infty \quad \text{as } \theta\to-\infty.\end{aligned}$$

Note that in this example, $C_2(\bar\sigma,\bar x)=\{\xi\in\mathbf{R}^3 \mid \xi_1=\xi_3,\ \xi_2=0,\ \xi_1\le 0\}\ne\{0\}$.

For any $\varepsilon>0$ and any $\sigma\in\Sigma$, point $x\in X$ is called *ε-solution* of problem (50) if $x\in D(\sigma)$ and $f(\sigma,x)\le v(\sigma)+\varepsilon$.

PROPOSITION 7.2. *Under the assumptions of Theorem 7.1, for any sequences $\{t_k\}\subset\mathbf{R}_+\backslash\{0\}$, $\{\rho^k\}\subset\Sigma$, and $\{x^k\}\subset X$ such that $\{t_k\}\to 0$, $\rho^k=o(t_k)$, $\{x^k\}\to\bar x$, and for each k the point x^k is an $o(t_k^{1/2})$-solution of problem (50) with $\sigma=\bar\sigma+t_kd+\rho^k$, any accumulation point of the sequence $\{(x^k-\bar x)/t_k^{1/2}\}$ is a solution of problem (54).*

PROOF. Without loss of generality, let us suppose that the entire sequence $\{(x^k-\bar x)/t_k^{1/2}\}$ converges to some $\bar\xi\in X$. Then $x^k=\bar x+t_k^{1/2}\bar\xi+o(t_k^{1/2})$, and since $x^k\in D(\bar\sigma+t_kd+\rho^k)$ for each k, from Assertion (i) of Proposition 7.1, it follows that $\bar\xi\in((\partial F/\partial x)(\bar\sigma,\bar x))^{-1}(T_Q(F(\bar\sigma,\bar x)))$, and (42) holds.

Furthermore, by the estimate (59), we obtain

$$\begin{aligned}\limsup_{k\to\infty}\frac{f(\bar\sigma+t_kd+\rho^k,x^k)-f(\bar\sigma,\bar x)}{t_k^{1/2}} &= \limsup_{k\to\infty}\frac{v(\bar\sigma+t_kd+\rho^k)-v(\bar\sigma)}{t_k^{1/2}}\\ &\le v_{1/2}(\bar\sigma,\bar x;d).\end{aligned}$$

On the other hand,

$$f(\bar\sigma+t_kd+\rho^k,x^k)=f(\bar\sigma,\bar x)+\left\langle\frac{\partial f}{\partial x}(\bar\sigma,\bar x),\bar\xi\right\rangle t_k^{1/2}+o\big(t_k^{1/2}\big),$$

and hence

$$\left\langle\frac{\partial f}{\partial x}(\bar\sigma,\bar x),\bar\xi\right\rangle\le v_{1/2}(\bar\sigma,\bar x;d).$$

Thus, by (55) and (58), we obtain that $\bar\xi\in\Xi_{1/2}(\bar\sigma,\bar x;d)$, and moreover, according to (56), $\bar\xi$ is a solution of problem (54). □

Throughout the rest of this section, we assume for simplicity that $\dim X<\infty$. From (51), (52), (55), and (56), it then easily follows that if

$$C_2(\bar\sigma,\bar x)=\{0\}, \tag{63}$$

then $v_{1/2}(\bar\sigma,\bar x;d)>-\infty$ (compare with Example 7.1).

In Arutyunov and Izmailov [2], it was shown that condition (63) implies the standard second-order sufficient optimality condition (e.g., Bonnans and Shapiro [6, (3.137)]), and moreover, if $\Lambda(\bar\sigma,\bar x)=\varnothing$, then the two conditions are equivalent.

Theorem 7.2. *Under the assumptions of Theorem 7.1, let* $\dim X < \infty$, *and let* (63) *hold. Then, for any mapping* ρ: $\mathbf{R}_+ \to \Sigma$ *such that* $\rho(t) = o(t)$, *equality*

$$v(\bar{\sigma} + td + \rho(t)) = v(\bar{\sigma}) + v_{1/2}(\bar{\sigma}, \bar{x}; d)t^{1/2} + o(t^{1/2}), \tag{64}$$

holds for $t \geq 0$, *and moreover, for any solution* $\bar{\xi}$ *of problem* (54), *problem* (50) *with* $\sigma = \bar{\sigma} + td + \rho(t)$ *has an* $o(t^{1/2})$*-solution of the form* $\bar{x} + t^{1/2}\bar{\xi} + o(t^{1/2})$.

Proof. Since the upper estimate (60) is already established in Theorem 7.1 (recall that under the assumption (63), it holds that $v_{1/2}(\bar{\sigma}, \bar{x}; d) > -\infty$) to establish (64), it remains to derive the lower estimate

$$v(\bar{\sigma} + td + \rho(t)) \geq v(\bar{\sigma}) + v_{1/2}(\bar{\sigma}, \bar{x}; d)t^{1/2} + o(t^{1/2}),$$

for $t \geq 0$. We argue by a contradiction. Suppose that there exist a sequence $\{t_k\} \subset \mathbf{R}_+ \backslash \{0\}$ and $\varepsilon > 0$ such that $\{t_k\} \to 0$ and $\forall\, k$,

$$\frac{v(\bar{\sigma} + t_k d + \rho(t_k)) - v(\bar{\sigma})}{t_k^{1/2}} \leq v_{1/2}(\bar{\sigma}, \bar{x}; d) - \varepsilon. \tag{65}$$

Consider an arbitrary sequence $\{x^k\} \subset B_\delta(\bar{x})$ such that $x^k \in D(\bar{\sigma} + t_k d + \rho(t_k))$ and $f(\bar{\sigma} + t_k d + \rho(t_k), x^k) = v(\bar{\sigma} + t_k d + \rho(t_k))\ \forall\, k$ (evidently, such sequence exists since $\dim X < \infty$, and hence $B_\delta(\bar{x})$ is compact).

For each k set $\xi^k = (x^k - \bar{x})/t_k^{1/2}$. Recall that (63) implies the second-order sufficient optimality condition. Thus, from Bonnans and Shapiro [6, Theorem 4.53], it follows that the sequence $\{\xi^k\}$ is bounded, and without loss of generality, we may suppose that this sequence converges to some $\bar{\xi} \in X$, and according to Proposition 7.2, $\bar{\xi}$ must be a solution of problem (54).

Furthermore,

$$\begin{aligned} v(\bar{\sigma} + t_k d + \rho(t_k)) - v(\bar{\sigma}) &= f(\bar{\sigma} + t_k d + \rho(t_k), x^k) - f(\bar{\sigma}, \bar{x}) \\ &= \left\langle \frac{\partial f}{\partial x}(\bar{\sigma}, \bar{x}), \bar{\xi} \right\rangle t_k^{1/2} + o\big(t_k^{1/2}\big), \end{aligned}$$

and since $\bar{\xi}$ is a solution of problem (54), it follows that

$$\begin{aligned} \lim_{k\to\infty} \frac{v(\bar{\sigma} + t_k d + \rho(t_k)) - v(\bar{\sigma})}{t_k^{1/2}} &= \left\langle \frac{\partial f}{\partial x}(\bar{\sigma}, \bar{x}), \bar{\xi} \right\rangle \\ &= v_{1/2}(\bar{\sigma}, \bar{x}; d). \end{aligned}$$

This contradicts (65), and this contradiction completes the proof of (64).

Let now $\bar{\xi}$ be an arbitrary solution of problem (54). By Assertion (ii) of Proposition 7.1, there exists a mapping r: $\mathbf{R}_+ \to X$ such that $r(t) = o(t^{1/2})$ and

$$\bar{x} + t^{1/2}\bar{\xi} + r(t) \in D(\bar{\sigma} + td + \rho(t)),$$

for all $t \geq 0$ small enough. Then, according to (64), for such t, we obtain

$$\begin{aligned} f(\bar{\sigma} + td + \rho(t), \bar{x} + t^{1/2}\bar{\xi} + r(t)) &= f(\bar{\sigma}, \bar{x}) + \left\langle \frac{\partial f}{\partial x}(\bar{\sigma}, \bar{x}), \bar{\xi} \right\rangle t^{1/2} + o(t^{1/2}) \\ &= v(\bar{\sigma}) + v_{1/2}(\bar{\sigma}, \bar{x}; d)t^{1/2} + o(t^{1/2}) \\ &\leq v(\bar{\sigma} + td + \rho(t)) + o(t^{1/2}), \end{aligned}$$

i.e., $\bar{x} + t^{1/2}\xi + r(t)$ is an $o(t^{1/2})$-solution of problem (50) with $\sigma = \bar{\sigma} + td + \rho(t)$. $\square$

Proposition 7.2 and Theorem 7.2 are strongly related to Bonnans and Shapiro [6, Theorem 4.120]. Note, however, that in the latter reference, the directional expansion of the optimal value function and the result similar to that of Proposition 7.2 are derived in terms of the auxiliary problem (54) rather than (61) used for the upper bound. At the same time, our analysis relies solely on auxiliary problem (54). Finally, Bonnans and Shapiro [6, Theorem 4.120] does not contain the last assertion of Theorem 7.2, which is an important characterization of solution sensitivity. Recall that the proof of this assertion relies on Proposition 7.1, which we believe to be new.

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