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Abnormal equality-constrained optimization problems: sensitivity theory

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Abstract. For the equality-constrained optimization problem, we consider the case when the customary regularity of constraints can be violated. Under the assumptions substantially weaker than those previously used in the literature, we develop a reasonably complete local sensitivity theory for this class of problems, including upper and lower bounds for the rate of change of the optimal value function subject to parametric perturbations, as well as the estimates and the description of asymptotic behavior of solutions of the perturbed problems.

Key words. sensitivity analysis – parametric optimization – equality-constrained problem – regularity – abnormal point – second-order sufficient conditions

1. Introduction. The normal case

We present the sensitivity analysis for constrained optimization problems under the CQ-type conditions substantially weaker than those previously used in the literature in this context. Our approach is within the framework of directional perturbations, in the spirit of [15], [7], [8]. Here, we consider a purely equality-constrained optimization problem only, and we develop a reasonably complete local sensitivity theory for this class of problems. Our results include the upper and lower bounds for the rate of change of the optimal value function subject to parametric perturbations, as well as the estimates and the description of asymptotic behavior of solutions of the perturbed problems. The case of mixed constraints (or more general abstract constraints) will be discussed by the authors elsewhere. The reason for this is twofold. First, the case of pure equality constraints certainly deserves a special consideration, as this is a classical setting which goes back to Lagrange and the foundations of nonlinear analysis and optimization. More importantly, pure equality constraints have strong specific properties which are not preserved in the general case, and which cannot be established as particular cases of the general theory. Roughly speaking, as will be shown below, some sensitivity properties of the abnormal equality-constrained problems turn out to be much similar to those available for the normal case. For general constraints, this is not at all the case (see the forthcoming paper [5]).

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Let Σ, X and Y be (real) Banach spaces, and consider the problem

$$\begin{array}{l} \text{minimize } f(\sigma, x) \\ \text{subject to } x \in D(\sigma) \end{array} \tag{1}$$

with

$$D(\sigma) = \{x \in X \mid F(\sigma, x) = 0\},$$

where $\sigma \in \Sigma$ is a parameter, $f : \Sigma \times X \to \mathbf{R}$ is a smooth function, and $F : \Sigma \times X \to Y$ is a smooth mapping.

Let $\bar{\sigma} \in \Sigma$ be a fixed (base) parameter value, and $\bar{x} \in X$ be a local solution of the unperturbed problem

$$\begin{array}{l} \text{minimize } f(\bar{\sigma}, x) \\ \text{subject to } x \in D(\bar{\sigma}). \end{array} \tag{2}$$

Then there exists a closed ball $B = \{x \in X \mid \|x - \bar{x}\| \le r\}$ with some $r > 0$, such that $\bar{x}$ is a global solution of the unperturbed problem restricted to B:

$$\begin{array}{l} \text{minimize } f(\bar{\sigma}, x) \\ \text{subject to } x \in D(\bar{\sigma}) \cap B. \end{array} \tag{3}$$

Consider the perturbed problem restricted to B:

$$\begin{array}{l} \text{minimize } f(\sigma, x) \\ \text{subject to } x \in D(\sigma) \cap B. \end{array} \tag{4}$$

Define the (*local*) *optimal value function* of problem (1) as the optimal value function of (4):

$$\omega(\sigma) = \inf_{x \in D(\sigma) \cap B} f(\sigma, x), \quad \sigma \in \Sigma.$$

With this definition, $\omega(\bar{\sigma}) = f(\bar{\sigma}, \bar{x})$, and if X is finite-dimensional (more precisely, if B is compact), then, as is well known, ω is lower semicontinuous at $\bar{\sigma}$. On the other hand, when X is infinite-dimensional, ω is not necessarily lower semicontinuous. A nice example of such behavior is presented below. Note that, in this example, the violation of lower semicontinuity of ω cannot be avoided by choosing r small enough.

Example 1. Let $X = l_2$, $Y = \mathbf{R}^2$, $\Sigma = \mathbf{R}$, $f(\sigma, x) = -\|x - e^1\|^2$, $F_1(\sigma, x) = \|x\|^2 - 1$, $F_2(\sigma, x) = (x_1 - 1)(\sum_{j=2}^{\infty} x_j^2/j - \sigma)$, $\bar{\sigma} = 0$, $\bar{x} = e^1$, where $e^1 = (1, 0, \ldots)$. It can be easily seen that $\bar{x}$ is an isolated point of $D(\bar{\sigma})$, and if $r < 2$, then $\bar{x}$ is a global solution of (3). For such r, $\omega(\bar{\sigma}) = \omega_r(\bar{\sigma}) = 0$.

At the same time, it can be directly verified that for every $r < \sqrt{2}$ and every $i = 2, 3, \ldots$, the system of equations

$$\|x\| = 1, \quad \|x - e^1\| = r \tag{5}$$

has a solution of the form $x^j = (\sqrt{1-\theta}, 0, \ldots, 0, \sqrt{\theta}, 0, \ldots)$, where $\sqrt{\theta}$ is on j-th place, and $\theta = \theta(r)$ is the unique solution of the equation

$$(\sqrt{1-\theta} - 1)^2 + \theta = r^2.$$

Furthermore, the solution set of (5) is connected, and if $\sigma \leq \theta/2$, then $F_2(\sigma, x^2) = (\sqrt{1-\theta} - 1)(\theta/2 - \sigma) \leq 0$, while $F_2(\sigma, x^j) = (\sqrt{1-\theta} - 1)(\theta/j - \sigma) \geq 0$ for all j large enough. We conclude that (5) has a solution $x(\sigma)$ such that $F_2(\sigma, x(\sigma)) = 0$, and in particular, $x(\sigma) \in D(\sigma)$. Obviously, $x(\sigma)$ is a global solution of (4), and $\omega(\sigma) = f(\sigma, x(\sigma)) = -r^2 \; \forall\, \sigma \in (0, \theta(r)/2)$.

Moreover, with a fixed r, it can even hold that $\omega(\sigma) = -\infty$ for $\sigma \in \Sigma$ arbitrary close to $\bar{\sigma}$. Indeed, if there exists a smooth function $\varphi : X \to \mathbf{R}$ unbounded from below on B, then take $\Sigma = \mathbf{R}$, $f(\sigma, x) = \sigma\varphi(x)$ and $D(\sigma) = X$. Then $\omega(0) = 0$, while $\omega(\sigma) = -\infty \; \forall\, \sigma > 0$.

On the other hand, even in the case of finite-dimensional X, further (and in particular, quantitative) analysis of the properties of the optimal value function and solutions of the perturbed problems requires further assumptions.

For every $\sigma \in \Sigma$, denote by $S(\sigma)$ the solution set of (4), and define the Lagrangian function

$$L(\sigma, x, \lambda) = f(\sigma, x) + \langle \lambda, F(\sigma, x) \rangle, \quad \sigma \in \Sigma, \; x \in X, \; \lambda \in Y^*,$$

where Y^* is the (topologically) dual space of Y, and $\langle \cdot, \cdot \rangle$ stands for duality pairing. Let

$$\Lambda(\bar{\sigma}, \bar{x}) = \left\{ \lambda \in Y^* \;\middle|\; \frac{\partial L}{\partial x}(\bar{\sigma}, \bar{x}, \lambda) = 0 \right\}$$

be the set of (normal) Lagrange multipliers associated with $\bar{x}$.

In our setting, we say that the point $\bar{x}$ is *normal*, if

$$\operatorname{im} \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x}) = Y, \tag{6}$$

where by $\operatorname{im} A$ we denote the image space (range) of a linear operator A, and *abnormal* otherwise. Under the normality assumption (which is the natural constraint qualification in this context), the following sensitivity results are well known:

- From the standard implicit function theorem, it follows that ω is upper semicontinuous at $\bar{\sigma}$. If X is finite-dimensional, then ω is continuous at $\bar{\sigma}$, $S(\sigma) \neq \emptyset$ for all $\sigma \in \Sigma$ close enough to $\bar{\sigma}$, and

$$\sup_{x \in S(\sigma)} \operatorname{dist}(x, S(\bar{\sigma})) \to 0 \text{ as } \sigma \to \bar{\sigma}, \tag{7}$$

  where $\operatorname{dist}(x, S) = \inf_{z \in S} \|x - z\|$ stands for the distance from $x \in X$ to $S \subset X$. In particular, if $\bar{x}$ is a strict local minimizer in (2), and B is chosen in such a way that $\bar{x}$ is a unique solution of (3), then

$$\sup_{x \in S(\sigma)} \|x - \bar{x}\| \to 0 \text{ as } \sigma \to \bar{\sigma}. \tag{8}$$

  This is the stability theorem; see, e.g., [15, Theorem 3.1 and Corollary 1].
- For $\sigma \in \Sigma$, the following linear upper bound holds:

$$\omega(\sigma) \leq \omega(\bar{\sigma}) + O(\|\sigma - \bar{\sigma}\|); \tag{9}$$

  see, e.g., [15, Proposition 3.1].

– For a given direction $d \in \Sigma$, bound (9) can be quantitatively sharpened in the following way: for $t \geq 0$

$$\omega(\bar{\sigma} + td) \leq \omega(\bar{\sigma}) + \left\langle \frac{\partial L}{\partial \sigma}(\bar{\sigma},\, \bar{x},\, \bar{\lambda}),\, d \right\rangle t + o(t) \tag{10}$$

with $\bar{\lambda} \in \Lambda(\bar{\sigma},\, \bar{x})$; see [15, Theorem 7.1], [7, Proposition 4.3].

Recall that in the normal case, the set of Lagrange multipliers is necessarily a singleton.

If $\bar{x}$ is a unique solution of (3), then the directional upper bound (10) is exact, and this fact can be expressed in the form of the equality

$$\omega'(\bar{\sigma};\, d) = \left\langle \frac{\partial L}{\partial \sigma}(\bar{\sigma},\, \bar{x},\, \bar{\lambda}),\, d \right\rangle \tag{11}$$

for the directional derivative of ω at $\bar{\sigma}$ with respect to any direction $d \in \Sigma$; see [14], [7, Theorem 4.5]. In particular, ω is Gâteaux differentiable at $\bar{\sigma}$.

However, in more general (including abnormal) settings, further assumptions are needed in order to obtain sharp lower bounds on ω and bounds on solutions of the perturbed problems. The most popular and natural in this context seems to be the *quadratic growth condition* (QGC) which consists of saying that there exists $\gamma > 0$ and a neighborhood V of $\bar{x}$ such that

$$f(\bar{\sigma},\, x) \geq f(\bar{\sigma},\, \bar{x}) + \gamma\|x - \bar{x}\|^2 \quad \forall\, x \in D(\bar{\sigma}) \cap V,$$

or some second order sufficient conditions guaranteeing QGC. Note that in the normal case, the natural second order sufficient condition is the following: there exists $\gamma > 0$ such that

$$\frac{\partial^2 L}{\partial x^2}(\bar{\sigma},\, \bar{x},\, \bar{\lambda})[\xi,\, \xi] \geq \gamma\|\xi\|^2 \quad \forall\, \xi \in \ker \frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x}),$$

where by $\ker A$ we denote the null space of a linear operator A, and $[\cdot,\, \cdot]$ stands for the arguments of a bilinear form (or, more generally, bilinear mapping). This condition actually leads to the so-called strong stability for the Lagrange optimality system [7, section 5], when the most complete answers to sensitivity questions are available via the standard implicit function theorem. For the abnormal case, the situation is by far more complex.

This paper contains further development of the preliminary results published in [4]. Here, we extend these results to the infinite-dimensional setting, and we complete our theory with numerous important additional facts (in particular, for the case when $\bar{x}$ is an isolated feasible point of the unperturbed problem). We emphasize that our results would be completely meaningful even in the finite-dimensional setting. In particular, we discuss application of our theory to the so-called chain problem [7, section 8], which is finite-dimensional. As an important example of possible infinite-dimensional application, we mention optimal control problems with phase constraints. This application requires an extensive discussion and will be the subject of our future work.

2. Preliminaries

We now consider the case when possibly the point $\bar{x}$ is abnormal, that is, normality condition (6) is not assumed to be *a priori* satisfied.

2.1. Some facts from nonlinear analysis

In order to proceed with sensitivity analysis, we need the following two implicit function (stability) theorems and the error bound theorem relevant in the context of abnormal problems. Let F be twice continuously differentiable near $(\bar{\sigma}, \bar{x})$ and three times differentiable at $(\bar{\sigma}, \bar{x})$ with respect to x. In most of the results presented below, these smoothness assumptions are extraneous, but we impose them throughout the paper just to be not distracted by non-essential details.

Let $\bar{x} \in D(\bar{\sigma})$ (in this section, $\bar{x}$ is not necessarily a local solution of (2)). Assume that $Y_1 = \operatorname{im} \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})$ is closed and has a closed complementary subspace Y_2 in Y. Let P be the projector onto Y_2 parallel to Y_1 in Y (P is continuous, by necessity). Define the mapping

$$\Phi : X \to Y, \quad \Phi(\xi) = \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})\xi + \frac{1}{2} P \frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi, \xi].$$

Note that

$$\Phi'(h)\xi = \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})\xi + P \frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[h, \xi] \quad \forall\, h, \xi \in X.$$

If Y is a Hilbert space, one can always take the orthogonal complement $Y_1^{\perp}$ as Y_2. With this choice, P is the orthogonal projector onto $Y_1^{\perp}$ in Y. Though, in any case, the 2-regularity property does not actually depend on the specific choice of Y_2.

Definition 1. *The mapping F is said to be 2-regular in x at the point $(\bar{\sigma}, \bar{x})$ with respect to a direction $\xi \in X$ if*

$$\operatorname{im} \Phi'(\xi) = Y.$$

The term "with respect to a direction" is justified by the observation that 2-regularity with respect to ξ implies 2-regularity with respect to $t\xi$ for every $t \neq 0$.

Define the cone

$$\begin{aligned} T(\bar{\sigma}, \bar{x}) &= \{\xi \in X \mid \Phi(\xi) = 0\} \\ &= \left\{ \xi \in \ker \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x}) \;\middle|\; \frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi, \xi] \in Y_1 \right\}. \end{aligned}$$

Theorem 1. *Assume that the subspace Y_1 is closed and has a closed complementary subspace Y_2 in Y, and the following hypothesis is satisfied:*

(H1) *There exists $h \in T(\bar{\sigma}, \bar{x})$ such that F is 2-regular in x at $(\bar{\sigma}, \bar{x})$ with respect to h.*

Then for all $\sigma \in \Sigma$ *close enough to* $\bar{\sigma}$ *it holds that* $D(\sigma) \neq \emptyset$, *and*

$$\operatorname{dist}(\bar{x},\ D(\sigma)) = O\left(\|\sigma - \bar{\sigma}\| + \left\| P\frac{\partial F}{\partial \sigma}(\bar{\sigma},\ \bar{x})(\sigma - \bar{\sigma})\right\|^{1/2}\right). \tag{12}$$

This theorem was proved in [9] basing on the covering theorem obtained in [6]. For an extensive discussion of 2-regularity and its applications in nonlinear analysis and optimization see [12], [13], [3], [10], [11] and references therein.

Another concept appropriate for treating abnormal problems applies to the case when $\operatorname{codim} Y_1 < \infty$. Define the cone $\mathcal{F}(\bar{\sigma},\ \bar{x})$ in Y^* consisting of all elements $\lambda \in Y_1^{\perp} \setminus \{0\}$ possessing the following property: there exists a linear subspace $\Pi = \Pi(\lambda) \subset X$ such that

$$\Pi \subset \ker \frac{\partial F}{\partial x}(\bar{\sigma},\ \bar{x}), \quad \operatorname{codim} \Pi \leq \operatorname{codim} Y_1,$$

$$\left\langle \lambda,\ \frac{\partial^2 F}{\partial x^2}(\bar{\sigma},\ \bar{x})[\xi,\ \xi]\right\rangle \geq 0 \quad \forall\, \xi \in \Pi.$$

Here $Y_1^{\perp}$ stands for the set of all $\lambda \in Y^*$ such that $\langle \lambda,\ y\rangle = 0\ \forall\, y \in Y_1$, and $\operatorname{codim} \Pi$ is computed with respect to $\ker \frac{\partial F}{\partial x}(\bar{\sigma},\ \bar{x})$ (not to the entire X!).

Theorem 2. *Assume that the subspace* Y_1 *is closed, and the following hypothesis is satisfied:*

(H2) *There exists* $h \in \ker \frac{\partial F}{\partial x}(\bar{\sigma},\ \bar{x})$ *such that*

$$\left\langle \lambda,\ \frac{\partial^2 F}{\partial x^2}(\bar{\sigma},\ \bar{x})[h,\ h]\right\rangle < 0 \quad \forall\, \lambda \in \mathcal{F}(\bar{\sigma},\ \bar{x}),$$

and either $\dim Y < \infty$, *or* $\operatorname{codim} Y_1 < \infty$ *and* $\ker \frac{\partial F}{\partial x}(\bar{\sigma},\ \bar{x})$ *has a closed complementary subspace in* X.

Then for all $\sigma \in \Sigma$ *close enough to* $\bar{\sigma}$ *it holds that* $D(\sigma) \neq \emptyset$, *and* (12) *is satisfied.*

Theorem 2 was proved in [1]. (We point out that, as a result of a misprint, assumption $\operatorname{codim} \ker \frac{\partial F}{\partial x}(\bar{\sigma},\ \bar{x}) < \infty$ appeared in [1, Remark 1] instead of the correct assumption consisting of saying that $\ker \frac{\partial F}{\partial x}(\bar{\sigma},\ \bar{x})$ has a closed complementary subspace in X.)

Relation between hypotheses (H1) and (H2) is not completely clarified yet. It seems quite likely that (H2) actually implies (H1), but by now, this conjecture is proved under the additional assumptions only [3, section 4.4]. That is why we consider Theorems 1 and 2 as complementary with respect to each other.

We proceed with the error bound for the zero set of a (strongly) 2-regular mapping, derived in [6].

Definition 2. *The mapping* F *is said to be* 2-regular in x at the point $(\bar{\sigma},\ \bar{x})$ *if it is 2-regular at this point with respect to every direction* $\xi \in T(\bar{\sigma},\ \bar{x}) \setminus \{0\}$.

Definition 3. *The mapping F is said to be* strongly 2-regular in x at the point $(\bar{\sigma}, \bar{x})$ *if there exists $\nu > 0$ such that*

$$\sup_{\substack{\xi \in T_\nu(\bar{\sigma}, \bar{x}), \\ \|\xi\|=1}} \sup_{\substack{y \in Y, \\ \|y\|=1}} \operatorname{dist}(0, (\Phi'(\xi))^{-1}(y)) < \infty,$$

where

$$T_\nu(\bar{\sigma}, \bar{x}) = \left\{ \xi \in X \;\middle|\; \left\| \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})\xi \right\| \leq \nu, \; \left\| P \frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi, \xi] \right\| \leq \nu \right\}.$$

Note that $T_0(\bar{\sigma}, \bar{x}) = T(\bar{\sigma}, \bar{x})$, and if $\dim X < \infty$, then strong 2-regularity is equivalent to 2-regularity. Some criteria of strong 2-regularity were derived in [2].

Theorem 3. *Assume that the subspace Y_1 is closed and has a closed complementary subspace Y_2 in Y, and F is strongly 2-regular in x at $(\bar{\sigma}, \bar{x})$.*

Then for $x \in X$, $x \neq \bar{x}$, it holds that

$$\operatorname{dist}(x, D(\bar{\sigma})) = O(\|(I - P)F(\bar{\sigma}, x)\| + \|PF(\bar{\sigma}, x)\| / \|x - \bar{x}\|), \tag{13}$$

where $I : Y \to Y$ is the identity mapping.

It is easy to see that hypotheses (H1) and (H2), as well as the strong 2-regularity condition, are weaker requirements than normality condition (6). Moreover, in the normal case, F is 2-regular in x at $(\bar{\sigma}, \bar{x})$ with respect to any direction $h \in X$, including $h = 0$, and both Theorems 1 and 2 reduce to the standard implicit function theorem, while Theorem 3 reduces to the standard (linear) error bound (see, e.g., [7, Proposition 3.3]).

Now we consider the directional perturbations for a given direction $d \in \Sigma$, that is, perturbations along the ray $\sigma(t) = \bar{\sigma} + td$, $t \geq 0$. Consider the arc of the form $x(t) = \bar{x} + t^{1/2}\xi + o(t^{1/2})$, $\xi \in X$. The form of the arc is suggested by the estimate (12). If $x(t) \in D(\sigma(t))$ for all $t \geq 0$ sufficiently small, then

$$\begin{aligned} 0 &= F(\bar{\sigma} + td, x(t)) \\ &= t^{1/2} \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})\xi + o(t^{1/2}), \end{aligned}$$

$$\begin{aligned} 0 &= PF(\bar{\sigma} + td, x(t)) \\ &= tP\left(\frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d + \frac{1}{2} \frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi, \xi] \right) + o(t), \end{aligned}$$

and it follows that necessarily $\xi \in T(\bar{\sigma}, \bar{x}; d)$, where

$$T(\bar{\sigma}, \bar{x}; d) = \left\{ \xi \in X \;\middle|\; \Phi(\xi) = -P \frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d \right\} \tag{14}$$

(clearly $T(\bar{\sigma}, \bar{x}; 0) = T(\bar{\sigma}, \bar{x})$). Under the additional assumption of 2-regularity of F in x at $(\bar{\sigma}, \bar{x})$ with respect to ξ, the converse implication is established in the following theorem.

Theorem 4. *Assume that the subspace Y_1 is closed and has a closed complementary subspace Y_2 in Y, and for a given $d \in \Sigma$, F is 2-regular in x at $(\bar{\sigma}, \bar{x})$ with respect to $\xi \in T(\bar{\sigma}, \bar{x}; d)$.*

Then there exist a neighborhood $\mathcal{U}$ of d and a mapping $\chi : \mathcal{U} \to X$ such that:

(i) *For $\zeta \in \mathcal{U}$*

$$\chi(\zeta) \in T(\bar{\sigma}, \bar{x}; \zeta), \quad \chi(\zeta) \to \xi \text{ as } \zeta \to d.$$

(ii) *There exists a mapping $r : \mathbf{R}_+ \times \mathcal{U} \to X$ such that for $t \geq 0$ and $\zeta \in \mathcal{U}$*

$$\bar{x} + t^{1/2}\chi(\zeta) + r(t, \zeta) \in D(\bar{\sigma} + t\zeta),$$

$$\|r(t, \zeta)\| = O(t) \text{ uniformly in } \zeta \in \mathcal{U}. \tag{15}$$

Proof. For a given $\zeta \in \Sigma$ consider the equation

$$\Phi(\chi) = -P\frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})\zeta$$

with respect to $\chi \in X$. Assertion (i) follows by the application of the standard implicit function theorem to this equation at $\chi = \xi$ with $\zeta = d$.

Now consider the mapping $G : \mathbf{R}_+ \times \mathcal{U} \times X \to Y$ defined as follows: for $\zeta \in \mathcal{U}$, $r \in X$ and $t > 0$

$$G(t, \zeta, x) = t^{-1/2}(I - P)F(\bar{\sigma} + t\zeta, \bar{x} + t^{1/2}(\chi(\zeta) + x)) \\ + t^{-1}PF(\bar{\sigma} + t\zeta, \bar{x} + t^{1/2}(\chi(\zeta) + x)),$$

while for $t = 0$ we define G by the equality

$$G(0, \zeta, x) = \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})(\chi(\zeta) + x) \\ + P\left(\frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[\chi(\zeta), x] + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[x, x]\right).$$

It can be directly verified that if we take $(t, \zeta) \in \mathbf{R}_+ \times \mathcal{U}$ as a parameter, the standard implicit function theorem can be applied to G at $(0, d, 0)$, and this results in (ii). □

Concerning this "directional stability" theorem, it is worth mentioning that the related results using the so-called directional regularity condition are available in the literature; see [7, Proposition 3.4], [8, Theorem 4.9 and Lemma 4.10]. However, for pure equality constraints, directional regularity coincides with normality, while the assumptions of Theorem 4 are substantially weaker.

Lemma 1. *Assume that the subspace Y_1 is closed and has a closed complementary subspace Y_2 in Y, and F is 2-regular in x at $(\bar{\sigma}, \bar{x})$ with respect to at least one $h \in T(\bar{\sigma}, \bar{x})$.*

Then for every $d \in \Sigma$ there exists $\xi \in T(\bar{\sigma}, \bar{x}; d)$ such that F is 2-regular in x at $(\bar{\sigma}, \bar{x})$ with respect to ξ.

Proof. For a given $d \in \Sigma$ and a real number t consider the equation

$$\Phi(\chi) = -tP\frac{\partial F}{\partial \sigma}(\bar{\sigma},\ \bar{x})\zeta$$

with respect to $\chi \in X$. Application of the standard implicit function theorem to this equation at $\chi = h$ with $t = 0$ results in the following: for every t close enough to zero, this equation has a solution $\chi(t)$ such that $\chi(t) \to h$ as $t \to 0$. Hence, if $t > 0$ is small enough, then F is 2-regular in x at $(\bar{\sigma},\ \bar{x})$ with respect to $\chi(t)$. We complete the proof by setting $\xi = t^{-1/2}\chi(t)$. □

2.2. *Second-order sufficient optimality conditions*

We next discuss some facts concerning unperturbed optimization problem (2). If the local solution $\bar{x}$ of this problem is an abnormal point, then $\Lambda(\bar{\sigma},\ \bar{x})$ can be empty. At the same time, necessarily $\Lambda_0(\bar{\sigma},\ \bar{x}) \neq \emptyset$, where

$$\Lambda_0(\bar{\sigma},\ \bar{x}) = \left\{(\lambda_0,\ \lambda) \in (\mathbf{R}_+ \times Y^*)\ \middle|\ \frac{\partial L_0}{\partial x}(\bar{\sigma},\ \bar{x},\ \lambda_0,\ \lambda) = 0,\ (\lambda_0,\ \lambda) \neq 0\right\}$$

is the set of generalized Lagrange multipliers associated with $\bar{x}$, and

$$L_0(\sigma,\ x,\ \lambda_0,\ \lambda) = \lambda_0 f(\sigma,\ x) + \langle\lambda,\ F(\sigma,\ x)\rangle, \quad \sigma \in \Sigma,\ x \in X,\ \lambda_0 \in \mathbf{R},\ \lambda \in Y^*,$$

is the generalized Lagrangian function (recall that Y_1 is supposed to be closed).

In the rest of this section, we are concerned with second-order sufficient optimality conditions of the form

$$\sup_{\substack{(\lambda_0,\lambda)\in\Lambda_0(\bar{\sigma},\bar{x}),\\ \lambda_0+\|\lambda\|=1}} \frac{\partial^2 L_0}{\partial x^2}(\bar{\sigma},\ \bar{x},\ \lambda_0,\ \lambda)[\xi,\ \xi] \geq \gamma\|\xi\|^2 \quad \forall\,\xi \in K \tag{16}$$

and

$$\sup_{\substack{\lambda\in\Lambda(\bar{\sigma},\bar{x}),\\ \|\lambda\|\leq M}} \frac{\partial^2 L}{\partial x^2}(\bar{\sigma},\ \bar{x},\ \lambda)[\xi,\ \xi] \geq \gamma\|\xi\|^2 \quad \forall\,\xi \in K, \tag{17}$$

where K is some cone in X, $M > 0$ and $\gamma > 0$ are some real numbers.

Definition 4. *The* weak second-order sufficient condition *(WSOSC) is said to hold at $\bar{x}$ if there exists $\gamma > 0$ such that* (16) *holds with* $K = \ker \frac{\partial F}{\partial x}(\bar{\sigma},\ \bar{x})$.

If $\dim X < \infty$, then WSOSC can be written in the form

$$\forall\,\xi \in \ker \frac{\partial F}{\partial x}(\bar{\sigma},\ \bar{x}),\ \xi \neq 0,$$

$$\exists\,(\lambda_0,\ \lambda) \in \Lambda_0(\bar{\sigma},\ \bar{x}) \text{ such that } \frac{\partial^2 L_0}{\partial x^2}(\bar{\sigma},\ \bar{x},\ \lambda_0,\ \lambda)[\xi,\ \xi] > 0.$$

Lemma 2. *WSOSC is equivalent to either of the following two conditions:*

(a) *There exist* $\nu > 0$ *and* $\gamma > 0$ *such that* (16) *holds with* $K = \{\xi \in X \mid \|\frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})\xi\| \leq \nu\|\xi\|\}$.

(b) *There exist* $\nu > 0$ *and* $\gamma > 0$ *such that* (16) *holds with* $K = \{\xi \in X \mid \xi \neq 0, \ \xi/\|\xi\| \in T_\nu(\bar{\sigma}, \bar{x})\}$.

If in addition F *is strongly 2-regular in* x *at* $(\bar{\sigma}, \bar{x})$, *or* $\dim X < \infty$, *then these conditions are further equivalent to the existence of* $\gamma > 0$ *such that* (16) *holds with* $K = T(\bar{\sigma}, \bar{x})$. *If* $\dim X < \infty$, *then the latter condition can be written in the form*

$$\forall \xi \in T(\bar{\sigma}, \bar{x}), \ \xi \neq 0,$$

$$\exists\, (\lambda_0, \lambda) \in \Lambda_0(\bar{\sigma}, \bar{x}) \text{ such that } \frac{\partial^2 L_0}{\partial x^2}(\bar{\sigma}, \bar{x}, \lambda_0, \lambda)[\xi, \xi] > 0.$$

Proof. The proof of the fact that WSOSC implies (a) can be obtained by the standard argument. The key observation is that for every sequence $\{\xi^k\} \subset X$ such that $\{\frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})\xi^k\} \to 0$ as $k \to \infty$, it follows from the Banach open mapping theorem (recall that Y_1 is assumed to be closed) that $\operatorname{dist}(\xi^k, \ker \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})) \to 0$ as $k \to \infty$. The fact that (a) implies WSOSC is evident.

Next, we prove that (b) implies (a) (the converse implication is evident in this case too). Let $\nu > 0$ and $\gamma > 0$ be taken from (b), and let $\xi \in X$ be such that $\|\xi\| = 1$ and $\|\frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})\xi\| \leq \nu$, while $\|P\frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi, \xi]\| \geq \nu$. Set

$$\eta = P\frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi, \xi] \Big/ \left\| P\frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi, \xi] \right\|.$$

With this choice, $\eta \in Y_2$ and $\|\eta\| = 1$. We assert that there exists $\lambda \in Y^*$ such that $\lambda \in Y_1^\perp$, $\langle \lambda, \eta \rangle = 1$ and $\|\lambda\| \leq \|P\|$. (The similar assertion in a more general setting can be found, for instance, in [16, Corollary 2, p. 139]. However, we provide the short proof for the sake of completeness.)

Indeed, consider the subspace $\tilde{Y} = Y_1 + \operatorname{span}\{\eta\}$ in Y, and define the linear functional $\tilde{\lambda} \in \tilde{Y}^*$ as follows: $\langle \tilde{\lambda}, \ y + t\eta \rangle = t$, $y \in Y_1$, $t \in \mathbf{R}$. Then, obviously, $\|\tilde{\lambda}\| \leq \|P\|$, and according to the Hahn–Banach theorem, there exists $\lambda \in Y^*$ such that $\langle \lambda, y \rangle = \langle \tilde{\lambda}, y \rangle$ $\forall\, y \in \tilde{Y}$ and, moreover, $\|\lambda\| = \|\tilde{\lambda}\| \leq \|P\|$. Clearly, λ possess all the properties required.

By the properties of λ, we have that $(0, \ \lambda/\|\lambda\|) \in \Lambda_0(\bar{\sigma}, \bar{x})$ and

$$\begin{aligned}
\frac{\partial^2 L_0}{\partial x^2}(\bar{\sigma}, \bar{x}, 0, \lambda/\|\lambda\|)[\xi, \xi] &= \left\langle \frac{\lambda}{\|\lambda\|}, \ \frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi, \xi] \right\rangle \\
&= \left\langle \frac{\lambda}{\|\lambda\|}, \ P\frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi, \xi] \right\rangle \\
&= \left\| P\frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi, \xi] \right\| \frac{\langle \lambda, \eta \rangle}{\|\lambda\|} \\
&\geq \nu/\|P\|.
\end{aligned}$$

Thus, in order to obtain (a), we need only to replace γ by $\min\{\gamma, \ \nu/\|P\|\}$. This completes the proof.

Let now F be strongly 2-regular in x at $(\bar{\sigma},\, \bar{x})$, and let there exists $\gamma > 0$ such that (16) holds with $K = T(\bar{\sigma},\, \bar{x})$. Let $\{\xi^k\} \subset X$ be any sequence such that $\{\frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})\xi^k\} \to 0$, $\{P\frac{\partial^2 F}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi^k,\, \xi^k]\} \to 0$ as $k \to \infty$. Applying Theorem 3 to the mapping Φ at 0, we find that that $\operatorname{dist}(\xi^k,\, T(\bar{\sigma},\, \bar{x})) \to 0$ as $k \to \infty$. By the standard argument we conclude that (b) holds. If $\dim X < \infty$, then one should employ the compactness argument instead of Theorem 3. □

Lemma 3. *If* $\Lambda(\bar{\sigma},\, \bar{x}) = \emptyset$, *then WSOSC is equivalent to the existence of* $\nu > 0$ *such that*

$$\{\xi \in T_\nu(\bar{\sigma},\, \bar{x}) \mid \|\xi\| = 1\} = \emptyset. \tag{18}$$

If $\dim X < \infty$, *then the latter condition is equivalent to the equality* $T(\bar{\sigma},\, \bar{x}) = \{0\}$.

Proof. By Lemma 2, the existence of $\nu > 0$ such that (18) holds implies WSOSC (even when $\Lambda(\bar{\sigma},\, \bar{x}) \neq \emptyset$).

From the equality $\Lambda(\bar{\sigma},\, \bar{x}) = \emptyset$ it follows that

$$\Lambda_0(\bar{\sigma},\, \bar{x}) = \left\{(0,\, \lambda) \in (\mathbf{R}_+ \times Y^*) \,\middle|\, \lambda \in Y_1^\perp,\, \lambda \neq 0\right\}.$$

Hence, if we assume that for every $\nu > 0$, there exists $\xi \in T_\nu(\bar{\sigma},\, \bar{x})$ such that $\|\xi\| = 1$, then for every $(\lambda_0,\, \lambda) \in \Lambda_0(\bar{\sigma},\, \bar{x})$ such that $\lambda_0 + \|\lambda\| = 1$ it holds that

$$\begin{aligned}\frac{\partial^2 L_0}{\partial x^2}(\bar{\sigma},\, \bar{x},\, \lambda_0,\, \lambda)[\xi,\, \xi] &= \left\langle \lambda,\, \frac{\partial^2 F}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, \xi]\right\rangle \\ &= \left\langle \lambda,\, P\frac{\partial^2 F}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, \xi]\right\rangle \\ &\leq \|P\|\nu.\end{aligned}$$

Taking into account Lemma 2, we arrive to a contradiction with WSOSC.

The last assertion is an easy exercise. □

It can be easily seen that in the existence of $\nu > 0$ such that (18) holds subsumes that $\bar{x}$ is an isolated feasible point of problem (2).

We now turn our attention to the case when $\Lambda(\bar{\sigma},\, \bar{x}) \neq \emptyset$. It is important to point out that this case certainly deserves special consideration, even if normality condition (6) is not necessarily satisfied. For instance, it is known that in the finite-dimensional setting, $\Lambda(\bar{\sigma},\, \bar{x}) \neq \emptyset$ generically, provided $\dim X$ is large enough with respect to $\dim Y$ [3, Theorem 10.2 in section 1.10].

Definition 5. *The* second-order sufficient condition *(SOSC) is said to hold at* $\bar{x}$ *if* $\Lambda(\bar{\sigma},\, \bar{x}) \neq \emptyset$ *and there exist* $M > 0$, $\nu > 0$ *and* $\gamma > 0$ *such that* (17) *holds with* $K = \ker \frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})$.

If $\dim X < \infty$, then SOSC can be written in the form

$$\forall\, \xi \in \ker \frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x}),\ \xi \neq 0,\ \exists\, \lambda \in \Lambda(\bar{\sigma},\, \bar{x}) \text{ such that } \frac{\partial^2 L}{\partial x^2}(\bar{\sigma},\, \bar{x},\, \lambda)[\xi,\, \xi] > 0.$$

Relation between WSOSC and SOSC is clarified by the following two lemmas.

Lemma 4. *If $\Lambda(\bar{\sigma},\ \bar{x}) \neq \emptyset$, then for an arbitrary cone K in X,* (16) *holds with some $\gamma > 0$ if and only if* (17) *holds with some $M > 0$ and $\gamma > 0$.*

Proof. Let (16) holds with some $\gamma > 0$. Fix $\tilde{\Lambda}(\bar{\sigma},\ \bar{x})$ and set

$$C = \sup_{\substack{\xi \in K,\\ \|\xi\|=1}} \left| \frac{\partial^2 L}{\partial x^2}(\bar{\sigma},\ \bar{x},\ \tilde{\lambda})[\xi,\ \xi] \right|, \quad \theta = \frac{\gamma}{2C}.$$

By (16) for any $\xi \in K$ we obtain

$$\begin{aligned}
&\sup_{\substack{(\lambda_0,\,\lambda)\in\Lambda_0(\bar{\sigma},\,\bar{x}),\\ \lambda_0+\|\lambda\|=1}} \frac{\partial^2 L_0}{\partial x^2}(\bar{\sigma},\ \bar{x},\ \lambda_0+\theta,\ \lambda+\theta\tilde{\lambda})[\xi,\ \xi]\\
&\quad= \sup_{\substack{(\lambda_0,\,\lambda)\in\Lambda_0(\bar{\sigma},\,\bar{x}),\\ \lambda_0+\|\lambda\|=1}} \left(\frac{\partial^2 L_0}{\partial x^2}(\bar{\sigma},\ \bar{x},\ \lambda_0,\ \lambda)[\xi,\ \xi] + \theta \frac{\partial^2 L}{\partial x^2}(\bar{\sigma},\ \bar{x},\ \tilde{\lambda})[\xi,\ \xi] \right)\\
&\quad\geq \sup_{\substack{(\lambda_0,\,\lambda)\in\Lambda_0(\bar{\sigma},\,\bar{x}),\\ \lambda_0+\|\lambda\|=1}} \frac{\partial^2 L_0}{\partial x^2}(\bar{\sigma},\ \bar{x},\ \lambda_0,\ \lambda)[\xi,\ \xi] - \theta C\\
&\quad\geq \gamma\|\xi\|^2 - \frac{\gamma}{2}\|\xi\|^2\\
&\quad= \frac{\gamma}{2}\|\xi\|^2.
\end{aligned}$$

Hence, if we set $M = 2C/\gamma + \|\tilde{\lambda}\|$, then

$$\begin{aligned}
&\sup_{\substack{\lambda\in\Lambda(\bar{\sigma},\,\bar{x}),\\ \|\lambda\|\leq M}} \frac{\partial^2 L}{\partial x^2}(\bar{\sigma},\ \bar{x},\ \lambda)[\xi,\ \xi]\\
&\quad\geq \sup_{\substack{(\lambda_0,\,\lambda)\in\Lambda_0(\bar{\sigma},\,\bar{x}),\\ \lambda_0+\|\lambda\|=1}} \frac{\partial^2 L}{\partial x^2}\left(\bar{\sigma},\ \bar{x},\ \frac{\lambda+\theta\tilde{\lambda}}{\lambda_0+\theta}\right)[\xi,\ \xi]\\
&\quad= \sup_{\substack{(\lambda_0,\,\lambda)\in\Lambda_0(\bar{\sigma},\,\bar{x}),\\ \lambda_0+\|\lambda\|=1}} \frac{1}{\lambda_0+\theta} \frac{\partial^2 L_0}{\partial x^2}(\bar{\sigma},\ \bar{x},\ \lambda_0+\theta, \lambda+\theta\tilde{\lambda})[\xi,\ \xi]\\
&\quad\geq \frac{\gamma}{2(1+2C/\gamma))}\|\xi\|^2.
\end{aligned}$$

This proves (17) with γ replaced by the constant in the right-hand side of the last relation.

The proof of the converse assertion is straightforward. ◻

The next lemma is a direct consequence of Lemmas 2 and 4.

Lemma 5. *If $\Lambda(\bar{\sigma},\ \bar{x}) \neq \emptyset$, then WSOSC is equivalent to either of the following conditions:*

(a) *SOSC.*
(b) *There exist $M > 0$, $\nu > 0$ and $\gamma > 0$ such that* (17) *holds with $K = \{\xi \in X \mid \|\frac{\partial F}{\partial x}(\bar{\sigma},\ \bar{x})\xi\| \leq \nu\|\xi\|\}$.*

(c) *There exist* $M > 0$, $\nu > 0$ *and* $\gamma > 0$ *such that* (17) *holds with* $K = \{\xi \in X \mid \xi \neq 0,\ \xi/\|\xi\| \in T_\nu(\bar{\sigma},\ \bar{x})\}$.

If in addition F *is strongly 2-regular in* x *at* $(\bar{\sigma},\ \bar{x})$, *or* $\dim X < \infty$, *then all these conditions are further equivalent to the existence of* $M > 0$ *and* $\gamma > 0$ *such that* (17) *holds with* $K = T(\bar{\sigma},\ \bar{x})$. *If* $\dim X < \infty$, *then the latter condition can be written in the form*

$$\forall\, \xi \in T(\bar{\sigma},\ \bar{x}),\ \xi \neq 0,\ \exists\, \lambda \in \Lambda(\bar{\sigma},\ \bar{x}) \text{ such that } \frac{\partial^2 L}{\partial x^2}(\bar{\sigma},\ \bar{x},\ \lambda)[\xi,\ \xi] > 0.$$

Theorem 5. *WSOSC implies QGC. If* $\Lambda(\bar{\sigma},\ \bar{x}) \neq \emptyset$ *and* F *is strongly 2-regular in* x *at* $(\bar{\sigma},\ \bar{x})$, *then the converse implication is also true. More precisely, under these assumptions, QGC implies that for every* $M > 0$, *there exist* $\nu > 0$ *and* $\gamma > 0$ *such that* (17) *holds with* $K = \{\xi \in X \mid \xi \neq 0,\ \xi/\|\xi\| \in T_\nu(\bar{\sigma},\ \bar{x})\}$.

Proof. The first assertion is well known [7, Theorem 6.3], [3, Theorem 8.1 in section 1.8], [8, Theorem 3.63].

Assume that QGC holds and F is strongly 2-regular in x at $(\bar{\sigma},\ \bar{x})$. For an arbitrary $\nu > 0$, consider an element $\xi \in T_\nu(\bar{\sigma},\ \bar{x})$ such that $\|\xi\| = 1$. Applying Theorem 3 with $x = \bar{x} + t\xi$, $t \geq 0$, we establish the existence of a mapping $r : \mathbf{R}_+ \to X$ such that for $t \geq 0$ it holds that

$$\bar{x} + t\xi + r(t) \in D(\bar{\sigma}), \quad \|r(t)\| \leq \nu t + o(t).$$

For arbitrary $M > 0$ and $\lambda \in \Lambda(\bar{\sigma},\ \bar{x})$ such that $\|\lambda\| \leq M$, from QGC it follows that for $t \geq 0$

$$\begin{aligned} \gamma t^2 + o(t^2) &\leq f(\bar{\sigma},\ \bar{x} + t\xi + r(t)) - f(\bar{\sigma},\ \bar{x}) \\ &= f(\bar{\sigma},\ \bar{x} + t\xi + r(t)) + \langle \lambda,\ F(\bar{\sigma},\ \bar{x} + t\xi + r(t))\rangle \\ &\quad - f(\bar{\sigma},\ \bar{x}) - \langle \lambda,\ F(\bar{\sigma},\ \bar{x})\rangle \\ &= L(\bar{\sigma},\ \bar{x} + t\xi + r(t),\ \lambda) - L(\bar{\sigma},\ \bar{x},\ \lambda) \\ &= \frac{1}{2}\frac{\partial^2 L}{\partial x^2}(\bar{\sigma},\ \bar{x},\ \lambda)[\xi,\ \xi]t^2 + (2\nu + \nu^2)O(t^2) + o(t^2). \end{aligned}$$

Hence, for every $\varepsilon > 0$,

$$\frac{\partial^2 L}{\partial x^2}(\bar{\sigma},\ \bar{x},\ \lambda)[\xi,\ \xi] \geq 2(\gamma - \varepsilon)$$

provided ν is small enough. Taking into account Lemma 5, this completes the proof. □

Without the additional assumptions such as the existence of a (normal) multiplier and 2-regularity of F, QGC does not necessarily imply WSOSC; see Example 4 below.

3. Upper bounds and the adjoint problems

We start with the following stability theorem.

Theorem 6. *Assume that (at least) one of hypotheses (H1) or (H2) is satisfied.*

Then ω is upper semicontinuous at $\bar{\sigma}$. If $\dim X < \infty$*, then ω is continuous at $\bar{\sigma}$, $S(\sigma) \neq \emptyset$ for all $\sigma \in \Sigma$ close enough to $\bar{\sigma}$, and* (7) *holds. In particular, if $\bar{x}$ is a unique solution of* (3)*, then* (8) *holds.*

Proof. The first assertion is an immediate consequence of Theorems 1 and 2. The rest can be proved by the standard argument (by contradiction). □

Note that the assertion of this theorem is exactly the same as the assertion of the stability theorem for the normal case (see, e.g., [15, Theorem 3.1 and Corollary 1]), while hypotheses (H1) and (H2) are both substantially weaker than normality. Perhaps more surprisingly, the same is true for the upper bound on ω, because of the specificity of pure equality constraints: the upper bound in the next theorem is exactly the same as for the normal case (see, e.g., [15, Proposition 3.1]).

Theorem 7. *Assume that* $\dim \Sigma < \infty$ *and hypothesis (H1) is satisfied.*

Then (9) *holds for $\sigma \in \Sigma$.*

Proof. Consider an arbitrary $d \in \Sigma$ satisfying the assumptions of Theorem 4. Then for $t \geq 0$ and $\zeta \in \mathcal{U}$

$$\begin{aligned}\omega(\bar{\sigma} + t\zeta) - \omega(\bar{\sigma}) &\leq f(\bar{\sigma} + t\zeta,\, \bar{x} + t^{1/2}\chi(\zeta) + r(t,\, \zeta)) - f(\bar{\sigma},\, \bar{x}) \\ &= \left\langle \frac{\partial f}{\partial x}(\bar{\sigma},\, \bar{x}),\ \chi(\zeta) \right\rangle t^{1/2} + O(t) \text{ uniformly in } \zeta \in \mathcal{U},\end{aligned} \tag{19}$$

where all the objects are defined as in Theorem 4. Observe that, in Theorem 4, ξ and $\chi(\zeta)$ can be always replaced by $-\xi$ and $-\chi(\zeta)$, respectively. The signs can be chosen in such a way that

$$\left\langle \frac{\partial f}{\partial x}(\bar{\sigma},\, \bar{x}),\ \chi(\zeta) \right\rangle \leq 0.$$

Hence, from (19) we obtain

$$\omega(\bar{\sigma} + t\zeta) \leq \omega(\bar{\sigma}) + O(t) \text{ uniformly in } \zeta \in \mathcal{U}.$$

The needed assertion follows now from Lemma 1 and the assumption $\dim \Sigma < \infty$ by the compactness argument. □

Under the additional assumption of the existence of a (normal) Lagrange multiplier, estimate (9) can be derived even for infinite-dimensional Σ, and in a much simpler way, directly from Theorems 1 and 2. Note that $\Lambda(\bar{\sigma},\, \bar{x}) \neq \emptyset$ holds automatically under hypothesis (H2); see [3, Theorem 4.1 in section 1.4 and Theorem 11.4 in section 1.11] for this fact and for some other conditions sufficient for the existence of a (normal) multiplier (we emphasize that hypothesis (H1) does not possess this property; see Examples 2–4 below).

Theorem 8. *Assume that $\Lambda(\bar{\sigma}, \bar{x}) \neq \emptyset$ and either hypothesis (H1), or hypothesis (H2) is satisfied.*

Then (9) *holds for $\sigma \in \Sigma$.*

Proof. According to Theorems 1 and 2, for every $\sigma \in \Sigma$ close enough to $\bar{\sigma}$, there exists $x(\sigma) \in D(\sigma)$ such that $\|x(\sigma) - \bar{x}\| = O(\|\sigma - \bar{\sigma}\|^{1/2})$. Then for any $\lambda \in \Lambda(\bar{\sigma}, \bar{x})$

$$\begin{aligned}
\omega(\sigma) - \omega(\bar{\sigma}) &\leq f(\sigma, x(\sigma)) - f(\bar{\sigma}, \bar{x}) \\
&= f(\sigma, x(\sigma)) + \langle \lambda, F(\sigma, x(\sigma)) \rangle - f(\bar{\sigma}, \bar{x}) - \langle \lambda, F(\bar{\sigma}, \bar{x}) \rangle \\
&= L(\sigma, x(\sigma), \lambda) - L(\bar{\sigma}, \bar{x}, \lambda) \\
&= \left\langle \frac{\partial L}{\partial \sigma}(\bar{\sigma}, \bar{x}, \lambda), \sigma - \bar{\sigma} \right\rangle \\
&\quad + \frac{1}{2} \frac{\partial^2 L}{\partial x^2}(\bar{\sigma}, \bar{x}, \lambda)[x(\sigma) - \bar{x}, x(\sigma) - \bar{x}] \\
&\quad + o(\|\sigma - \bar{\sigma}\|) + o(\|x(\sigma) - \bar{x}\|^2) \\
&= O(\|\sigma - \bar{\sigma}\|).
\end{aligned}$$

□

Estimate (9) cannot be qualitatively improved, even in the normal case. In order to improve it quantitatively, we turn our attention to directional perturbations. By Theorem 4, we arrive at the following directional version of stability theorem.

Proposition 1. *For a given $d \in \Sigma$, assume that there exists $\xi \in T(\bar{\sigma}, \bar{x}; d)$ such that F is 2-regular in x at $(\bar{\sigma}, \bar{x})$ with respect to ξ.*

Then

$$\limsup_{t \to 0+} \omega(\bar{\sigma} + td) \leq \omega(\bar{\sigma}).$$

If $\dim X < \infty$, *then*

$$\lim_{t \to 0+} \omega(\bar{\sigma} + td) = \omega(\bar{\sigma}),$$

$S(\bar{\sigma} + td) \neq \emptyset$ *for all* $t \geq 0$ *small enough, and*

$$\sup_{x \in S(\bar{\sigma} + td)} \operatorname{dist}(x, S(\bar{\sigma})) \to 0 \text{ as } t \to 0+.$$

In particular, if $\bar{x}$ is a unique solution of (3), *then*

$$\sup_{x \in S(\bar{\sigma} + td)} \|x - \bar{x}\| \to 0 \text{ as } t \to 0+.$$

Consider the so-called *first adjoint problem*

$$\begin{aligned}
&\text{minimize } \left\langle \tfrac{\partial f}{\partial x}(\bar{\sigma}, \bar{x}), \xi \right\rangle \\
&\text{subject to } \xi \in T(\bar{\sigma}, \bar{x}; d).
\end{aligned} \tag{20}$$

Recall that

$$T(\bar{\sigma},\, \bar{x};\, d) = \left\{ \xi \in \ker \frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x}) \,\middle|\, \frac{\partial F}{\partial \sigma}(\bar{\sigma},\, \bar{x})d + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, \xi] \in Y_1 \right\};$$

see (14). Auxiliary problem (20) can be found in the literature [7, problem $(\mathcal{P}S_d)$], [8, Proposition 4.117], though it was previously used in the different context (not for abnormal equality constraints). Let

$$\omega_1(d) = \inf_{\xi \in T(\bar{\sigma},\, \bar{x};\, d)} \left\langle \frac{\partial f}{\partial x}(\bar{\sigma},\, \bar{x}),\, \xi \right\rangle \tag{21}$$

be the optimal value of problem (20).

Proposition 2. *For a given $d \in \Sigma$, assume that F is 2-regular in x at $(\bar{\sigma},\, \bar{x})$ with respect to a direction $\xi \in T(\bar{\sigma},\, \bar{x};\, d)$.*

Then for $t \geq 0$

$$\omega(\bar{\sigma} + td) \leq \omega(\bar{\sigma}) - \left| \left\langle \frac{\partial f}{\partial x}(\bar{\sigma},\, \bar{x}),\, \xi \right\rangle \right| t^{1/2} + O(t).$$

In particular, if the set of $\xi \in T(\bar{\sigma},\, \bar{x};\, d)$ such that F is 2-regular in x at $(\bar{\sigma},\, \bar{x})$ with respect to ξ is dense in $T(\bar{\sigma},\, \bar{x};\, d)$, then

$$\omega(\bar{\sigma} + td) \leq \omega(\bar{\sigma}) + \omega_1(d)t^{1/2} + o(t^{1/2}),$$

and $\omega_1(d) \leq 0$ provided $T(\bar{\sigma},\, \bar{x};\, d) \neq \emptyset$.

The proof follows the line of the proof of Theorem 7 for $\zeta = d$. Note that if $-P\frac{\partial F}{\partial \sigma}(\bar{\sigma},\, \bar{x})d$ is not a critical value for Φ, then F is 2-regular in x at $(\bar{\sigma},\, \bar{x})$ with respect to *every* direction $\xi \in T(\bar{\sigma},\, \bar{x};\, d)$. On the other hand, according to the Sard theorem, in the finite-dimensional setting the set of critical values of Φ is of measure zero.

According to Lemma 1, hypothesis (H1) guarantees that $T(\bar{\sigma},\, \bar{x};\, d) \neq \emptyset\ \forall\, d \in \Sigma$. At the same time, without further assumptions, hypothesis (H1) does not guarantee the last estimate in Proposition 2 (see Example 3). On the other hand, under the assumptions of this proposition, this estimate cannot be sharpened (see Example 2).

The assertion of Proposition 2 is very close to that of [7, Theorem 7.5], [8, Proposition 4.117]. Recall, however, that for pure equality constraints, the latter references deal with the normal case only, and hence, the assumptions of [7, Theorem 7.5], [8, Proposition 4.117] just cannot be satisfied in this case. Indeed, the set of (normal) Lagrange multipliers is necessarily nonempty under the normality condition.

If $\Lambda(\bar{\sigma},\, \bar{x}) \neq \emptyset$, then necessarily $\omega_1(d) = 0$ for every $d \in \Sigma$ such that $T(\bar{\sigma},\, \bar{x};\, d) \neq \emptyset$, but the estimates in Proposition 2 can be sharpened. For $\lambda \in \Lambda(\bar{\sigma},\, \bar{x})$, consider the *second adjoint problem*

$$\begin{array}{l} \text{minimize } \left\langle \frac{\partial L}{\partial \sigma}(\bar{\sigma},\, \bar{x},\, \lambda),\, d \right\rangle + \frac{1}{2}\frac{\partial^2 L}{\partial x^2}(\bar{\sigma},\, \bar{x},\, \lambda)[\xi,\, \xi] \\ \text{subject to } \xi \in T(\bar{\sigma},\, \bar{x};\, d), \end{array} \tag{22}$$

with understanding that the values of the objective function of this problem at its feasible points do not depend on the choice of $\lambda \in \Lambda(\bar{\sigma},\, \bar{x})$. This problem can also be found in the literature [7, problem $(\mathcal{P}L_d)$ and its dual interpretation], [8, Proposition 4.113], but also in the context of normal equality constraints only. Let

$$\omega_2(d) = \left\langle \frac{\partial L}{\partial \sigma}(\bar{\sigma},\, \bar{x},\, \lambda),\, d \right\rangle + \frac{1}{2} \inf_{\xi \in T(\bar{\sigma},\, \bar{x};\, d)} \frac{\partial^2 L}{\partial x^2}(\bar{\sigma},\, \bar{x},\, \lambda)[\xi,\, \xi] \tag{23}$$

be the optimal value of problem (22).

Proposition 3. *Let $\Lambda(\bar{\sigma},\, \bar{x}) \neq \emptyset$. For a given $d \in \Sigma$, assume that F is 2-regular in x at $(\bar{\sigma},\, \bar{x})$ with respect to a direction $\xi \in T(\bar{\sigma},\, \bar{x};\, d)$.*

Then for $t \geq 0$

$$\omega(\bar{\sigma} + td) \leq \omega(\bar{\sigma}) + \left(\left\langle \frac{\partial L}{\partial \sigma}(\bar{\sigma},\, \bar{x},\, \lambda),\, d \right\rangle + \frac{1}{2} \inf_{\xi \in T(\bar{\sigma},\, \bar{x};\, d)} \frac{\partial^2 L}{\partial x^2}(\bar{\sigma},\, \bar{x},\, \lambda)[\xi,\, \xi] \right) t + o(t)$$

with any $\lambda \in \Lambda(\bar{\sigma},\, \bar{x})$. In particular, if the set of $\xi \in T(\bar{\sigma},\, \bar{x};\, d)$ such that F is 2-regular in x at $(\bar{\sigma},\, \bar{x})$ with respect to ξ is dense in $T(\bar{\sigma},\, \bar{x};\, d)$, then

$$\omega(\bar{\sigma} + td) \leq \omega(\bar{\sigma}) + \omega_2(d)t + o(t). \tag{24}$$

Proof. From Theorem 4 for $\zeta = d$ it follows that for $t \geq 0$ and any $\lambda \in \Lambda(\bar{\sigma},\, \bar{x})$

$$\begin{aligned}
\omega(\bar{\sigma} + td) - \omega(\bar{\sigma}) &\leq f(\bar{\sigma} + td,\, \bar{x} + t^{1/2}\xi + r(t,\, d)) - f(\bar{\sigma},\, \bar{x}) \\
&= f(\bar{\sigma} + td,\, \bar{x} + t^{1/2}\xi + r(t,\, d)) \\
&\quad + \langle \lambda,\, F(\bar{\sigma} + td,\, \bar{x} + t^{1/2}\xi + r(t,\, d)) \rangle \\
&\quad - f(\bar{\sigma},\, \bar{x}) - \langle \lambda,\, F(\bar{\sigma},\, \bar{x}) \rangle \\
&= L(\bar{\sigma} + td,\, \bar{x} + t^{1/2}\xi + r(t,\, d),\, \lambda) - L(\bar{\sigma},\, \bar{x},\, \lambda) \\
&= \left(\left\langle \frac{\partial L}{\partial \sigma}(\bar{\sigma},\, \bar{x},\, \lambda),\, d \right\rangle + \frac{1}{2} \frac{\partial^2 L}{\partial x^2}(\bar{\sigma},\, \bar{x},\, \lambda)[\xi,\, \xi] \right) t + o(t).
\end{aligned}$$

This immediately implies the needed assertion. $\square$

Under the assumptions of Proposition 3, estimate (24) cannot be sharpened (see Example 5). Hypothesis (H1) guarantees that $\forall\, d \in \Sigma\ T(\bar{\sigma},\, \bar{x};\, d) \neq \emptyset$, and hence, $\omega_2(d) < +\infty$.

In the normal case, Proposition 3 corresponds formally to [7, Theorem 7.3], [8, Theorem 4.113]. However, in this case $T(\bar{\sigma},\, \bar{x};\, d) = \ker \frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})$, and according to the second-order necessary optimality condition

$$\frac{\partial^2 L}{\partial x^2}(\bar{\sigma},\, \bar{x},\, \bar{\lambda})[\xi,\, \xi] \geq 0 \quad \forall\, \xi \in \ker \frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x}),$$

for the Lagrange multiplier $\bar{\lambda}$ it holds that

$$\begin{aligned}
\omega_2(d) &= \left\langle \frac{\partial L}{\partial \sigma}(\bar{\sigma},\, \bar{x},\, \bar{\lambda}),\, d \right\rangle + \frac{1}{2} \inf_{\xi \in \ker \frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})} \frac{\partial^2 L}{\partial x^2}(\bar{\sigma},\, \bar{x},\, \bar{\lambda})[\xi,\, \xi] \\
&= \left\langle \frac{\partial L}{\partial \sigma}(\bar{\sigma},\, \bar{x},\, \bar{\lambda}),\, d \right\rangle.
\end{aligned}$$

Hence, estimate (24) takes the form (10). This is in agreement with [15, Theorem 7.1], [7, Proposition 4.3].

4. Lower bounds and asymptotic behavior of solutions

We start this section with the following lower bound on ω which was pointed out for us by E. R. Avakov.

Theorem 9. *Assume that hypothesis (H1) is satisfied and F is strongly 2-regular in x at $(\bar{\sigma},\ \bar{x})$. Moreover, assume that $\bar{x}$ is a unique solution of* (3).

Then for $\sigma \in \Sigma$ it holds that

$$\omega(\sigma) \geq \omega(\bar{\sigma}) + O(\|\sigma - \bar{\sigma}\| + (\|P\|\|\sigma - \bar{\sigma}\|)^{1/2}).$$

Proof. Consider arbitrary sequences $\{\sigma^k\} \subset \Sigma$ and $\{x^k\} \subset X$ such that $\{\sigma^k\} \to \bar{\sigma}$, $\{x^k\} \to \bar{x}$ as $k \to \infty$, $x^k \in S(\sigma^k)$, $x^k \neq \bar{x}\ \forall\, k$.

From Theorem 3 for every k we obtain the estimate

$$\begin{aligned}
\operatorname{dist}(x^k,\ D(\bar{\sigma})) &= O(\|F(\bar{\sigma},\ x^k)\| + \|PF(\bar{\sigma},\ x^k)\|/\|x^k - \bar{x}\|)\\
&= O(\|F(\sigma^k,\ x^k) - F(\bar{\sigma},\ x^k)\|\\
&\quad + \|P(F(\sigma^k,\ x^k) - F(\bar{\sigma},\ x^k))\|/\|x^k - \bar{x}\|)\\
&= O(\|\sigma^k - \bar{\sigma}\| + \|P\|\|\sigma^k - \bar{\sigma}\|/\|x^k - \bar{x}\|).
\end{aligned}$$

Evidently, this leads to the estimate

$$\operatorname{dist}(x^k,\ D(\bar{\sigma})) = O(\|\sigma^k - \bar{\sigma}\| + (\|P\|\|\sigma^k - \bar{\sigma}\|)^{1/2}).$$

Define a sequence $\{\tilde{x}^k\} \subset X$ such that

$$\|x^k - \tilde{x}^k\| = \operatorname{dist}(x^k,\ D(\bar{\sigma})) + o(\operatorname{dist}(x^k,\ D(\bar{\sigma}))).$$

Then $\{\tilde{x}^k\} \to \bar{x}$ as $k \to \infty$, and hence, $\tilde{x}^k \in B$ and $f(\bar{\sigma},\ \tilde{x}^k) - f(\bar{\sigma},\ \bar{x}) \geq 0$ for all k large enough. We now find that

$$\begin{aligned}
\omega(\sigma^k) - \omega(\bar{\sigma}) &= f(\sigma^k,\ x^k) - f(\bar{\sigma},\ \bar{x})\\
&= f(\bar{\sigma},\ \tilde{x}^k) - f(\bar{\sigma},\ \bar{x}) + O(\|\sigma^k - \bar{\sigma}\| + (\|P\|\|\sigma^k - \bar{\sigma}\|)^{1/2}),
\end{aligned}$$

and this gives the needed estimate. □

We proceed with the estimates on solutions of the perturbed problem. The corresponding lower bounds on ω will be derived next. To begin with, the following interesting fact takes place.

Theorem 10. *Assume that* $\dim \Sigma < \infty$*, hypothesis (H1) is satisfied, and F is strongly 2-regular in x at $(\bar{\sigma},\ \bar{x})$. Moreover, assume that QGC holds.*

Then there exists a neighborhood U of $\bar{x}$ such that for $\sigma \in \Sigma$

$$\begin{aligned}
\|x - \bar{x}\| &= O\left(\left(\|\sigma - \bar{\sigma}\| + \left\|P\frac{\partial F}{\partial \sigma}(\bar{\sigma},\ \bar{x})(\sigma - \bar{\sigma})\right\| / \|x - \bar{x}\|\right)^{1/2}\right)\\
&= O(\|\sigma - \bar{\sigma}\|^{1/3}) \text{ uniformly in } x \in S(\sigma) \cap U.
\end{aligned} \tag{25}$$

In particular, if $\dim X < \infty$ *and $\bar{x}$ is a unique solution of* (3)*, then*

$$\sup_{x \in S(\sigma)} \|x - \bar{x}\| = O(\|\sigma - \bar{\sigma}\|^{1/3}).$$

We emphasize that in this theorem, we neither assert that $S(\sigma)$ is nonempty for $\sigma \in \Sigma$ close to $\bar{\sigma}$, nor that this set tends to $S(\bar{\sigma})$ as $\sigma \to \bar{\sigma}$, unless $\dim X < \infty$. The same comment concerns all infinite-dimensional results in the rest of the paper.

Proof. Consider arbitrary sequences $\{\sigma^k\} \subset \Sigma$ and $\{x^k\} \subset X$ such that $\{\sigma^k\} \to \bar{\sigma}$, $\{x^k\} \to \bar{x}$ as $k \to \infty$, $x^k \in S(\sigma^k)$, $x^k \neq \bar{x}\ \forall\, k$, and

$$\|\sigma^k - \bar{\sigma}\| = O(\|x^k - \bar{x}\|). \tag{26}$$

(If sequences possessing these properties do not exist, much stronger estimate than (25) holds, and we are done).

From Theorem 7 for every k we obtain the estimate

$$\begin{aligned} O(\|\sigma^k - \bar{\sigma}\|) &\geq \omega(\sigma^k) - \omega(\bar{\sigma}) \\ &= f(\sigma^k,\, x^k) - f(\bar{\sigma},\, \bar{x}) \\ &= f(\bar{\sigma},\, x^k) - f(\bar{\sigma},\, \bar{x}) + O(\|\sigma^k - \bar{\sigma}\|). \end{aligned}$$

Hence,

$$f(\bar{\sigma},\, x^k) - f(\bar{\sigma},\, \bar{x}) \leq O(\|\sigma^k - \bar{\sigma}\|). \tag{27}$$

Define a sequence $\{\tilde{x}^k\} \subset X$ such that

$$\|x^k - \tilde{x}^k\| = \operatorname{dist}(x^k,\, D(\bar{\sigma})) + o(\operatorname{dist}(x^k,\, D(\bar{\sigma}))).$$

According to QGC and (27),

$$\begin{aligned} \gamma\|\tilde{x}^k - \bar{x}\|^2 &\leq f(\bar{\sigma},\, \tilde{x}^k) - f(\bar{\sigma},\, \bar{x}) \\ &\leq f(\bar{\sigma},\, \tilde{x}^k) - f(\bar{\sigma},\, x^k) + f(\bar{\sigma},\, x^k) - f(\bar{\sigma},\, \bar{x}) \\ &\leq O(\|\tilde{x}^k - x^k\| + \|\sigma^k - \bar{\sigma}\|) \\ &= O(\operatorname{dist}(x^k,\, D(\bar{\sigma})) + \|\sigma^k - \bar{\sigma}\|). \end{aligned} \tag{28}$$

Furthermore, $x^k \in D(\sigma^k)$, hence by Theorem 3 it holds that

$$\begin{aligned} \operatorname{dist}(x^k,\, D(\bar{\sigma})) &= O(\|F(\bar{\sigma},\, x^k)\| + \|PF(\bar{\sigma},\, x^k)\|/\|x^k - \bar{x}\|) \\ &= O(\|F(\sigma^k,\, x^k) - F(\bar{\sigma},\, x^k)\| \\ &\quad + \|P(F(\sigma^k,\, x^k) - F(\bar{\sigma},\, x^k))\|/\|x^k - \bar{x}\|) \\ &= O\left(\|\sigma^k - \bar{\sigma}\| + \left\| P\frac{\partial F}{\partial \sigma}(\bar{\sigma},\, \bar{x})(\sigma^k - \bar{\sigma})\right\| / \|x^k - \bar{x}\|\right), \end{aligned}$$

where (26) was taken into account. The last relation combined with (28) results in the estimate

$$\|\tilde{x}^k - \bar{x}\|^2 = O\left(\|\sigma^k - \bar{\sigma}\| + \left\| P\frac{\partial F}{\partial \sigma}(\bar{\sigma},\, \bar{x})(\sigma^k - \bar{\sigma})\right\| / \|x^k - \bar{x}\|\right),$$

and this further implies

$$\begin{aligned}\|x^k - \bar{x}\| &\le \|x^k - \tilde{x}^k\| + \|\tilde{x}^k - \bar{x}\| \\ &= O\left(\left(\|\sigma^k - \bar{\sigma}\| + \left\| P\frac{\partial F}{\partial \sigma}(\bar{\sigma},\, \bar{x})(\sigma^k - \bar{\sigma})\right\| / \|x^k - \bar{x}\|\right)^{1/2}\right) \\ &= O\left(\left(\|\sigma^k - \bar{\sigma}\| / \|x^k - \bar{x}\|\right)^{1/2}\right).\end{aligned}$$

This gives (25).

The last assertion of the theorem follows from the first one, and from Theorem 6. □

The question is open whether estimate (25) can be improved or not under the assumptions of Theorem 10 (when $\Lambda(\bar{\sigma},\, \bar{x}) = \emptyset$). On the other hand, 2-regularity condition or QGC cannot be omitted in Theorem 10 (Examples 4, 6, 7).

As mentioned above, QGC can be replaced by stronger sufficient optimality conditions. In particular, the passage from QGC to the condition of the existence of $\nu > 0$ such that (18) holds makes it possible to sharpen the bound (25). Recall that the latter condition is equivalent to WSOSC provided $\Lambda(\bar{\sigma},\, \bar{x}) = \emptyset$, and to the equality $T(\bar{\sigma},\, \bar{x}) = \{0\}$ provided $\dim X < \infty$; see Lemma 3.

Theorem 11. *Assume that there exists $\nu > 0$ such that* (18) *holds.*

Then there exists a neighborhood U of $\bar{x}$ such that for $\sigma \in \Sigma$

$$\|x - \bar{x}\| = O(\|\sigma - \bar{\sigma}\|^{1/2}) \textit{ uniformly in } x \in D(\sigma) \cap U.$$

Proof. The proof is by contradiction: suppose that there exist sequences $\{\sigma^k\} \subset \Sigma$ and $\{x^k\} \subset X$ such that $\{\sigma^k\} \to \bar{\sigma}$, $\{x^k\} \to \bar{x}$ as $k \to \infty$, $x^k \in D(\sigma^k)$, $x^k \ne \bar{x}\ \forall\, k$, and

$$\|\sigma^k - \bar{\sigma}\|^{1/2} / \|x^k - \bar{x}\| \to 0 \text{ as } k \to \infty. \tag{29}$$

For every k

$$\begin{aligned}0 &= F(\sigma^k,\, x^k) \\ &= \frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})(x^k - \bar{x}) + O(\|\sigma^k - \bar{\sigma}\|) + o(\|x^k - \bar{x}\|),\end{aligned}$$

$$\begin{aligned}0 &= PF(\sigma^k,\, x^k) \\ &= \frac{1}{2}P\frac{\partial^2 F}{\partial x^2}(\bar{\sigma},\, \bar{x})[x^k - \bar{x},\, x^k - \bar{x}] + O(\|\sigma^k - \bar{\sigma}\|) + o(\|x^k - \bar{x}\|^2).\end{aligned}$$

Hence, from (29), for $\xi^k = (x^k - \bar{x})/\|x^k - \bar{x}\|$ we have

$$\frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})\xi^k \to 0, \qquad P\frac{\partial^2 F}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi^k,\, \xi^k] \to 0 \text{ as } k \to \infty.$$

Thus, for every $\nu > 0$ it holds that $\xi^k \in T_\nu(\bar{\sigma},\, \bar{x})$ and $\|\xi^k\| = 1$ for all k large enough. This contradicts (18). □

In the framework of directional perturbations, the following quantitative result takes place.

Proposition 4. *For a given $d \in \Sigma$, assume that there exists $\xi \in T(\bar{\sigma}, \bar{x}; d)$ such that F is 2-regular in x at $(\bar{\sigma}, \bar{x})$ with respect to ξ. Moreover, assume that $T(\bar{\sigma}, \bar{x}) = \{0\}$,* $\dim X < \infty$*, and $\bar{x}$ is a unique solution of* (3).

Then for $t \geq 0$

$$\sup_{x \in S(\bar{\sigma}+td)} \|x - \bar{x}\| = O(t^{1/2}), \tag{30}$$

$$\omega(\bar{\sigma} + td) \geq \omega(\bar{\sigma}) + \omega_1(d)t^{1/2} + o(t^{1/2}). \tag{31}$$

Moreover, if $\bar{\xi} \in X$ is a solution of (20)*, and there exists $\xi \in T(\bar{\sigma}, \bar{x}; d)$ arbitrary close to $\bar{\xi}$ and such that F is 2-regular in x at $(\bar{\sigma}, \bar{x})$ with respect to ξ, then, for $\sigma = \bar{\sigma} + td$,* (4) *has an $o(t^{1/2})$-solution of the form $\bar{x} + t^{1/2}\bar{\xi} + o(t^{1/2})$.*

In addition, suppose that the set of $\xi \in T(\bar{\sigma}, \bar{x}; d)$ such that F is 2-regular in x at $(\bar{\sigma}, \bar{x})$ with respect to ξ is dense in $T(\bar{\sigma}, \bar{x}; d)$. Then

$$\omega(\bar{\sigma} + td) = \omega(\bar{\sigma}) + \omega_1(d)t^{1/2} + o(t^{1/2}), \tag{32}$$

and the solution set of (20) *coincides with the set consisting of $\xi \in X$ such that, for $\sigma = \bar{\sigma} + td$,* (4) *has an $o(t^{1/2})$-solution of the form $\bar{x} + t^{1/2}\xi + o(t^{1/2})$.*

Here for every $\sigma \in \Sigma$ and $\varepsilon > 0$, *ε-solution* of (4) is a point $x \in D(\sigma) \cap B$ such that $f(\sigma, x) \leq \omega(\sigma) + \varepsilon$.

Proof. Estimate (30) follows from Proposition 1 and Theorem 11; in particular, $S(\bar{\sigma} + td) \neq \emptyset$ if t is small enough.

Consider an arbitrary sequence $\{t_k\}$ of positive real numbers such that $\{t_k\} \to 0$ as $k \to \infty$, and an arbitrary sequence $\{x^k\} \subset X$ such that $x^k \in S(\bar{\sigma} + t_k d)\ \forall\, k$. It follows from (30) that for every k

$$\begin{aligned} 0 &= F(\bar{\sigma} + t_k d,\, x^k) \\ &= \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})(x^k - \bar{x}) + o(t_k^{1/2}), \end{aligned}$$

$$\begin{aligned} 0 &= PF(\bar{\sigma} + t_k d,\, x^k) \\ &= P\left(t_k \frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[x^k - \bar{x},\, x^k - \bar{x}]\right) + o(t_k). \end{aligned}$$

Set $\xi^k = (x^k - \bar{x})/t_k^{1/2}$. According to (30), the sequence $\{\xi^k\}$ us bounded, and from the last two equalities it follows that every limit point of this sequence belongs to $T(\bar{\sigma}, \bar{x}; d)$. Furthermore,

$$\begin{aligned} \omega(\bar{\sigma} + t_k d) - \omega(\bar{\sigma}) &= f(\bar{\sigma} + t_k d,\, x^k) - f(\bar{\sigma}, \bar{x}) \\ &= \left\langle \frac{\partial f}{\partial x}(\bar{\sigma}, \bar{x}),\, x^k - \bar{x} \right\rangle + o(t_k^{1/2}) \\ &= \left\langle \frac{\partial f}{\partial x}(\bar{\sigma}, \bar{x}),\, \xi^k \right\rangle t_k^{1/2} + o(t_k^{1/2}), \end{aligned}$$

hence

$$\liminf_{k\to\infty}(\omega(\bar\sigma+t_k d)-\omega(\bar\sigma))/t_k^{1/2}\geq\omega_1(d).$$

This proves (31).

Let $\bar\xi\in X$ be a solution of (20), let there exist a sequence $\{\xi^k\}\subset T(\bar\sigma,\ \bar x;\ d)$ such that $\{\xi^k\}\to\bar\xi$ as $k\to\infty$, and let F be 2-regular in x at $(\bar\sigma,\ \bar x)$ with respect to $\xi^k\ \forall k$. We next prove that there exists a mapping $r:\mathbf{R}_+\to X$ such that for $t\geq 0$ it holds that

$$\bar x+t^{1/2}\bar\xi+r(t)\in D(\bar\sigma+td),\quad \|r(t)\|=o(t^{1/2}). \tag{33}$$

Indeed, assume that there exist a sequence $\{t_j\}$ of positive real numbers such that $\{t_j\}\to 0$ as $j\to\infty$, and a number $\delta>0$ such that for all j

$$\|\bar x+t_j^{1/2}\bar\xi-x\|\geq\delta t_j^{1/2}\quad\forall\, x\in D(\bar\sigma+t_j d).$$

Fix k such that $\|\xi^k-\bar\xi\|\leq\delta/2$. Then

$$\begin{aligned}\|\bar x+t_j^{1/2}\xi^k-x\|&\geq\|\bar x+t_j^{1/2}\bar\xi-x\|-t_j^{1/2}\|\xi^k-\bar\xi\|\\&\geq\frac{1}{2}\delta t_j^{1/2}\quad\forall\, x\in D(\bar\sigma+t_j d).\end{aligned}$$

But this is in contradiction with the fact that, according to Theorem 4, for $t\geq 0$

$$\operatorname{dist}(\bar x+t^{1/2}\xi^k,\ D(\bar\sigma+td))=o(t^{1/2}).$$

From (31) and (33) it follows that for $t\geq 0$

$$\begin{aligned}f(\bar\sigma+td,\ \bar x+t^{1/2}\bar\xi+r(t))&=f(\bar\sigma,\ \bar x)+\left\langle\frac{\partial f}{\partial x}(\bar\sigma,\ \bar x),\ \bar\xi\right\rangle t^{1/2}+o(t^{1/2})\\&=\omega(\bar\sigma)+\omega_1(d)t^{1/2}+o(t^{1/2})\\&\leq\omega(\bar\sigma+td)+o(t^{1/2}),\end{aligned}$$

that is, $\bar x+t^{1/2}\bar\xi+r(t)$ is an $o(t^{1/2})$-solution of problem (4) with $\sigma=\bar\sigma+td$.

Finally, under the assumptions of the last assertion of the proposition, (32) follows from Proposition 2 and (31). Suppose that, for some $\xi\in X$ and for $\sigma=\bar\sigma+td$, $t\geq 0$, problem (4) has an $o(t^{1/2})$-solution of the form $\bar x+t^{1/2}\xi+o(t^{1/2})$. Then according to the discussion preceding Theorem 4, $\xi\in T(\bar\sigma,\ \bar x;\ d)$. Moreover, according to Proposition 2,

$$\begin{aligned}f(\bar\sigma+td,\ \bar x+t^{1/2}\xi+o(t^{1/2}))&\leq\omega(\bar\sigma+td)+o(t^{1/2})\\&\leq\omega(\bar\sigma)+\omega_1(d)t^{1/2}+o(t^{1/2}).\end{aligned}$$

At the same time

$$f(\bar\sigma+td,\ \bar x+t^{1/2}\xi+o(t^{1/2}))=\omega(\bar\sigma)+\left\langle\frac{\partial f}{\partial x}(\bar\sigma,\ \bar x),\ \xi\right\rangle t^{1/2}+o(t^{1/2}),$$

and hence,

$$\left\langle\frac{\partial f}{\partial x}(\bar\sigma,\ \bar x),\ \xi\right\rangle\leq\omega_1(d),$$

that is, ξ is a solution of (20). □

The assertion of Proposition 4 is very close to that of [7, Theorem 7.6] (see also [8, section 4.8.3]). However, the assumptions in the latter reference just cannot be satisfied in this case; see our comments following Proposition 2.

If $\Lambda(\bar{\sigma},\, \bar{x}) \neq \emptyset$, the passage from QGC to SOSC in Theorem 10 makes it possible to drop the strong 2-regularity condition, to sharpen the bound on solutions and establish the unimprovable lower bound on ω.

Theorem 12. *Assume that (at least) one of hypotheses (H1) or (H2) is satisfied. Moreover, assume that SOSC holds.*

Then there exists a neighborhood U of $\bar{x}$ such that for $\sigma \in \Sigma$

$$\|x - \bar{x}\| = O(\|\sigma - \bar{\sigma}\|^{1/2}) \text{ uniformly in } x \in S(\sigma) \cap U. \tag{34}$$

In particular, if $\dim X < \infty$ *and $\bar{x}$ is a unique solution of* (3)*, then*

$$\sup_{x \in S(\sigma)} \|x - \bar{x}\| = O(\|\sigma - \bar{\sigma}\|^{1/2}), \tag{35}$$

$$\omega(\sigma) = \omega(\bar{\sigma}) + O(\|\sigma - \bar{\sigma}\|). \tag{36}$$

Proof. Consider arbitrary sequences $\{\sigma^k\} \subset \Sigma$ and $\{x^k\} \subset X$ such that $\{\sigma^k\} \to \bar{\sigma}$, $\{x^k\} \to \bar{x}$ as $k \to \infty$, and $x^k \in S(\sigma^k)$, $x^k \neq \bar{x}\ \forall\, k$. We argue by contradiction. Assume that (29) holds. Set $\xi^k = (x^k - \bar{x})/\|x^k - \bar{x}\|$, i.e., $\|\xi^k\| = 1$. Following the line of the proof of Theorem 11, we conclude that for all k sufficiently large, $\xi^k \in T_\nu(\bar{\sigma},\, \bar{x})$, where $\nu > 0$ is chosen as in condition (c) of Lemma 5. Hence, there exist $M > 0$ and $\gamma > 0$ such that, for every sufficiently large k, there is $\lambda^k \in \Lambda(\bar{\sigma},\, \bar{x})$ satisfying $\|\lambda^k\| \leq M$ and

$$\frac{\partial^2 L}{\partial x^2}(\bar{\sigma},\, \bar{x},\, \lambda^k)[\xi^k,\, \xi^k] \geq \gamma. \tag{37}$$

From Theorem 8 we obtain the estimate

$$\begin{aligned} O(\|\sigma^k - \bar{\sigma}\|) &\geq \omega(\sigma^k) - \omega(\bar{\sigma}) \\ &= f(\sigma^k,\, x^k) - f(\bar{\sigma},\, \bar{x}) \\ &= f(\sigma^k,\, x^k) + \langle \lambda^k,\, F(\sigma^k,\, x^k)\rangle - f(\bar{\sigma},\, \bar{x}) - \langle \lambda^k,\, F(\bar{\sigma},\, \bar{x})\rangle \\ &= L(\sigma^k,\, x^k,\, \lambda^k) - L(\bar{\sigma},\, \bar{x},\, \lambda^k) \\ &= \frac{1}{2}\frac{\partial^2 L}{\partial x^2}(\bar{\sigma},\, \bar{x},\, \lambda^k)[x^k - \bar{x},\, x^k - \bar{x}] \\ &\quad + O(\|\sigma^k - \bar{\sigma}\|) + o(\|x^k - \bar{x}\|^2). \end{aligned} \tag{38}$$

By (37), it follows that for every $\varepsilon > 0$

$$O(\|\sigma^k - \bar{\sigma}\|) \geq (\gamma - \varepsilon)\|x^k - \bar{x}\|^2$$

for all k large enough. But this contradicts (29), and (34) is proved.

Under the assumption that $\dim X < \infty$ and $\bar{x}$ is a unique solution of (3), estimate (35) follows from (34) and Theorem 6. Estimate (36) is by (35) and the intermediate relations in (38). $\square$

In the normal case, Theorem 10 reduces to Theorem 12, and both correspond to [7, Proposition 6.4].

The following directional version of Theorem 12 holds:

Proposition 5. *For a given $d \in \Sigma$, assume that there exists $\xi \in T(\bar{\sigma}, \bar{x}; d)$ such that F is 2-regular in x at $(\bar{\sigma}, \bar{x})$ with respect to ξ. Moreover, assume that SOSC holds.*

Then there exists a neighborhood U of $\bar{x}$ such that for $t \geq 0$

$$\|x - \bar{x}\| = O(t^{1/2}) \text{ uniformly in } x \in S(\bar{\sigma} + td) \cap U.$$

In particular, if $\dim X < \infty$ *and $\bar{x}$ is a unique solution of* (3), *then* (30) *holds.*

The proof follows the line of the proof of Theorem 12. The difference is that we should refer to Proposition 1 instead of Theorem 6, and to Proposition 3 instead of Theorem 8.

Finally, the following quantitative result takes place.

Proposition 6. *For a given $d \in \Sigma$, assume that there exists $\xi \in T(\bar{\sigma}, \bar{x}; d)$ such that F is 2-regular in x at $(\bar{\sigma}, \bar{x})$ with respect to ξ. Moreover, assume that SOSC holds,* $\dim X < \infty$, *and $\bar{x}$ is a unique solution of* (3).

Then for $t \geq 0$

$$\omega(\bar{\sigma} + td) \geq \omega(\bar{\sigma}) + \omega_2(d)t + o(t). \tag{39}$$

Moreover, if $\bar{\xi} \in X$ is a solution of (22), *and there exists $\xi \in T(\bar{\sigma}, \bar{x}; d)$ arbitrary close to $\bar{\xi}$ and such that F is 2-regular in x at $(\bar{\sigma}, \bar{x})$ with respect to ξ, then, for $\sigma = \bar{\sigma} + td$,* (4) *has an $o(t)$-solution of the form $\bar{x} + t^{1/2}\bar{\xi} + o(t^{1/2})$.*

In addition, suppose that the set of $\xi \in T(\bar{\sigma}, \bar{x}; d)$ such that F is 2-regular in x at $(\bar{\sigma}, \bar{x})$ with respect to ξ is dense in $T(\bar{\sigma}, \bar{x}; d)$. Then

$$\omega(\bar{\sigma} + td) = \omega(\bar{\sigma}) + \omega_2(d)t + o(t), \tag{40}$$

and the solution set of (22) *coincides with the set consisting of $\xi \in X$ such that* (4) *with $\sigma = \bar{\sigma} + td$ has an $o(t)$-solution of the form $\bar{x} + t^{1/2}\xi + o(t^{1/2})$.*

Proof. Consider an arbitrary sequence $\{t_k\}$ of positive real numbers such that $\{t_k\} \to 0$ as $k \to \infty$, and an arbitrary sequence $\{x^k\} \subset X$ such that $x^k \in S(\bar{\sigma} + t_k d)\ \forall\, k$. For every k set $\xi^k = (x^k - \bar{x})/t_k^{1/2}$. In the proof of Proposition 4 it was derived from (30) that every limit point of the sequence $\{\xi^k\}$ belongs to $T(\bar{\sigma}, \bar{x}; d)$. Furthermore, for an arbitrary fixed $\lambda \in \Lambda(\bar{\sigma}, \bar{x})$

$$\begin{aligned}
\omega(\bar{\sigma} + t_k d) - \omega(\bar{\sigma}) &= f(\bar{\sigma} + t_k d,\, x^k) - f(\bar{\sigma},\, \bar{x}) \\
&= f(\bar{\sigma} + t_k d,\, x^k) + \langle \lambda,\, F(\bar{\sigma} + t_k d,\, x^k) \rangle \\
&\quad - f(\bar{\sigma},\, \bar{x}) - \langle \lambda,\, F(\bar{\sigma},\, \bar{x}) \rangle \\
&= L(\bar{\sigma} + t_k d,\, x^k,\, \lambda) - L(\bar{\sigma},\, \bar{x},\, \lambda) \\
&= \left(\left\langle \frac{\partial L}{\partial \sigma}(\bar{\sigma},\, \bar{x},\, \lambda),\, d \right\rangle + \frac{1}{2} \frac{\partial^2 L}{\partial x^2}(\bar{\sigma},\, \bar{x},\, \lambda)[\xi^k,\, \xi^k] \right) t_k \\
&\quad + o(t_k),
\end{aligned}$$

hence

$$\liminf_{k\to\infty}(\omega(\bar{\sigma}+t_kd)-\omega(\bar{\sigma}))/t_k \geq \omega_2(d).$$

This proves (39).

The rest of the proof follows the line of the proof of Proposition 4. The difference is that instead of Proposition 2 and (31), one should refer to Proposition 3 and (39), respectively. □

Hypotheses (H1) or (H2) combined with SOSC imply that $\forall\, d \in \Sigma$ (22) has a solution, and in particular, $-\infty < \omega_2(d) < +\infty$. The estimate (40) means that ω is directionally differentiable at $\bar{\sigma}$ with respect to a direction d, and $\omega'(\bar{\sigma};\, d) = \omega_2(d)$.

The estimates in Theorem 12 and Proposition 5 cannot be sharpened (Example 5, which also illustrates the equality (40) in Proposition 6). Moreover, SOSC cannot be dropped in these results (Examples 6, 7).

In the normal case, Proposition 6 formally corresponds to [7, Theorem 7.4] (see also [8, section 4.8.2]). Note, however, that according to our discussion above, in the normal case equality (40) reduces to (11).

5. Examples

In this section, we illustrate the results obtained above by some finite-dimensional examples. Set $\Sigma = \mathbf{R}^s$, $X = \mathbf{R}^n$, and $Y = \mathbf{R}^l$, where the dimensions s, n, and l, will be specified for each example below.

Example 2. Let $s = 1$, $n = 3$, $l = 2$, $f(\sigma,\, x) = x_1$, $F_1(\sigma,\, x) = x_1^2 + x_2^2 - x_3^2$, $F_2(\sigma,\, x) = x_1x_3 - \sigma$, $\bar{\sigma} = 0$, $\bar{x} = 0$. The mapping F is 2-regular in x at $(\bar{\sigma},\, \bar{x})$ with respect to every direction in $T(\bar{\sigma},\, \bar{x};\, d)\ \forall\, d \in \Sigma$, $d \neq 0$.

The first adjoint problem (20) coincides with the original problem when $d = \sigma$, and, for example, $d > 0$ implies $\omega_1(d) = -d^{1/2}$, $\omega(\bar{\sigma}+td) = -d^{1/2}t^{1/2}\ \forall\, t \geq 0$ for any choice of B. (Recall that $\omega_1(d)$ is defined in (21).)

Example 2 illustrates that the last estimate in Proposition 2 cannot be sharpened.

Example 3. Let $s = 1$, $n = 4$, $l = 3$, $f(\sigma,\, x) = x_1$, $F_1(\sigma,\, x) = x_1^2 + x_2^2 - x_3^2$, $F_2(\sigma,\, x) = x_1(x_1 - x_3 - x_3^3)$, $F_3(\sigma,\, x) = x_3x_4 - \sigma$, $\bar{\sigma} = 0$, $\bar{x} = 0$. The mapping F is 2-regular in x at $(\bar{\sigma},\, \bar{x})$, for instance, with respect to $h = (0,\ 1,\ 1,\ 0) \in T(\bar{\sigma},\, \bar{x})$.

At the same time, $\omega_1(d) = -\infty\ \forall\, d \in \Sigma$, $d \neq 0$, but $\omega(\sigma) = 0\ \forall\, \sigma \in \Sigma$ for any choice of B.

Example 3 demonstrates that hypothesis (H1) does not guarantee the last estimate in Proposition 2.

Example 4. Let $s = 1$, $n = 3$, $l = 2$, $f(\sigma,\, x) = x_1 + x_2^2$, $F_1(\sigma,\, x) = x_1^2 + x_2^2 - x_3^2 - \sigma$, $F_2(\sigma,\, x) = x_1(x_1 - x_3 - x_3^p)$, $p \geq 3$ is an odd integer, $\bar{\sigma} = 0$, $\bar{x} = 0$. Here $D(\bar{\sigma}) = \{x \in X \mid x_1 = 0,\ x_2^2 = x_3^2\}$, and it is easy to see that the mapping F is 2-regular in x at $(\bar{\sigma},\, \bar{x})$ with respect to some $h \in T(\bar{\sigma},\, \bar{x})$, and QGC holds. At the same time, 2-regularity of F in x at $(\bar{\sigma},\, \bar{x})$ is violated, for example, with respect to $h = (1,\ 0,\ 1) \in T(\bar{\sigma},\, \bar{x})$.

It can be shown that for any choice of B and any $\sigma > 0$ small enough the set $S(\sigma)$ is a singleton $\{(x_1, 0, x_3)\}$, where

$$x_1 = -(\sigma/2)^{1/(p+1)} + o((\sigma/2)^{1/(p+1)}), \quad x_3 = -(\sigma/2)^{1/(p+1)} + o((\sigma/2)^{1/(p+1)}),$$

$$\omega(\sigma) = -(\sigma/2)^{1/(p+1)} + o((\sigma/2)^{1/(p+1)}).$$

Example 4 illustrates that QGC does not necessarily imply WSOSC, and that 2-regularity condition cannot be omitted in Theorem 10. Moreover, Examples 2–4 show that hypothesis (H1) does not imply that $\Lambda(\bar{\sigma}, \bar{x}) \neq \emptyset$.

Example 5. Let $s = 1, n = 2, l = 1, \alpha$ be a fixed real, $f(\sigma, x) = (x_1^2 + 2\alpha x_1 x_2 + x_2^2)/2$, $F(\sigma, x) = x_1 x_2 - \sigma$, $\bar{\sigma} = 0$, $\bar{x} = 0$. All the assumptions of Theorem 12 are satisfied here, and moreover, F is 2-regular in x at $(\bar{\sigma}, \bar{x})$ with respect to every direction $\xi \in X$ such that $\xi \neq 0$, and hence, with respect to every direction in $T(\bar{\sigma}, \bar{x}; d)$ $\forall d \in \Sigma$, $d \neq 0$.

It is easy to see that for any choice of B and any $\sigma \in \Sigma$ close enough to zero, for every point $x \in S(\sigma)$ it holds that

$$|x_1| = |x_2| = |\sigma|^{1/2}.$$

The second adjoint problem (22) coincides with the original problem when $d = \sigma$, and $\omega(\bar{\sigma} + td) = \omega_2(td) = \omega_2(d)t$ $\forall d \in \Sigma, \forall t \geq 0$. (Recall that $\omega_2(d)$ is defined in (23).) Moreover, if $|\alpha| \neq 0$ and $d \neq 0$, then $\omega_2(d) \neq 0$. Note that if $|\alpha| > 1$, then $\omega_2(d)$ can take both positive and negative values, depending on the signs of α and d. Hence, $\omega(\sigma)$ can grow, as well as decrease, under the perturbations of $\sigma \in \Sigma$.

Example 5 demonstrates that estimate (24) in Proposition 3, as well as the estimates in Theorem 12 and Proposition 5 cannot be sharpened.

Example 6. Let $s = l = 1$, $n = 2$, $f(\sigma, x) = |x_1|^p/p + |x_2|^q/q$, $p, q \geq 2$ be fixed integers, constraints be the same as in Example 5, $\bar{\sigma} = 0$, $\bar{x} = 0$. All the assumptions of Theorem 12 are satisfied here, except SOSC, which is violated when $\max\{p, q\} > 2$.

It is easy to see that for any choice of B, any $\sigma \in \Sigma$ and any $x \in S(\sigma)$ sufficiently close to zero it holds that

$$|x_1| = |\sigma|^{q/(p+q)}, \quad |x_2| = |\sigma|^{p/(p+q)}, \quad \omega(\sigma) = (1/p + 1/q)|\sigma|^{pq/(p+q)}.$$

In particular, if $p > 2$, $q = 2$, then estimate (35) does not hold. At the same time, estimate (36) holds for every $p, q \geq 2$.

Example 7. Let $s = 1$, $n = 3$, $l = 2$, $f(\sigma, x) = -\alpha x_1^2 + x_2^2 + x_3^2$, $\alpha > 1$ be a fixed real, constraints be the same as in Example 4, $\bar{\sigma} = 0$, $\bar{x} = 0$.

All the assumptions of Theorem 12 are satisfied here, except SOSC. It can be shown that for any choice of B both estimates (35) and (36) fail to hold here.

Examples 6, 7 demonstrate that QGC cannot be omitted in Theorem 10, and SOSC cannot be omitted in Theorem 12 and Proposition 5.

6. The chain problem

Consider the so-called chain problem (for the discussion of the underlying problem in statics see [7, section 2]). Let $s = 2$, $n = 2m$, $l = m+2$, $m \geq 2$, and for $(\sigma,\ x) \in \Sigma \times X$

$$f(\sigma,\ x) = \frac{1}{2}\sum_{i=1}^{m}\left(\sum_{j=1}^{i} v_j + \sum_{j=1}^{i-1} v_j\right) = \sum_{i=1}^{m} \alpha_i v_i,$$

where $\alpha_i = m - i + 1/2$, $i = 1,\ \ldots,\ m$;

$$F(\sigma,\ x) = (f_1(x),\ \ldots,\ f_m(x),\ g_1(\sigma_1,\ u),\ g_2(\sigma_2,\ v)),$$

$$f_i(x) = u_i^2 + v_i^2 - 1, \quad i = 1,\ \ldots,\ m,$$

$$g_1(\sigma_1,\ u) = \sum_{i=1}^{m} u_i - \sigma_1, \quad g_2(\sigma_2,\ v) = \sum_{i=1}^{m} v_i - \sigma_2.$$

Here $x = (u,\ v)$, $u,\ v \in \mathbf{R}^m$ (it is convenient to consider X as $\mathbf{R}^m \times \mathbf{R}^m$).

For $\bar{\sigma}^1 = (m,\ 0)$, we have $D(\bar{\sigma}^1) = \{\bar{x}^1\}$, where $\bar{x}^1 = ((1,\ \ldots,\ 1),\ 0)$ corresponds to the horizontal position of the chain. Another important parameter value is $\bar{\sigma}^2 = (0,\ m)$. With this value, $D(\bar{\sigma}^2) = \{\bar{x}^2\}$, where $\bar{x}^2 = (0,\ (1,\ \ldots,\ 1))$ corresponds to the vertical position of the chain.

Define the Lagrangian function

$$L(\sigma,\ x,\ \lambda,\ \mu_1,\ \mu_2) = \sum_{i=1}^{m} \alpha_i v_i + \sum_{i=1}^{m} \lambda_i (u_i^2 + v_i^2 - 1) + \mu_1 \left(\sum_{i=1}^{m} u_i - \sigma_1\right) + \mu_2 \left(\sum_{i=1}^{m} v_i - \sigma_2\right), \quad \lambda \in \mathbf{R}^m,\ \mu_1,\ \mu_2 \in \mathbf{R}.$$

For the horizontal position, by direct computations we obtain:

$$\Lambda(\bar{\sigma}^1,\ \bar{x}^1) = \left\{ (\lambda,\ \mu_1,\ \mu_2) \in \mathbf{R}^m \times \mathbf{R} \times \mathbf{R} \;\middle|\; \begin{array}{l} 2\lambda_i + \mu_1 = 0, \\ \alpha_i + \mu_2 = 0, \\ i = 1,\ \ldots,\ m \end{array} \right\} = \emptyset,$$

$$\ker \frac{\partial F}{\partial x}(\bar{\sigma}^1,\ \bar{x}^1) = \left\{ \xi = (0,\ \nu) \in \mathbf{R}^m \times \mathbf{R}^m \;\middle|\; \sum_{i=1}^{m} \nu_i = 0 \right\},$$

$$\operatorname{rank} \frac{\partial F}{\partial x}(\bar{\sigma}^1,\ \bar{x}^1) = l - 1,$$

$$\operatorname{im} \frac{\partial F}{\partial x}(\bar{\sigma}^1,\ \bar{x}^1) = \left\{ y = (\eta,\ \zeta_1,\ \zeta_2) \in \mathbf{R}^m \times \mathbf{R} \times \mathbf{R} \;\middle|\; \frac{1}{2}\sum_{i=1}^{m} \eta_i = \zeta_1 \right\},$$

$$\left(\operatorname{im} \frac{\partial F}{\partial x}(\bar{\sigma}^1,\, \bar{x}^1)\right)^{\perp} = \left\{ y = (\eta,\, \zeta_1,\, 0) \in \mathbf{R}^m \times \mathbf{R} \times \mathbf{R} \,\middle|\, \begin{array}{l} 2\eta_i + \zeta_1 = 0, \\ i = 1,\, \ldots,\, m \end{array} \right\},$$

$$Py = (m/4 + 1)^{-1}(p/2,\, \ldots,\, p/2,\, -p,\, 0),$$

$$p = p(\eta,\, \zeta_1) = \frac{1}{2}\sum_{i=1}^{m} \eta_i - \zeta_1, \quad y = (\eta,\, \zeta_1,\, \zeta_2) \in \mathbf{R}^m \times \mathbf{R} \times \mathbf{R}.$$

Furthermore, for an arbitrary $d \in \Sigma$

$$T(\bar{\sigma}^1,\, \bar{x}^1;\, d) = \left\{ \xi = (0,\, \nu) \in \mathbf{R}^m \times \mathbf{R}^m \,\middle|\, \sum_{i=1}^{m} \nu_i = 0,\; d_1 + \frac{1}{2}\sum_{i=1}^{m} \nu_i^2 = 0 \right\},$$

and in particular $T(\bar{\sigma}^1,\, \bar{x}^1) = T(\bar{\sigma}^1,\, \bar{x}^1;\, 0) = \{0\}$. Moreover, 2-regularity of F in x at $(\bar{\sigma}^1,\, \bar{x}^1)$ with respect to $\xi \in \mathbf{R}^m \times \mathbf{R}^m$ consists of saying that the linear map $x \to P\frac{\partial^2 F}{\partial x^2}(\bar{\sigma}^1,\, \bar{x}^1)[\xi,\, x] : \ker \frac{\partial F}{\partial x}(\bar{\sigma}^1,\, \bar{x}^1) \to (\operatorname{im} \frac{\partial F}{\partial x}(\bar{\sigma}^1,\, \bar{x}^1))^{\perp}$ is onto. For $\xi = (0,\, \nu)$, the latter reduces to the following condition: there exists $v \in \mathbf{R}^m$ such that $\sum_{i=1}^{m} v_i = 0$, but $\sum_{i=1}^{m} \nu_i v_i \neq 0$, which is always satisfied with e.g. $v = \nu$ provided $d_1 < 0$ and $\xi \in T(\bar{\sigma}^1,\, \bar{x}^1;\, d)$. Summarizing, we have:

- If $d_1 < 0$, then $T(\bar{\sigma}^1,\, \bar{x}^1;\, d) \neq \emptyset$, and F is 2-regular in x at $(\bar{\sigma}^1,\, \bar{x}^1)$ with respect to every $\xi \in T(\bar{\sigma}^1,\, \bar{x}^1;\, d)$.
- If $d_1 \geq 0$, $d \neq 0$, then $T(\bar{\sigma}^1,\, \bar{x}^1;\, d) = \emptyset$.

Take $d_1 = -1$. The first adjoint problem (20) takes the form

$$\begin{array}{l} \text{minimize } \sum_{i=1}^{m} \alpha_i \nu_i \\ \text{subject to } \xi = (0,\, \nu) \in \mathbf{R}^m \times \mathbf{R}^m :\; \sum_{i=1}^{m} \nu_i = 0,\; \frac{1}{2}\sum_{i=1}^{m} \nu_i^2 = 1. \end{array}$$

It can be easily seen that this problem has a unique solution $\bar{\xi} = (0,\, \bar{\nu}) \in \mathbf{R}^m \times \mathbf{R}^m$, and $\omega_1(d) < 0$. According to Proposition 4, for $t \geq 0$

$$\omega(\bar{\sigma}^1 + td) = \omega(\bar{\sigma}^1) + \omega_1(d)t^{1/2} + o(t^{1/2}),$$

and, for $\sigma = \bar{\sigma}^1 + td$, (4) has an $o(t^{1/2})$-solution of the form $\bar{x} + t^{1/2}\xi + o(t^{1/2})$ if and only if $\xi = \bar{\xi}$.

Next, we consider the vertical position of the chain:

$$\Lambda(\bar{\sigma}^2,\, \bar{x}^2) = \left\{ (\lambda,\, \mu_1,\, \mu_2) \in \mathbf{R}^m \times \mathbf{R} \times \mathbf{R} \,\middle|\, \begin{array}{c} \mu_1 = 0, \\ \alpha_i + 2\lambda_i + \mu_2 = 0, \\ i = 1,\, \ldots,\, m \end{array} \right\} \neq \emptyset,$$

$$\ker \frac{\partial F}{\partial x}(\bar{\sigma}^2,\, \bar{x}^2) = \left\{ \xi = (\nu,\, 0) \in \mathbf{R}^m \times \mathbf{R}^m \,\middle|\, \sum_{i=1}^{m} \nu_i = 0 \right\},$$

$$\operatorname{rank} \frac{\partial F}{\partial x}(\bar{\sigma}^2,\, \bar{x}^2) = l - 1,$$

$$\operatorname{im} \frac{\partial F}{\partial x}(\bar{\sigma}^2,\ \bar{x}^2) = \left\{ y = (\eta,\ \zeta_1,\ \zeta_2) \in \mathbf{R}^m \times \mathbf{R} \times \mathbf{R} \,\middle|\, \frac{1}{2}\sum_{i=1}^{m} \eta_i = \zeta_2 \right\},$$

$$\left(\operatorname{im} \frac{\partial F}{\partial x}(\bar{\sigma}^2,\ \bar{x}^2) \right)^{\perp} = \left\{ y = (\eta,\ 0,\ \zeta_2) \in \mathbf{R}^m \times \mathbf{R} \times \mathbf{R} \,\middle|\, \begin{array}{l} 2\eta_i + \zeta_2 = 0, \\ i = 1,\ \ldots,\ m \end{array} \right\},$$

$$Py = (m/4 + 1)^{-1}(p/2,\ \ldots,\ p/2,\ 0,\ -p),$$

$$p = p(\eta,\ \zeta_2) = \frac{1}{2}\sum_{i=1}^{m} \eta_i - \zeta_2, \quad y = (\eta,\ \zeta_1,\ \zeta_2) \in \mathbf{R}^m \times \mathbf{R} \times \mathbf{R}.$$

For an arbitrary $d \in \Sigma$

$$T(\bar{\sigma}^2,\ \bar{x}^2;\ d) = \left\{ \xi = (\nu,\ 0) \in \mathbf{R}^m \times \mathbf{R}^m \,\middle|\, \sum_{i=1}^{m} \nu_i = 0,\ d_2 + \frac{1}{2}\sum_{i=1}^{m} \nu_i^2 = 0 \right\},$$

and in particular $T(\bar{\sigma}^2,\ \bar{x}^2) = T(\bar{\sigma}^2,\ \bar{x}^2;\ 0) = \{0\}$. It can be seen that:

- If $d_2 < 0$, then $T(\bar{\sigma}^2,\ \bar{x}^2;\ d) \neq \emptyset$, and F is 2-regular in x at $(\bar{\sigma}^2,\ \bar{x}^2)$ with respect to every $\xi \in T(\bar{\sigma}^2,\ \bar{x}^2;\ d)$.
- If $d_2 \geq 0$, $d \neq 0$, then $T(\bar{\sigma}^2,\ \bar{x}^2;\ d) = \emptyset$.

Take $d_2 = -1$. After some manipulations, we come to the following form of the second adjoint problem (22):

$$\begin{array}{l} \text{minimize } -\frac{1}{2}\sum_{i=1}^{m} \alpha_i \nu_i^2 \\ \text{subject to } \xi = (\nu,\ 0) \in \mathbf{R}^m \times \mathbf{R}^m :\ \sum_{i=1}^{m} \nu_i = 0,\ \frac{1}{2}\sum_{i=1}^{m} \nu_i^2 = 1. \end{array}$$

It can be shown that this problem has two symmetric solutions $\pm\bar{\xi}$, $\bar{\xi} = (\bar{\nu},\ 0) \in \mathbf{R}^m \times \mathbf{R}^m$, and $\omega_2(d) < 0$. According to Proposition 6, for $t \geq 0$

$$\omega(\bar{\sigma}^2 + td) = \omega(\bar{\sigma}^2) + \omega_2(d)t + o(t),$$

and, for $\sigma = \bar{\sigma}^2 + td$, (4) has an $o(t)$-solution of the form $\bar{x} + t^{1/2}\xi + o(t^{1/2})$ if and only if $\xi = \pm\bar{\xi}$.

Our conclusions about the sensitivity properties of the chain problem are the same as in [7]. The advantage is that our approach makes it possible to treat this problem directly, without any preliminary transformation of constraints.

7. Concluding remarks

We presented the local sensitivity theory for abnormal equality-constrained optimization problems. In this section, we complete our discussion with some comments on the results obtained above.

The question is open if it is possible to obtain a quantitative directional lower bound on ω without QGC or stronger conditions. It is known that in the normal case, the answer is positive [14]. Closely related is the problem of stability of Lagrange multipliers in abnormal problems.

Very promising (though not obvious) is the possibility to extend the results on lower bounds and solution estimates to the case when $\Lambda(\bar{\sigma},\, \bar{x}) = \emptyset$ using second-order sufficient conditions based on 2-regularity theory [13], [3].

The assumption that F is three times differentiable in x at $(\bar{\sigma},\, \bar{x})$ is essential in Theorems 4, 7 and Proposition 2 only. In the rest of the paper, it suffices to assume that F is twice continuously differentiable in x near $(\bar{\sigma},\, \bar{x})$. Moreover, in some of the results above, smoothness assumptions can be further relaxed. In particular, in the case of finite-dimensional Y, it would be enough to assume that f and F are smooth with respect to the so-called finite topology in X, and the second-order sufficient conditions can also be relaxed accordingly (see [3, pp. 13–15]).

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