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TWO-DIMENSIONAL TOMOGRAPHY PROBLEMS AND THE THEORY OF *A*-ANALYTIC FUNCTIONS

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Abstract

We reduce the inverse problem of finding the right-hand side of the stationary one-velocity transport equation to the boundary value problem for an elliptic equation with operator coefficients on the plane. Particular cases of these inverse problems are the problem of inverting the Radon transform in the fan-beam statement and the problem of emission tomography (the Radon problem with absorption). We present the Cauchy- and Poisson-type integral formulas for solutions to the corresponding boundary value problems; in the case of incomplete data an analog of the Carleman-type formula is given.

Key words and phrases: inverse problems, emission tomography problem.

In the present article, we study the connection between the inverse problems of finding the right-hand side of the stationary one-velocity transport equation and the theory of elliptic equations with operator coefficients on the plane. Particular cases of such inverse problems are the problem of inverting the Radon transform in the fan-beam statement and the emission tomography problem (the Radon problem with absorption). The productivity of the complex interpretation of the planar tomography problems was first observed in the monograph [4]. The systematic study of the tomography problems from this viewpoint was initiated in [1, 5–7]. In the present article, we expose some new results in this field. The article comprises four sections. In Section 1, we describe reduction of the inverse problems of finding

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the right-hand side of the transport equation to the boundary value problems for elliptic equations with operator coefficients. In particular, we show that the differential statement of the Radon problem reduces to the Cauchy problem for the Beltrami-type operator equation

$$\overline{\partial}_A \mathbf{u} = \overline{\partial}\mathbf{u} - A\partial\mathbf{u} = \mathbf{0}.$$

Thus, the inversion formula for the Radon transform amounts to the A-analog of the Cauchy formula for the operator $\overline{\partial}_A$. If we know the integrals of the sought function over all straight lines passing through a given set M then we naturally arrive at using the operator analog of the Carleman-type formula. It was demonstrated in [1] that the straight line tomography problem with single reflection from the boundary reduces to the Riemann–Hilbert boundary value problem and, in the case of a half-plane, to the Dirichlet problem for the operator $\Delta_A = 4\overline{\partial}_A\partial_A$ whose solution is given by the A-analog of the Poisson formula. The corresponding Cauchy and Poisson formulas are presented in Section 2. The Carleman-type formula is derived in Section 3. In Section 4, using the analogs of the representation theorems, we prove that more complicated tomography problems with variable absorption reduce to the corresponding problems without absorption.

1. Reduction of inverse problems to boundary value problems

Let Ω be a simply connected open set in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Consider the following stationary one-velocity transport equation in the domain Ω:

$$\begin{aligned} &u_x(x,y,\varphi)\cos\varphi + u_y(x,y,\varphi)\sin\varphi + \mu(x,y)u(x,y,\varphi) \\ &\quad - \frac{1}{2\pi}\int_{-\pi}^{\pi} \gamma(x,y,\cos(\varphi-\varphi'))u(x,y,\varphi')d\varphi' = a(x,y). \end{aligned} \tag{1.1}$$

As is well known, the transport equation has the following physical meaning: the function $u(x,y,\varphi)$ is the density of particles at a point (x,y) moving in the direction $\nu = \{\cos\varphi, \sin\varphi\}$; the function $a(x,y)$ is the density of particle sources in the domain Ω, and the functions $\mu(x,y)$ and $\gamma(x,y,\cos\varphi)$ characterize the absorption and scattering properties of the medium. The function $\mu(x,y) \geq 0$ is called the *absorption* and the function $\gamma(x,y,\cos\varphi)$, the ***dispersion index***. If the source function $a(x,y)$ is known then, to guarantee uniqueness of a solution to the direct problem, we need to know the incident flow

$$u\big|_{\Sigma_-} = f_-,$$

where $\Sigma = \partial\Omega \times [-\pi,\pi]$, $\Sigma_+ = \{(x,y,\varphi) \in \Sigma : \langle n,\nu\rangle \geq 0)\}$, $\Sigma_- = \Sigma \setminus \Sigma_+$, n is the outward unit normal of $\partial\Omega$, and $\langle\,,\rangle$ stands for the inner product in $\mathbb{R}^2$.

In particular, the absence of the particle flow incident from the exterior of the domain Ω means that $f_- \equiv 0$.

The basic object of our study is the inverse problem of finding the right-hand side $a(x,y)$, provided that we know the outgoing particle flow for the equation (1.1) in the domain Ω; i.e., the values of $u(x,y,\varphi)$ on the manifold Σ_+,

$$u\big|_{\Sigma_+} = f_+,$$

or some linear functionals of $f_\pm$. Considering such a statement of the inverse problem, we henceforth assume that we simply know the trace of the function u on the whole boundary Σ,

$$u\big|_{\Sigma} = f, \tag{1.2}$$

or the corresponding functionals of f.

We now turn to the complex interpretation of the problem (1.1), (1.2). To this end, we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ by putting $z = x + iy$, $i^2 = -1$. Passing to the complex variable z, we preserve the former notations for functions; i.e., we put $a(z) = a(z,\bar{z}) = a(x,y)$, $u(z,\varphi) = u(x,y,\varphi)$, etc. Defining the operators

$$\partial u = \frac{\partial u}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right), \quad \bar{\partial} u = \frac{\partial u}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right),$$

we rewrite the problem (1.1), (1.2) in the complex form

$$\begin{aligned} & u_{\bar{z}}(z,\varphi)e^{-i\varphi} + u_z(z,\varphi)e^{i\varphi} + \mu(z)u(z,\varphi) \\ & \qquad - \frac{1}{2\pi}\int_{-\pi}^{\pi} \gamma\big[z,\cos(\varphi-\varphi')\big]u(z,\varphi')\,d\varphi' = a(z), \\ & \qquad z \in \Omega, \quad \varphi \in [-\pi,\pi]; \end{aligned} \tag{1.3}$$

$$u\big|_{\Sigma} = f(t,\varphi). \tag{1.4}$$

Now, expanding the functions u, f, and γ in Fourier series in the variable φ and recalling that all functions are real-valued, we obtain

$$u(z,\varphi) = u_0(z) + 2\,\mathrm{Re}\left\{\sum_{k=1}^{\infty} u_k(z)e^{-ik\varphi}\right\}, \tag{1.5}$$

$$f(z,\varphi) = f_0(z) + 2\,\mathrm{Re}\left\{\sum_{k=1}^{\infty} f_k(z)e^{-ik\varphi}\right\}, \tag{1.6}$$

$$\gamma(z,\cos\varphi) = \gamma_0(z) + \sum_{k=1}^{\infty} \gamma_k(z)\big(e^{-ik\varphi} + e^{ik\varphi}\big), \tag{1.7}$$

where $u_0(z)$, $f_0(z)$, and $\gamma_k(z)$, $k = 0, 1, 2, \ldots$, are real-valued functions. Inserting the expansions (1.5)–(1.7) into (1.3) and making the corresponding transformations, we obtain the following countable system of first-order differential equations in the Fourier coefficients $\{u_k(z)\}_{k=0}^{\infty}$:

$$(u_k)_{\bar z} + (u_{k+2})_z + \mu(z)u_{k+1} = \gamma_{k+1}u_{k+1}, \quad k = 0, 1, 2, \ldots . \tag{1.8}$$

Moreover, the right-hand side $a(z)$ is determined by the formula

$$a(z) = 2\,\mathrm{Re}\{(u_1)_z\} + \mu(z)u_0(z) - \gamma_0(z)u_0(z). \tag{1.9}$$

Make the so-obtained Fourier coefficients into the vectors

$$\mathbf{u}(z) = \{u_0(z), u_1(z), \ldots\}, \quad \mathbf{f}(z) = \{f_0(z), f_1(z), \ldots\}$$

which we consider, for definiteness, in the Hilbert space $X = l_2$ constituted by complex-valued vectors $\mathbf{u} = \{u_0, u_1, \ldots\}$, $u_j \in \mathbb{C}$, with the norm

$$\|\mathbf{u}\|^2 = \sum_{j=0}^{\infty} |u_j|^2.$$

Consider the right shift U in the space X:

$$U \colon \{u_0, u_1, \ldots\} \to \{0, u_0, u_1, \ldots\}.$$

Then the adjoint operator U^* in l_2 is the left shift

$$U^* \colon \{u_0, u_1, \ldots\} \to \{u_1, u_2, \ldots\}.$$

Moreover, introduce the weighted left shift

$$\Gamma(z) \colon \{u_0, u_1, \ldots\} \to \{\gamma_1 u_1, \gamma_2 u_2, \gamma_3 u_3, \ldots\}.$$

Using the above notations, we can rewrite the system (1.8) in operator form

$$\mathbf{u}_{\bar z} + U^*U^*\mathbf{u}_z + \mu U^*\mathbf{u} - \Gamma(z)\mathbf{u} = \mathbf{0}.$$

Putting, for brevity,

$$A = -U^*U^*, \quad \overline{\partial}_A = \overline{\partial} - A\partial, \quad A_0(z) = \mu(z)U^* - \Gamma(z),$$

we obtain the following generalized Beltrami-type equation with operator coefficients:

$$\overline{\partial}_A \mathbf{u} + A_0(z)\mathbf{u} = 0, \quad z \in \Omega. \tag{1.10}$$

The data (1.4) transform into the Cauchy data

$$\mathbf{u}\big|_{\partial\Omega} = \mathbf{f}; \tag{1.11}$$

consequently, to solve the original inverse problem (1.3), (1.4), it suffices to obtain an analog of the Cauchy formula for (1.10); afterwards, the right-hand side $a(z)$ is determined by (1.9).

The main result of the present article is the Cauchy formula for the operator in (1.10) with $A_0(z) = \mu(z)B$ (the case $[A, B] = 0$ is considered in Section 4 (Theorem 4.3) under the assumption that $\mu(z) \in C^2(\overline{\Omega})$, with Ω a strictly convex domain with smooth boundary). The main difficulty in studying (1.10) is that, unlike the classical Beltrami equation, in our case the equality $\|A\| = 1$ holds not only in the space l_2 but also in the whole scale of the spaces l_2^m with the norm

$$\|\mathbf{u}\|_m^2 = \sum_{j=0}^{\infty}(1+j)^{2m}|u_j|^2, \quad m > 0.$$

It is well known (see, for instance, [14]) that the resolvent $R(\lambda) = (A - \lambda E)^{-1}$ of the operator $A = U^*$, as a bounded operator in l_2^m for $|\lambda| > 1$, extends to the unit circle as a bounded operator from l_2^{m+1} into l_2^m; i.e., $R(\lambda) \in L(l_2^{m+1}, l_2^m)$ for $|\lambda| = 1$ and $m > -1/2$; moreover, $R(\lambda)$ is strongly continuous in λ for $m > 1/2$. In Sections 2–4, we use this property to study the equation like (1.10) in the abstract situation by postulating the indicated property of the resolvent of the operator A in an arbitrary discrete scale of Banach spaces (Condition A). In the derivation of the Carleman formulas for A-analytic functions, we have to require that $\|A\| < 1$. As regards the inverse problem (1.3), (1.4), we can fulfill this condition by considering the operator $A = U^*$ in the space $l_{2,s}$ with the norm

$$\|\mathbf{u}\|_s^2 = \sum_{j=0}^{\infty} s^{-2j}|u_j|^2, \quad s \in (0,1).$$

In this case we have $\|A\|_s \le s < 1$.

2. The Cauchy and Poisson formulas

Suppose that X is a complex Banach space and we are given a chain $X^m : X^{m+1} \subseteq X^m \subseteq X^0 = X$ of Banach spaces densely embedded into X, where $m \ge 0$ is integer:

$$\|\mathbf{u}\|_{X^m} = \|\mathbf{u}\|_m \le \|\mathbf{u}\|_{m+1} \text{ for all } \mathbf{u} \in X^{m+1}; \quad \|\mathbf{u}\|_0 = \|\mathbf{u}\|.$$

We state the following condition on the operator A:

Condition A. Suppose that $A \in \mathcal{L}(X^m)$ for each $m \ge 0$, the operator $A\big|_{X^m}$ has spectral radius $\rho\big(A\big|_{X^m}\big)$ equal to 1, and the resolvent $R(\lambda)$ extends by strong continuity in X^m to the circle $|\lambda| = 1$ as an operator in $\mathcal{L}(X^{m+1}, X^m)$.

Here $\mathcal{L}(X)$ is the space of bounded linear operators in X, $\mathcal{L}(X^{m+1};X^m)$ is the space of bounded linear operators acting from X^{m+1} into X^m, and $A\big|_{X^m}$ is the restriction of the operator A onto the subspace X^m.

The first part of Condition A implies that the resolvent $R(\lambda) = (A-\lambda E)^{-1}$ of the operator A belongs to the space $\mathcal{L}(X^m)$ for $\lambda > 1$ and is an analytic operator function outside the unit disk. The second part of Condition A implies existence of the following operator function for $z \neq 0$:

$$K(z) = (\bar{z})^{-1}R\big(-e^{2i\varphi}\big) = (\bar{z})^{-1}\big(A + e^{2i\varphi}E\big)^{-1} = (zE + \bar{z}A)^{-1}, \quad \varphi = \arg(z),$$

with values in the space $\mathcal{L}(X^{m+1}, X^m)$.

The basic properties of the operator function $K(z)$ are stated in the following theorem:

Theorem 2.1. *Suppose that the operator A satisfies Condition* A *in the discrete scale $\{X^m\}_{m=0}^{\infty}$. Then the following assertions are valid:*

1) $K(z) \in \mathcal{L}\big(X^{m+1}, X^m\big)$ *and* $\|K(z)\|_{\mathcal{L}(X^{m+1},X^m)} \leq \frac{C_m}{|z|}$, $z \neq 0$;

2) $K(z_1)K(z_2) = K(z_2)K(z_1) \in \mathcal{L}\big(X^{m+1}, X^m\big)$, $z_1 \neq z_2$;

3) $K(z_1)K(z_2) = K(z_2 - z_1)\big[K(z_1) - K(z_2)\big]$, $z_1 \neq z_2$;

4) *the strongly continuous derivatives $K_{\bar{z}}$ and K_z belong to $\mathcal{L}\big(X^{m+2}, X^m\big)$ for $z \neq 0$ and the equality*

$$K_{\bar{z}} - AK_z = 0$$

holds. Moreover, for $z \neq 0$, the highest-order derivatives have the form

$$K_z^{(n)} = (-1)^n n! K^{n+1}, \quad K_{\bar{z}}^{(n)} = (-1)^n n! A^n K^{n+1} \in \mathcal{L}\big(X^{m+n+1}, X^m\big).$$

Remark 1. We fulfill Condition A for a given operator A by choosing the corresponding scale $\{X^m\}_{m=0}^{\infty}$ of Banach spaces. Once such a scale is already found for the operator A, it is the same for the operator A^2; i.e., the operator $(zE + \bar{z}A^2)^{-1} \in \mathcal{L}\big(X^{m+1}, X^m\big)$ is defined. This follows from assertion 2) of Theorem 2.1.

Suppose that Ω is an open set in the complex plane $\mathbb{C}$ and a function $\mathbf{u}\colon \Omega \to X$ with values in a complex Banach space X belongs to the class $C^1(\Omega; X)$, where $C^k(\Omega; X)$ is the space of k times strongly continuously differentiable functions with values in X.

Definition. A function $\mathbf{u}(z) \in C^1(\Omega; X)$ is called *A-analytic* in the domain Ω if the equality

$$\mathbf{u}_{\bar{z}} - A\mathbf{u}_z = 0 \tag{2.1}$$

is valid for all $z \in \Omega$. Denote the set of all A-analytic functions $\mathbf{u}\colon \Omega \to X^m$ by the symbol $A(\Omega; X^m)$.

As an example of an A-analytic function in the domain $\mathbb{C}\setminus\{0\}$ we may take the function $\mathbf{u}(z) = K(z)\mathbf{h}$, where $\mathbf{h}$ is some fixed vector in the space X^{m+1} and $K(z)$ is the operator function of Theorem 2.1.

From Theorem 2.1 and the Stokes formula we derive the following theorem:

Theorem 2.2. *Assume that* $\mathbf{u}(z) \in C^1(\overline{\Omega}; X^{m+1})$. *Then, for all* $\zeta \in \Omega$,

$$\mathbf{u}(\zeta) = \frac{1}{2\pi i}\int_\Gamma K(z-\zeta)(dz + Ad\bar{z})\mathbf{u}(z) - \frac{1}{\pi}\int_\Omega K(z-\zeta)(\mathbf{u}_{\bar{z}} - A\mathbf{u}_z)\,dx\,dy.$$

This theorem yields an analog of the Cauchy-type integral formula for A-analytic functions.

Theorem 2.3. *Suppose that* $\mathbf{u}(z) \in C(\overline{\Omega}; X^{m+1}) \cap A(\Omega; X^m)$. *Then, for all* $\zeta \in \Omega$,

$$\mathbf{u}(\zeta) = \frac{1}{2\pi i}\int_\Gamma K(z-\zeta)\big(dz + Ad\bar{z}\big)\mathbf{u}(z).$$

Detailed proofs of Theorems 2.2 and 2.3 are given in [5, 7].

To obtain analogs of the Poisson and Cauchy formulas in the half-plane $\Pi = \{\operatorname{Im} z > 0\}$, we suppose, for simplicity, that X is the complexification of the real Hilbert space X'; i.e., $X = X' \oplus iX'$. Respectively, the scale $\{X^m\}$ consists of Hilbert spaces too; moreover, alongside Condition A, we suppose that the operator A is real; i.e., $AX' \subset X'$.

Definition. The vector-functions $\mathbf{v}(x,y) \in C^2(\Pi; X)$ such that

$$\Delta_A \mathbf{v}(x,y) = (E-A)^2\partial_{xx}\mathbf{v}(x,y) + (E+A)^2\partial_{yy}\mathbf{v}(x,y) = 0$$

are called *A-harmonic* in the half-plane. We denote the set of such functions by $h_A(\Pi; X)$.

Consider the operator function

$$P_{A,z}(t) = P_{A,y}(x-t) = \frac{1}{\pi}\frac{y}{(x-t)^2+y^2}(E-A^2)(\mu-A)^{-1}(\bar{\mu}-A)^{-1},$$

where $z = x+iy$ and $\mu = \frac{y-i(x-t)}{y+i(x-t)}$. Using this function, we define the A-harmonic Poisson integral by the formula

$$\mathbf{v}(z) = \int_{-\infty}^{\infty} P_{A,y}(x-t)\mathbf{g}(t)\,dt = (P_{A,y} * \mathbf{g})(x). \tag{2.2}$$

The immediate calculations carried out in [5] show that $P_{A,z}(t)$, with $t \in \mathbb{R}$ fixed, is an A-harmonic function of z in Π. Moreover,

$$P_{A,z}(t) \in \mathcal{L}(X^{m+2}, X^m)$$

and

$$\|P_{A,z}(t)\mathbf{e}\|_{X^m} \leqslant \frac{c(z,m)}{1+t^2}\|\mathbf{e}\|_{X^{m+2}}$$

for all $\mathbf{e} \in X^{m+2}$. Consequently, $P_{A,z}(t) \in L_q(\mathbb{R}, \mathcal{L}(X^{m+2}, X^m))$, $1 \leqslant q \leqslant \infty$, and $\mathbf{v}(z) = (P_{A,y} * \mathbf{g})(x) \in h_A(\Pi; X^m)$ for $\mathbf{g} \in L_p(\mathbb{R}, X^{m+2})$, $1 \leqslant p \leqslant \infty$.

The following assertions are valid for the A-harmonic Poisson integral (2.2):

Theorem 2.4. a) *Suppose that* $\mathbf{g}(x) \in L_\infty(\mathbb{R}; X^{m+2})$ *and* x_0 *is a point of continuity of* $\mathbf{g}(x)$. *Then* $(P_{A,y} * \mathbf{g})(x) \to \mathbf{g}(x_0)$ *in* X^m *as* $z = x + iy \to x_0$.

b) *Suppose that* $\mathbf{g}(x) \in L_p(\mathbb{R}; X^{m+2})$, $1 \leqslant p < \infty$. *Then*

$$\|P_{A,y} * \mathbf{g} - \mathbf{g}\|_{L_p(\mathbb{R}; X^m)} \to 0 \quad \text{as} \quad y \to 0.$$

Remark 2. If the function $\mathbf{g}(x)$ is uniformly continuous and bounded on $\mathbb{R}$ then the convergence is uniform.

Corollary. a) *Suppose that a function* $\mathbf{g}(x)$ *with values in* X^{m+2} *is bounded and uniformly continuous on* $\mathbb{R}$ *and*

$$\mathbf{v}(x,y) = \begin{cases} (P_{A,y} * \mathbf{g})(x) & \text{for } y > 0, \\ \mathbf{g}(x) & \text{for } y = 0. \end{cases}$$

Then $\mathbf{v}(x,y) \in h_A(\Pi; X^m) \cap C(\overline{\Pi}; X^m)$.

b) *If* $\mathbf{v}(x,y) \in h_A(\Pi; X^{m+2}) \cap C(\overline{\Pi}; X^{m+2})$ *then* $\mathbf{v}(x,y)$ *is representable as the A-harmonic Poisson integral of its boundary values:*

$$\mathbf{v}(x,y) = \int_{-\infty}^{\infty} P_{A,y}(x-t)\mathbf{v}(t)\,dt.$$

The following analog of Fatou's theorem holds for A-harmonic functions in the upper half-plane:

Theorem 2.5. *Suppose that* $\mathbf{v}(z) \in h_A(\Pi; X^{m+2})$ *and*

$$\sup_y \int_{-\infty}^{\infty} \|\mathbf{v}(x+iy)\|^p_{X^{m+2}}\,dx < \infty$$

for some $p \in [1, \infty]$.

Then the following limit over nontangent directions exists for almost all $t \in \mathbb{R}$:

$$\lim_{\substack{z\to t \\ z \in \Gamma_\alpha(t)}} \mathbf{v}(z) = \mathbf{g}(t) \in L_p(\mathbb{R}; X^{m+2}),$$

where $\Gamma_\alpha(t) = \{z = x + iy : |x - t| < c(\alpha)y\}$.

If $p > 1$ *then* $\mathbf{v}(z) = (P_{A,y} * \mathbf{g})(x)$ *and*

$$\lim_{y\to 0} \|\mathbf{v}(x+iy) - \mathbf{g}(x)\|_{L_p(\mathbb{R}; X^m)} = 0$$

for every $1 < p < \infty$.

If $p = 1$ then $\mathbf{v}(z) = (P_{A,y} * \omega)(x)$, where ω is the finite measure on $\mathbb{R}$ connected with the boundary values of $\mathbf{g}(t)$ by the formula $d\omega = \mathbf{g}(t)\,dt + d\nu$ and $d\nu$ is a measure taking values in the space X^{m+2} and concentrated on a set of Lebesgue measure zero.

The proofs of all the above assertions proceed by the same scheme as in [9, 10] in the classical case of $A = 0$. Moreover, if $\|A\| < 1$ and the initial function $\mathbf{v}(z)$ is A-analytic then the Poisson formula is also valid in the case $p = 1$. A precise statement of this assertion is given below.

We say that a vector-function $\mathbf{u}(z)$ belongs to $H_A^p(\Pi; X)$, $1 \leqslant p < \infty$, if $\mathbf{u}(z) \in A(\Pi; X)$ and

$$\|\mathbf{u}\|_{H_A^p}^p = \sup_y \int_{-\infty}^{\infty} \|\mathbf{u}(x+iy)\|_X^p \, dx < \infty.$$

The space $H_A^\infty(\Pi; X)$ is defined to be the space of bounded A-analytic functions in Π with the norm

$$\|\mathbf{u}\|_\infty = \sup_y \|\mathbf{u}(\cdot + iy)\|_{L_\infty(R;X)}\cdot$$

Theorem 2.6. a) *Suppose that $\mathbf{u}(z) \in H_A^p(\Pi; X^{m+2})$, $1 < p \leqslant \infty$. Then the following limit over nontangent directions exists for almost all $t \in \mathbb{R}$:*

$$\lim_{z \not\to t} \mathbf{u}(z) = \mathbf{f}(t) \in L_p(\mathbb{R}; X^{m+2})$$

and $\mathbf{u}(z) = (P_{A,y} * \mathbf{f})(x)$. For $p \in (0, \infty)$, we have

$$\|\mathbf{u}(x+iy) - \mathbf{f}(x)\|_{L_p(\mathbb{R};X^m)} \to 0$$

as $y \to 0$.

b) *Suppose that $\mathbf{u}(z) \in H_A^1(\Pi; X)$ and $\|A\| < 1$. Then the following limit over nontangent directions exists for almost all $t \in \mathbb{R}$:*

$$\lim_{z \not\to t} \mathbf{u}(z) = \mathbf{f}(t) \in L_1(\mathbb{R}; X),$$

$\mathbf{u}(z) = (P_{A,y} * \mathbf{f})(x)$, and $\|\mathbf{u}(x+iy) - \mathbf{f}(x)\|_{L_1(\mathbb{R};X)} \to 0$ as $y \to 0$.

Moreover, for such functions, we can also obtain an analog of the Cauchy formula.

Theorem 2.7. a) *Suppose that $\mathbf{u}(z) \in H_A^p(\Pi; X^{m+1})$ and $1 < p < \infty$. Then*

$$\mathbf{u}(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} (E + A) K(t - z) \mathbf{u}(t)\, dt, \tag{2.3}$$

$$0 = \int_{-\infty}^{\infty} (E + A) K(t - \bar{z}) \mathbf{u}(t)\, dt, \tag{2.4}$$

where $K(z) = (z + \bar{z}A)^{-1}$.

b) *Suppose that* $\mathbf{u}(z) \in H^1_A(\Pi; X)$ *and* $\|A\| < 1$. *Then formulas* (2.3) *and* (2.4) *are valid.*

3. The Carleman formulas for A-analytic functions in the half-plane

Suppose that $\mathbf{u}(z)\colon \overline{\Pi} \to X$ is an A-analytic function and we know the values of $\mathbf{u}(z)$ on some set $M \subset \partial\Pi$ of positive Lebesgue measure. If $\|A\| < 1$ then we can obtain some formulas which enable us to find $\mathbf{u}(z)$ in the whole half-plane Π from its values on the set M. To this end, as in the classical case $A = 0$ (see [2]), we construct a weight "quenching" function $\Phi(z)$ (in our case, it is an operator function) whose norm is small on $\partial\Pi \setminus M$ and is large on M.

Theorem 3.1. *Suppose that* $\|A\| < 1$ *and*

$$\psi(z, A) = \frac{1}{\pi} \int_M \frac{y}{(x-t)^2 + y^2} (E - A^2)(\mu - A)^{-1}(\bar{\mu} - A)^{-1}\, dt,$$

$$\chi(z, A) = \frac{1}{\pi} \int_M \left(\frac{x - t}{(x-t)^2 + y^2} (E + A)^2 (\mu - A)^{-1}(\bar{\mu} - A)^{-1} + \frac{t}{1 + t^2} \right) dt,$$

$$\Phi(z) = \exp\bigl[\psi(z, A) + i\chi(z, A)\bigr], \tag{3.1}$$

where the operator exponential function is defined by the formula $e^U = \sum_{k=0}^{\infty} \frac{1}{k!} U^k$. *Then the function* $\Phi(z) \in \mathcal{L}(X)$ *possesses the following properties:*

1) $\overline{\partial}_A \Phi(z) = 0$ *in* Π;
2) $\bigl[\Phi(z), \Phi(\zeta)\bigr] = 0$;
3) $\bigl[\Phi(z), A\bigr] = 0$;
4) $\Phi^{-1}(z)$ *is a continuous linear operator for all* $z \in \Pi$;
5) $\bigl\|\Phi^{-1}(z)\Phi(t)\bigr\| < 1$ *for all* $z \in \Pi$ *and* $t \in \partial\Pi \setminus M$.

Using the analogs of the Cauchy and Poisson formulas obtained in Theorems 2.6 and 2.7 and the above-constructed quenching function $\Phi(z)$, we obtain some Carleman-type formulas which yield a solution to the problem of A-analytic continuation.

Theorem 3.2. *Suppose that* $\|A\| < 1$, $\mathbf{u}(z) \in H^p_A(\Pi; X)$, $1 \leqslant p < \infty$, *and the function* $\Phi(z)$ *is defined by* (3.1). *Then*

$$\mathbf{u}(z) = \lim_{n\to\infty} \frac{1}{2\pi i} \int_M (E + A) K(t - z) \bigl[\Phi^{-1}(z)\Phi(t)\bigr]^n \mathbf{u}(t)\, dt \tag{3.2}$$

for all $z \in \Pi$.

Theorem 3.3. *Suppose that* $\|A\| < 1$, $\mathbf{u}(z) \in H_A^p(\Pi; X)$, $1 \leqslant p \leqslant \infty$, *and the function* $\Phi(z)$ *is defined by* (3.1). *Then*

$$\mathbf{u}(z) = \lim_{n\to\infty} \frac{1}{\pi} \int_M \frac{y}{(x-t)^2+y^2}(E-A^2)(\mu-A)^{-1}(\bar{\mu}-A)^{-1}\left[\Phi^{-1}(z)\Phi(t)\right]^n \mathbf{u}(t)\,dt \tag{3.3}$$

for all $z \in \Pi$.

The above results readily yield the following theorem which is an analog of the second part of F. and M. Rieszs' theorem:

Theorem 3.4. *Suppose that* $\|A\| < 1$, $\mathbf{u}(z) \in H_A^p(\Pi; X)$, $1 \leqslant p \leqslant \infty$, *and* $\mathbf{u}(t) = 0$ *on some set* $M \subset \partial\Pi$ *of positive Lebesgue measure. Then* $\mathbf{u}(z) \equiv 0$.

The Carleman-type formulas (3.2) and (3.3) enable us to obtain a conditional stability estimate for the problem of A-analytic continuation from a part of the boundary.

Theorem 3.5. *Suppose that* $\|A\| = s < 1$, $M \subset \partial\Pi$ *is a set of positive Lebesgue measure,* $\mathbf{u}(z) \in H_A^p(\Pi; X)$, $1 \leqslant p \leqslant \infty$, *and* $l(\mathbf{u}) = \|\mathbf{u}\|_{L_p(\partial\Pi\setminus M;X)}$. *Then*

$$\|\mathbf{u}(z)\|_X \leqslant \varepsilon l(\mathbf{u}) + c(\varepsilon)\|\mathbf{u}\|_{L_p(M;X)}$$

for all $z = x + iy \in \Pi$ *and all* $\varepsilon > 0$, *where* $c(\varepsilon) = e\big(c_{s,p}(y)\big)^{1/\gamma}\varepsilon^{1-1/\gamma}$, $c_{s,p}(y)$ *is a constant (depending on* s, p, y), *and* $\gamma = \frac{1-s}{\pi}\int_M \frac{y}{(x-t)^2+y^2}\,dt$.

A. P. Soldatov [2] demonstrated that the Bitsadze representation [17] for solutions to second-order elliptic systems with constant coefficients

$$\mathcal{A}\partial_{xx}V(x,y) + \mathcal{B}\partial_{xy}V(x,y) + \mathcal{C}\partial_{yy}V(x,y) = 0,$$

with $V = (V_1,\dots,V_n)$, can be written as $V(x,y) = \operatorname{Re}\Theta\mathbf{u}(z)$, where $\mathbf{u}(z)$ is an A-analytic function and Θ and A are matrices expressible in terms of the coefficients of the system. Thus, formulas (3.2) and (3.3) may also be applied for solving the Cauchy problems for second-order elliptic systems in a half-plane. As an example, we consider the Cauchy problem for the Lamé system of equations,

$$\mu\Delta V + (\lambda+\mu)\operatorname{grad}\operatorname{div} V = 0, \tag{3.4}$$

$$V\big|_M = G(x), \tag{3.5}$$

$$T_\partial V\big|_M = H(x), \tag{3.6}$$

where λ and μ are positive constants, $V = (V_1, V_2)$ is the displacement vector, T_∂ is the stress operator defined by the equality

$$T_\partial = \begin{pmatrix} (\lambda+2\mu)\nu_1 & \mu\nu_2 \\ \lambda\nu_2 & \mu\nu_1 \end{pmatrix}\partial_x + \begin{pmatrix} \mu\nu_2 & \lambda\nu_1 \\ \mu\nu_1 & (\lambda+2\mu)\nu_2 \end{pmatrix}\partial_y,$$

and $\nu = (\nu_1, \nu_2)$ is the outward unit normal.

The following theorem is valid:

Theorem 3.6. *Suppose that a function $V(x,y) \in C^2(\Pi;\mathbb{R}^2) \cap C^1(\overline{\Pi};\mathbb{R}^2)$ is a solution to the problem (3.4)–(3.6) and possesses the property*

$$V(x,0) \in L_p(\partial\Pi;\mathbb{R}^2), \quad \int_0^x (T_\partial V)(t,0)\,dt \in L_p(\partial\Pi;\mathbb{R}^2).$$

Then $V(x,y)$ is determined by the formula

$$V = \operatorname{Re}\Theta\mathbf{u},$$

where

$$\Theta = \begin{pmatrix} i & i \\ -1 & 2\varkappa - 1 \end{pmatrix}, \quad \varkappa = \frac{\lambda + 3\mu}{\lambda + \mu},$$

and $\mathbf{u}(z)$ is a function of the class $H_A^p(\Pi;\mathbb{R}^2)$, with $A = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$, $1 \leqslant p \leqslant \infty$, which takes the values $\mathbf{u}\big|_M = \mathbf{f} = \mathbf{g} + i\mathbf{h}$ on the set M, with the functions $\mathbf{g}(t)$ and $\mathbf{h}(t)$ determined by the relations

$$\operatorname{Re}\Theta(\mathbf{g} + i\mathbf{h})(t) = G(t),$$
$$\operatorname{Re}\Theta'(\mathbf{g} + i\mathbf{h})(t) = \int_{t_1}^t H(\tau)d\tau, \quad \Theta' = \begin{pmatrix} 1 & 2 - \varkappa \\ i & -i\varkappa \end{pmatrix}.$$

The values of the function $\mathbf{u}(z)$, $z = x + iy \in \Pi$, are determined by the following Carleman-type formulas:

$$\mathbf{u}(z) = \frac{1}{\pi} \lim_{n\to\infty} e^{-n\varphi(z)} \int_{t_1}^{t_2} \frac{e^{n\varphi(t)}}{(x-t)^2 + y^2} \begin{pmatrix} 1 & -n\varphi_0(z) + 2\frac{(x-t)^2 - y^2}{(x-t)^2 + y^2} \\ 0 & 1 \end{pmatrix} \mathbf{f}(t)\,dt \quad (3.7)$$

for $1 \leqslant p \leqslant \infty$ and

$$\mathbf{u}(z) = \frac{1}{2\pi i} \lim_{n\to\infty} e^{-n\varphi(z)} \int_{t_1}^{t_2} \frac{e^{n\varphi(t)}}{t - z} \begin{pmatrix} 1 & -n\varphi_0(z) + 2\frac{iy}{t-z} \\ 0 & 1 \end{pmatrix} \mathbf{f}(t)\,dt \quad (3.8)$$

for $1 \leqslant p < \infty$, where

$$\varphi(z) = \frac{1}{\pi}\left(\alpha + i \ln \frac{|z - t_1|}{|z - t_2|}\right), \quad \alpha = \arctan \frac{x - t_1}{|z - t_1|} + \arctan \frac{t_2 - x}{|z - t_2|} = \alpha_1 + \alpha_2,$$
$$\varphi_0(z) = -\frac{2}{\pi} e^{i(\alpha_1 - \alpha_2)} \sin\alpha.$$

I. È. Niezov [13] obtained a solution to the Cauchy problem for the Lamé system of equations in domains of special shape by means of some other Carleman functions. The above results represent another approach to solution of this problem.

Using (3.7) and (3.8), we can obtain estimates which characterize conditional stability of the Cauchy problem for the Lamé system. The following inequality is valid for the quenching function $\Phi(z)$ in (3.7) and (3.8):

$$\left\|\left(\Phi^{-1}(z)\Phi(t)\right)^n\right\| \leqslant \begin{cases} e^{n(1-\frac{\alpha}{\pi})}\left(n|\varphi_0(z)|+2\right), & t \in M, \\ e^{-n\frac{\alpha}{\pi}}\left(n|\varphi_0(z)|+2\right), & t \in \partial\Pi \setminus M, \end{cases}$$

and $|\varphi_0(z)| \leqslant \frac{2}{\pi}$. Therefore,

$$\begin{aligned} \left\|\mathbf{u}(z)\right\| &\leqslant e^{-n\frac{\alpha}{\pi}}\left(n|\varphi_0(z)|+2\right)\left\|\int_{\mathbb{R}\setminus M} \frac{y}{(x-t)^2+y^2}\mathbf{f}(t)\,dt\right\| \\ &\quad + e^{n(1-\frac{\alpha}{\pi})}\left(n|\varphi_0(z)|+2\right)\left\|\int_M \frac{y}{(x-t)^2+y^2}\mathbf{f}(t)\,dt\right\| \\ &\leqslant 2c_p^i(y)e^{-n\frac{\alpha}{\pi}}\left(1+\frac{n}{\pi}\right)\|\mathbf{f}\|_{L_p(\partial\Pi\setminus M;\mathbb{C}^2)} \\ &\quad + 2c_p^i(y)e^{n(1-\frac{\alpha}{\pi})}\left(1+\frac{n}{\pi}\right)\|\mathbf{f}\|_{L_p(M;\mathbb{C}^2)}, \quad i=1,2. \end{aligned}$$

Here $c_p^1(y) = \left\|\frac{y}{x^2+y^2}\right\|_{L_q(dx)}$ for (3.7) and $c_p^2(y) = \left\|\frac{1}{z}\right\|_{L_q(dx)}$ for (3.8), $\frac{1}{p}+\frac{1}{q}=1$.

We can standardly demonstrate that the inequality

$$2c_p^i e^{-n\frac{\alpha}{\pi}}\left(1+\frac{n}{\pi}\right) < \varepsilon$$

is valid for arbitrary $\varepsilon > 0$ and

$$n \geqslant n_\varepsilon = \frac{\pi}{\alpha}\frac{e}{1-e}\ln\frac{\dot{\alpha}^2}{8c_p^i(y)}\varepsilon.$$

Therefore, the following conditional stability estimate holds for the problem of A-analytic continuation in the half-plane with the operator $A = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$:

$$\left\|\mathbf{u}(z)\right\| \leqslant \varepsilon l(\mathbf{u}) + c(\varepsilon)\|\mathbf{f}\|_{L_p(M;\mathbb{C}^2)},$$

where $l(\mathbf{u}) = \|\mathbf{u}\|_{L_p(\partial\Pi\setminus M;\mathbb{C}^2)}$ and $c(\varepsilon) = c_{\alpha,p}\varepsilon^{1-\frac{e}{e-1}\frac{\pi}{\alpha}}$.

4. The Cauchy-type integral formulas for generalized A-analytic functions

In this section, we study the generalized Beltrami-type equation

$$\overline{\partial}_A \mathbf{u} + \mu(z) B \mathbf{u} = 0, \tag{4.1}$$

where B is a perturbing operator.

We consider the theory of generalized A-analytic functions and their connection with A-analytic functions (the representation theorem). Various versions of the representation theorem yield inversion formulas and, consequently, uniqueness and stability theorems for the two-dimensional problem of emission tomography with variable absorption $\mu(z)$.

We suppose that the operator A and the scale $\{X^m\}_{m=0}^{\infty}$ of Banach spaces satisfy the assumptions of Section 2; i.e., Condition A is satisfied. Moreover, we suppose that the perturbing operator B belongs to $\mathcal{L}(X^m)$ for $m = 0, 1, \dots$ and $[A, B] = AB - BA = 0$.

Definition. A *generalized A-analytic* function in a domain Ω is a function $\mathbf{u}(z) \in C^1(\Omega; X^m)$ satisfying the equation (4.1) in Ω.

If the equation (4.1) has a solution $\mathbf{u}(z) \in C^1(\Omega; X^m)$ and the operator function $G(z)$ satisfies the equation

$$G_{\bar{z}} - AG_z + \mu(z) B = 0 \tag{4.2}$$

then the function

$$\mathbf{v}(z) = \exp(-G)\mathbf{u}(z) \tag{4.3}$$

is A-analytic in the domain Ω, i.e., is a solution to the equation

$$\mathbf{v}_{\bar{z}} - A\mathbf{v}_z = 0.$$

Therefore, solving (4.2), we obtain the representation (4.3) for generalized A-analytic functions which is an operator analog of the representation for the usual generalized analytic functions [18].

In the theorems below we state sufficient conditions for existence of an operator solution $G(z)$ to the equation (4.2) and indicate how to find it.

Theorem 4.1. *Suppose that μ is a real-analytic function such that the following estimate holds for all $z = x + iy \in \overline{\Omega}$:*

$$\left| \frac{\partial^k \mu}{\partial x^i \partial y^j}(z) \right| \le c \frac{i!j!}{R^k}, \quad i + j = k, \quad k = 0, 1, 2, \dots,$$

where $\operatorname{diam}(\Omega) < 2R$. Then the operator function

$$G(z) = -\sum_{n=1}^{\infty} \frac{(\bar{z} - \bar{z}_0)^n}{n!} \left(-\overline{\partial}_A\right)^{n-1} \mu B \in C^1\left(\overline{\Omega}; L(X^m)\right),$$

where z_0 is the midpoint of the diameter of the domain Ω, satisfies the operator equation (4.2).

Proof. Introduce the operators

$$G_1 = \mu B, \quad G_{n+1} = \left(-\bar{\partial}_A\right)^n \mu B, \quad n = 1, 2, \ldots.$$

Then

$$G(z) = -\sum_{n=1}^{\infty} \frac{(\bar{z} - \bar{z}_0)^n}{n!} G_n(z), \quad G_{n+1} = -\left[(G_n)_{\bar{z}} - A(G_n)_z\right], \quad (4.4)$$

$$G_z(z) = -\sum_{n=1}^{\infty} \frac{(\bar{z} - \bar{z}_0)^n}{n!} (G_n)_z,$$

$$G_{\bar{z}}(z) = -\sum_{n=1}^{\infty} \frac{(\bar{z} - \bar{z}_0)^n}{n!} (G_n)_{\bar{z}} - \sum_{n=0}^{\infty} \frac{(\bar{z} - \bar{z}_0)^n}{n!} G_{n+1}. \quad (4.5)$$

The operator G satisfies formally the operator equation (4.2). We show that the series (4.4) and its derivatives (4.5) converge in the uniform operator topology (i.e., in the norm of the space $\mathcal{L}(X^m)$) and the equalities (4.4), (4.5), and (4.1) hold in the strict sense. Calculate the operator $\left(\bar{\partial}_A\right)^n \mu B$ in the variables x and y. We have

$$\begin{aligned}(\bar{\partial}_A)^n \mu B &= 2^{-n}\left((E - A)\partial_x + i(E + A)\partial_y\right)^n \mu B \\ &= 2^{-n} B \sum_{p=0}^{n} i^p C_n^p (E - A)^{n-p} (E + A)^p \frac{\partial^n \mu}{\partial x^{n-p} \partial y^p}.\end{aligned}$$

Recalling the inequality $\|A\| \le 1$ and the conditions of the theorem, we obtain the estimate

$$\|G_{n+1}\|_m = \left\|\left(\bar{\partial}_A\right)^n \mu B\right\|_m \le c\|B\|_m \sum_{p=0}^{n} C_n^p \frac{p!(n-p!)}{R^n} = c\|B\|_m \frac{(n+1)!}{R^n}$$

which implies that $G, e^{-G} \in \mathcal{L}(X^m)$. Similarly, we can prove that $G_z, G_{\bar{z}} \in \mathcal{L}(X^m)$.

Remark. We can write down the operator $\exp(-G)$, using the Bell polynomials $Y_n(x_1, x_2, \ldots, x_n)$, $n \ge 1$, in n variables [15] that are employed in combinatorial analysis. By definition, we have

$$Y_n(x_1, x_2, \ldots, x_n) = \sum_{\pi(n)} \frac{n!}{k_1! k_2! \cdots k_n!} \left(\frac{x_1}{1!}\right)^{k_1} \left(\frac{x_2}{2!}\right)^{k_2} \cdots \left(\frac{x_n}{n!}\right)^{k_n},$$

where summation is carried out over all unordered partitions $\pi(n)$ of the number n, i.e., over all representations of n as the sum of positive integers:

$$\pi(n) = \left\{(k_1, k_2, \ldots, k_n) : k_1 + 2k_2 + 3k_3 + \cdots + nk_n = n, \ k_j \ge 0\right\}.$$

Using the exponential generating function

$$1+\sum_{n=1}^{\infty}\frac{t^n}{n!}Y_n(x_1,x_2,\dots,x_n)=\exp\left[\sum_{n=1}^{\infty}\frac{t^n}{n!}x_n\right]$$

for the Bell polynomials, we see that

$$\exp\bigl[-G(z)\bigr]=E+\sum_{n=1}^{\infty}\frac{(\bar z-\bar z_0)^n}{n!}Y_n(G_1,G_2,\dots,G_n).$$

We give some examples of construction of the operators G and e^{-G}. Suppose that the point z_0 coincides with the origin.

Example 1. Let $\mu(z)=\text{const}$. Then $G_1=\mu B$ and $G_k=0$ for $k\geq 2$. Therefore, $G(z)=-\bar z\mu B$ and

$$e^{-G(z)}=e^{\bar z\mu B}.$$

Example 2. Suppose that $\mu(z)=x^2+y^2=z\bar z$. Then $G_1=z\bar zB$, $G_2=-\overline{\partial}_A(z\bar zB)=-(zE-\bar zA)B$, $G_3=-2AB$, and $G_k=0$ for $k\geq 4$. Therefore,

$$G(z)=-\bar z^2\left(\frac{z}{2!}+\frac{\bar z}{3!}A\right)B.$$

Example 3. Suppose that $\mu(z)$ is an analytic function in the domain Ω (i.e., $\mu_{\bar z}=0$) which satisfies the conditions of Theorem 4.1 (the condition of real analyticity in Theorem 4.1 is inessential). In this case, $G_{n+1}=\mu^{(n)}BA^n$ and, by the Remark to Theorem 4.1, we infer that

$$\exp\bigl[-G(z)\bigr]=1+\sum_{n=1}^{\infty}\frac{\bar z^n}{n!}Y_n\left(\mu B,\mu^{(1)}AB,\mu^{(2)}A^2B,\dots,\mu^{(n-1)}A^{n-1}B\right).$$

Another way of solving the equation (4.2) is connected with solution of the direct problem for the transport equation; therefore, we will make use of the condition of real analyticity of the function $\mu(z)$ and convexity of the domain Ω. Given a point z in the strictly convex domain $\overline{\Omega}$ and a direction $e^{i\varphi}$, define the function $t=\gamma(z,\varphi)$ so that t be the point of intersection of the ray starting at the point z in the direction $-e^{i\varphi}$ with the boundary Γ of the domain Ω. Assume that $z\in\Omega$. Then the function

$$m(z,\varphi)=\int_{\gamma(z,\varphi)}^{z}\mu(\zeta)|d\zeta|$$

is a solution to the transport equation

$$e^{-i\varphi}m_{\bar z}+e^{i\varphi}m_z=\mu(z). \tag{4.6}$$

Expanding the function $m(z, \varphi)$ in the Fourier series

$$m(z, \varphi) = \sum_{k=-\infty}^{\infty} m_k e^{-ik\varphi},$$

we obtain the following system of equations in the odd harmonics (see Section 1):

$$(m_{2k+1})_{\bar{z}} + (m_{2k+3})_z = 0, \ k = 0, 1, \dots,$$
$$2(m_1)_z = \mu(z).$$

Using these relations, we easily see that a formal solution to the equation (4.2) can be written as

$$G(z) = -2 \sum_{k=0}^{\infty} \overline{m}_{2k+1}(z) B(-A)^k.$$

We can also find this solution by solving the nonhomogeneous equation (4.2) by the formula

$$G(\zeta) = \frac{1}{\pi} \iint_{\Omega} K(z - \zeta) \mu(z) B \, dx \, dy.$$

The following theorem shows that the function $G(z)$ is a smooth solution to the equation (4.2).

Theorem 4.2. *Suppose that a strictly convex domain Ω has smooth boundary Γ of class C^2 and $\mu \in C^2(\overline{\Omega})$ is a real-valued function. Then the operator function $G(z)$ equals*

$$-2B \sum_{k=0}^{\infty} \overline{m}_{2k+1}(z)(-A)^k \in C^1(\Omega; \mathcal{L}(X^m)) \cap C(\overline{\Omega}; \mathcal{L}(X^m))$$

and satisfies the equation (4.2).

To prove Theorem 4.2, we need some lemma on smoothness of the function χ, a solution to the transport equation (4.6) with the right-hand side equal to 1.

Lemma 4.1. *Suppose that the boundary Γ of a strictly convex domain Ω is of the class C^2 and χ is a solution to the following boundary value problem:*

$$\chi_{\bar{z}}(z, \varphi) e^{-i\varphi} + \chi_z(z, \varphi) e^{i\varphi} = 1, \quad z \in \Omega,$$
$$\chi\big|_{\Sigma_-} = 0.$$

Then the functions χ_z and $\chi_{\bar{z}}$ are continuously differentiable with respect to φ at each interior point $z \in \Omega$. Moreover, the following formulas hold:

$$\chi_z(z, \varphi) = \frac{e^{-i\varphi}}{2}\big(1 + i \cot(\beta - \varphi)\big), \quad \chi_{\bar{z}}(z, \varphi) = \frac{e^{i\varphi}}{2}\big(1 - i \cot(\beta - \varphi)\big). \quad (4.7)$$

Here $e^{i\beta(z,\varphi)}$ is the vector tangent to the boundary Γ at the point $t = \gamma(z, \varphi)$.

We can prove Lemma 4.1 by straightforwardly calculating the partial derivatives χ_x and χ_y.

Proof of Theorem 4.2. Show that the functions m, m_z, and $m_{\bar z}$ are continuously differentiable with respect to φ at each point $z \in \Omega$. Then the vectors

$$\{m_1, m_3, \dots\}, \quad \{m_{1z}, m_{3z}, \dots\}, \quad \{m_{1\bar z}, m_{3\bar z}, \dots\}$$

belong to l_2^1, which guarantees convergence of the series

$$G(z) = -2B \sum_{k=0}^{\infty} \overline{m}_{2k+1}(z)(-A)^k$$

and its derivatives. Continuous differentiability of the function m with respect to φ for all $z \in \overline{\Omega}$ follows from the representation

$$m(z,\varphi) = \int_{\gamma(z,\varphi)}^{z} \mu(\zeta)\,|d\zeta| = \int_0^{\chi(z,\varphi)} \mu(z - \rho e^{i\varphi})\,d\rho.$$

Differentiating this formula with respect to z and φ, we find that

$$\begin{aligned} m_z &= \mu\big(\gamma(z,\varphi)\big)\chi_z + \int_0^{\chi(z,\varphi)} \mu_z\,d\rho, \\ (m_z)'_\varphi &= \mu_z\big(\gamma(z,\varphi)\big)\chi'_\varphi + \frac{d}{d\varphi}\Big(\mu\big(\gamma(z,\varphi)\big)\chi_z\Big) \\ &\qquad + i\int_0^{\chi(z,\varphi)} \big(-\mu_{zz}e^{i\varphi} + \mu_{z\bar z}e^{-i\varphi}\big)\rho\,d\rho, \end{aligned} \tag{4.8}$$

where $\gamma(z,\varphi) = z - \chi e^{i\varphi}$ and $\gamma'_\varphi = -i\chi e^{i\varphi} - \chi'_\varphi e^{i\varphi}$. Recalling the smoothness condition and Lemma 4.1, from (4.8) we infer that the function m_z is continuously differentiable with respect to φ for all $z \in \Omega$. Continuity of the function $(m_{\bar z})'_\varphi$ is verified straightforwardly. Theorem 4.2 is proven.

Theorems 4.1 and 4.2 enable us to transfer the properties of A-analytic functions to solutions to the equation (4.1). In particular, we have the integral formula with the generalized Cauchy kernel for solutions to the equation (4.1).

Theorem 4.3. *Suppose that the conditions of Theorems 4.1 or 4.2 are satisfied. If $\mathbf{u}(z) \in C^1(\Omega; X^m) \cap C(\overline{\Omega}; X^{m+1})$ is a solution to the equation (4.1) then the following integral formula with the generalized Cauchy kernel is valid:*

$$\mathbf{u}(\zeta) = \frac{1}{2\pi i}\int_\Gamma K_B(z,\zeta)\big(dz + A\,d\bar z\big)\mathbf{u}(z), \quad \zeta \in \Omega,$$

where $K_B(z,\zeta) = K(z-\zeta)e^{G(\zeta)-G(z)}$ is an operator function with values in $\mathcal{L}(X^{m+1}; X^m)$ continuous in both variables $z, \zeta \in \overline{\Omega}$ for $z \neq \zeta$.

In conclusion, we observe a possible approach to determination of the absorption $\mu(z)$ when the source function $a(z)$ is unknown. For example, if a solution $\mathbf{u}(z)$ to the equation (4.1) is continuous up to the boundary then the following equality holds [7]:

$$\frac{1}{2}\mathbf{f}(t) = \frac{1}{2\pi i}\int_\Gamma K(z-t)e^{G(t)-G(z)}(dz + Ad\bar{z})\mathbf{f}(z), \quad t \in \Gamma,$$

which can be considered as a nonlinear equation in the function $\mu(z)$. If $\mu =$ const and $G(z) = -\mu\bar{z}B$ then for the coefficient μ we obtain the relation

$$\frac{1}{2}\mathbf{f}(t) = \frac{1}{2\pi i}\sum_{k=0}^{\infty}\frac{\mu^k B^k}{k!}\int_\Gamma K(z-t)\left(\bar{z}-\bar{t}\right)^k(dz + Ad\bar{z})\mathbf{f}(z), \quad t \in \Gamma.$$

Observe that the results known so far on uniqueness and stability of a solution to the problem of emission tomography with absorption of finite smoothness require that the gradient of the absorption be small (see [11, 12, 16]). In Theorems 4.2 and 4.3, we only require that the function μ belong to the class $C^2(\overline{\Omega})$. A detailed exposition of the applications of the theory of A-analytic functions to the emission tomography problems is presented in the article [8].

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