# International Mathematical Internet Olympiad

Alexander Domoshnitsky,Vadim Bugaenko,Roman Yavich
adom@ariel.ac.il, romany@ariel.ac.il
(Ariel University Center, Ariel, Israel)
Alexei Kannel-Belov (Bar Ilan University, Ramat Gan, Israel)

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We start the project Mathematical Internet Olympiads in 2006. Usually they lasts for four hours during which the web site with the Olympiad problems has more than four thousand hits. Thousands students from more than 30 countries took part in these competitions. They represented countries from Brazil and the USA in the west to Vietnam in the east. Among the Olympiad laureates there are prize winners of national and international competitions from Russia, Ukraine, Armenia, Brazil, Georgia, and Israel. The headquarters of the Olympiad were located in the Ariel University Center, Israel.

There are a lot of students' contests today on different subjects, including math. Generally speaking, these competitions belong to one of the two types. First, there are contests targeted at `professionals', one should have certain advance training to solve problems there. A beginner has almost no chance to succeed. Even the wording of a problem could be unfamiliar for him or her. Most probably a novice will not be able to solve anything, and the effect will be negative. Rather than to arouse interest in math the experience will only lead to disappointment and diffidence. A contest of the other type is organized with mass participation in mind. Its problems are accessible for a beginner, to solve them one just needs some level of quick-wittedness. Moreover, some problems are not that much different from exercises which students solve in class. However, such contests are usually of no interest for strong students. Olympiad `professionals' do not care to participate, it is `not cool' for them.

Is it possible to come up with a contest format that is attractive and beneficial for an Olympiad professional and a beginner alike? We tried to find such an optimal format. We proposed to the participants a large number of problems. Their list included simple enough problems as well as quite difficult ones. So each participant could choose problems of appropriate level. However problems were evaluated not equally: the score for the solution of an individual problem was inversely proportional to the number of participants who solved it. Thus actually only difficult problems influenced the results of the winners. On the other hand, when problems are selected in such a way, there are almost no empty papers, because even weaker students can find some problems feasible for them. Thus the result of a beginner who managed to solve, say four problems does not look like a failure in comparison with the result of a winner who solved six or seven problems.

Another characteristic feature of our Olympiad is the use of the Internet. Thus students from different parts of the world get a chance to participate. The Olympiad starts with the opening of the web site where the problems are listed, and after the allotted time the contestants must send their solutions to the jury by email. In the course of the Olympiad the participants can ask questions about the problems online. Audio and video connection was established with some of the locations at which the Olympiad was held. The same technical tools are used at the closing ceremony.

Our first Olympiad was held in 2006, with the participation of Israeli students only. It came as a surprise for the organizers that not only future mathematicians, physicists and engineers responded to the invitation to participate, but also students of other specializations, even nursing students. The number of participants grows each year, and the Olympiad became international already in its second year: universities of Russia, Ukraine, Romania, Bulgaria and Germany joined it. Students from 35 universities of 19 countries participated in the last Olympiad.

Due to collaboration with the Russian National Accreditation Agency in the Sphere of Education and its head V.G.Navodnov the final round of the Olympiads in 2009- 2011 were combined with the final round of the All-Russian students' mathematical Internet-Olympiad. We view such collaboration to be quite effective and we plan to continue and extend it.

Below we present the problems proposed to the participants of the Olympiad round of December 17, 2009 and the statistics of solutions. More info, in particular the solutions of the previous International Mathematical Internet Olympiads, can be found at the Olympiad site www.i-olymp.net .

**Problems:**

1. At a primeval market one could exchange a mammoth skin for two saber toothed tiger skins, and a peacock feather skirt for three stone spears. At another market, located one day away from the first one, one could exchange a mammoth skin for three peacock skirts, and a tiger skin for four spears. All exchanges could be done both ways. A hunter brings a mammoth skin to the first market and wants to exchange it for four tiger skins. Could he do this in 33 days?

2. Find $\lim\limits_{x\to\infty} x^{\frac{1}{\ln x}}$ .

3. Is the function $\ln\left(x+\sqrt{x^2+1}\right)$ even, odd or neither of these two?

4. Bassil executes a parachute jump with constant speed $v$, while his friend Michael rides a Ferris wheel which rotates with constant speed. (The initial point of Bassil's jump is higher than the uppermost point of the Ferris wheel). At the moment of Bassil's landing the cabin in which Michael sits is at the lowest point (at the ground level). It is known that during Bassil's descent the friends were at the same height precisely four times. Find the vertical component of Michael's speed at the moment when they found themselves at the same height for the first time. (You may consider the two boys to be points).

5. How many solutions does the equation $x^2 = 2^x$ have?

6. Let $A$ and $B$ be arbitrary non-zero matrices of the second order. Prove that there exists a matrix $C$ such that $A\cdot C\cdot B \neq 0$

7. Find the minimal value of the expression $\sqrt{(x-1960)^2+y^2}+\sqrt{x^2+(y-441)^2}$

8. Find a non-zero polynomial with integer coefficients with a root $\sqrt[5]{\sqrt{2}+1}-\sqrt[5]{\sqrt{2}-1}$ .

9. Two numbers are chosen randomly in the segment [0,1]. Calculate the probability that the square of the first number is greater than the second one.

10. The axes of two cylinders, the radiuses of which equal 1, intersect and are perpendicular. Find the

volume of the intersection of the cylinders.

11. A professor has formulated $n$ statements $A_1, A_2, \ldots, A_n$. He gives his postgraduate students subjects for dissertations: "$A_i$ implies $A_j$". No dissertation should be a direct logical consequence of previously given ones. What is the maximal number of students that the professor could have?

12. Let $f(x,y)$ be an infinitely differentiable function of two variables with a local minimum at the origin, and let this function have no other {\it critical} points. Is it true that the minimum is global? (A point is called critical if both partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at it equal zero).

**Solutions**

1. *Answer: Yes.* Firstly, let us note that the hunter can exchange a mammoth skin for 9 spears in two days. In order to do so, he has to first exchange it for three peacock-feather skirts at the second market, and then exchange each of the skirts for three spears at the first market. Note also that in two days the hunter can get mammoth skin for only 8 spears. First, he has to exchange the spears for 2 tiger skins at the second market, and then exchange the tiger skins for one mammoth skin at the first. Thus, every four days the hunter can add one spear to his belongings. If the hunter continues repeating the above cycle, then on the thirty first day (the third day of the eighth cycle) he will find himself at the second market, with 16 spears. These spears can be exchanged for 4 tiger skins. Therefore, the hunter can achieve his goal in 31 days.

2. *Answer:* $e$ . Let us transform the function

$$x^{\frac{1}{\ln x}} = \left(e^{\ln x}\right)^{\frac{1}{\ln x}} = e^{\frac{\ln x}{\ln x}} = e^1 = e$$

Therefore, $\lim\limits_{x\to\infty} x^{\frac{1}{\ln x}} = \lim\limits_{x\to\infty} e = e$ .

3. *Answer: The function is odd.* Denote given function by $f(x)$. We have

$$f(-x) = \ln(-x + \sqrt{(-x)^2+1}) = \ln\frac{(-x+\sqrt{x^2+1})}{x+\sqrt{x^2+1}} =$$

$$\ln\frac{1}{x+\sqrt{x^2+1}} = -\ln\left(x+\sqrt{x^2+1}\right) = -f(x)$$

4. Equality $f(-x) = -f(x)$ denotes that the function $f(x)$ is odd.

5. *Answer:* $v$ . The graph that depicts the changes in Michael's altitude is asinusoid $h_m = R\left(1+\sin(\varphi_0+\omega t)\right)$, while the graph that depicts the changes in Bassil's altitude is a straight line $h_v = h_0 - vt$ . The two graphs are shown together in the figure below. It can be seen clearly that this is the only natural position of the graphs for which the first intersection point is the point of tangency. Therefore, at this point the derivatives of these two functions are equal, which means that the sought for speed is equal to the speed of Bassil's descent.

6. *Answer:* 3 *roots*. In the interval $(-\infty, 0]$ the function $x^2$ decreases from $\infty$ to 0, and the function $2^x$

increases from 0 to 1. Therefore, in this interval the equation has only one root. If $x \in (0,1]$, then $x^2 < 1$, $2^x > 1$, and therefore in this interval the equation has no roots. Let us examine the interval $(1, +\infty)$. Our equation is equivalent to the equation $x^{1/x} = 2^{1/2}$. Let us examine the behavior of the function $f(x) = x^{1/x}$ regarding monotony and extrema. Its derivative equals to $\frac{(1-\ln x)x^{1/x}}{x^2}$. Therefore, the function increases in the interval $[0, e]$, has its maximum at $x = e$, and decreases in the interval $[e, +\infty)$. In addition, $\lim_{x \to 1} f(x) = \lim_{x \to 1} f(x) = 1$. That means that in this interval the function takes each value from the range of values, except the maximal one, twice – once in the interval $(0, e)$ and once in the interval $(e, +\infty)$. Therefore, the equation has two positive roots, one in each of these intervals. They are $x = 2$, and $x = 4$.

7. Remember that in a two-dimensional space every second-order matrix is a linear operator. The following three statements are equivalent: "The operator matrix does not equal zero"; "There exists a vector which does not belong to the kernel of the operator"; "The image of the operator contains a nonzero vector". The product of the matrices corresponds to the composition of the operators, starting with the last factor and ending with the first. Let us examine the vector $\vec{w} \notin KerA$, then $A(\vec{w}) \neq \vec{0}$. Let us also examine the nonzero vector $\vec{v} \in \operatorname{Im} B$, then a vector $\vec{u}$ must exist, such that $B(\vec{u}) = \vec{v}$. Let us choose an operator $C$, such that $C(\vec{v}) = \vec{w}$. Then, $ACB(\vec{u}) = AC(\vec{v}) = A(\vec{w}) \neq \vec{0}$. Thus, the kernel of the operator $ABC$ contains a nonzero vector, which means that the operator's matrix does not equal zero.

   *Note:* This problem can also be solved by composing and solving a system of linear equations, where the elements of the matrix $C$ are the unknowns, and the elements of the matrices $A$ and $B$ are parameters.

8. *Answer:* 2009. This expression is equal to the sum of the distances from point $M(x, y)$, to points $A(1960,0)$ and $B(0,441)$. According to inequality of triangle this expression takes the minimal value at all points of the segment $AB$, and this minimum is equal to the distance between points $A$ and $B$, i.e. $\sqrt{1960^2 + 441^2} = 2009$.

9. *Answer: One of such a polynomials is* $x^5 + 5x^3 + 5x - 2 = 0$. Let us use the following identities:

$$(a-b)^3 = a^3 - b^3 - 3ab(a-b). \quad (1)$$

$$(a-b)^5 = a^5 - b^5 - 5ab\left(a^3 - b^3\right) + 10a^2b^2(a-b). \quad (2)$$

Assume $= \sqrt[5]{\sqrt{2}+1}$, $b = \sqrt[5]{\sqrt{2}-1}$. It can be easily shown that $ab = 1$, $a^5 - b^5 = 2$. Let $x = a - b$. It follows from (1) that $a^3-b^3=x^3+3x$ , and if follows from (2) that

$x^5=2-5(a^3-b^3)+10x=2-5(x^3+3x)+10x$
Hence $x^5+5x^3+5x-2=0$ .

10. Answer: $\frac{1}{3}$. The two numbers that have been chosen, $x$ and $y$ , can be seen as coordinates of a point situated within a unit square on the coordinate plane. The sought for probability is equal to the ratio of the area of the part of the square where $y<x^2$ to the area of the whole square. The area of the square equals 1, while the area of the part of the square equals $\int_0^1 x^2dx=\frac{1}{3}$. Therefore, the sought for probability equals $\frac{1}{3}$.

11. *Answer:* $\frac{16}{3}$ . Let us introduce a coordinate axis perpendicular to the axes of the cylinders, with origin at the point where the cylinders intersect. The cross-section of the body we are examining by a plane perpendicular to our axis, which intersects our axis at a point with coordinate $x$ , is a square, the side of which equals $2\sqrt{1-x^2}$ (where $-1\leq x\leq 1$ ). The required volume can be expressed as an integral of the cross-section area: $\int_{-1}^{1}4(1-x^2)\,dx=4\left(x-\frac{x^3}{3}\right)\Big|_{-1}^{1}=\frac{16}{3}$

*Note.* This problem can also be solved without integration, using Cavalieri's principle which can be formulated in the following way: if for two regions in three-space (solids) included between two parallel planes, every plane parallel to these two planes intersects both regions in cross-sections of equal area, then the two regions have equal volumes.

Let us compare the cross-sections of the body which we are examining to those of a sphere, the radius of which equals 1, and the center of which is at the intersection of the cylinders. The cross-section of this sphere by a plane perpendicular to our axis, which intersects our axis at a point with coordinate $x$ , is a circle, the radius of which equals $\sqrt{1-x^2}$. Therefore, the ratio of the cross-section areas always equals $\pi/4$. Thus, the volume of the body can be obtained by dividing the volume of the sphere (which equals $\frac{4\pi}{3}$) by $\pi/4$ .

11. *Answer:* $\frac{(n-1)(n+2)}{2}$. Let us begin by demonstrating that the professor can formulate $\frac{(n-1)(n+2)}{2}$ acceptable subjects for dissertations. Let us assume that the professor asks that all other statements be derived from $A_1$, and statements with greater numbers be derived from $A_2$, and then $A_3$, etc. As a result he will have $\frac{(n-1)n}{2}$ subjects. After that he can assign another $n-1$ subjects by deriving all the previous statements from the last one in descending order. Thus, he will now have $\frac{(n-1)(n+2)}{2}$ subjects. The condition that none of the dissertations must be a direct logical consequence of previously defended ones can be checked easily. Now, let us assume that the professor has formulated a list of subjects, and prove that

it contains no more than $\frac{(n-1)(n+2)}{2}$ items. Let us denote all the statements as points on a plane, and for each line $A_i$ to point $A_j$. Let us prove that the number of double-headed arrows will be no greater than $n-1$. In order to do that we examine an undirected graph, the edges of which are double-headed arrows. Suppose that it contains a cycle. If so, the last dissertation of this cycle will be a consequence of those that were defended earlier, which contradicts the condition. Therefore, the graph must be a tree. All that we have to do now is use the known theorem, according to which the number of edges of any tree is smaller than the number of its apexes. Let us calculate the maximal possible number of arrows. We have $\frac{n(n-1)}{2}$ pairs of points $[(A]_i, A_j)$. No more than $n-1$ of them are connected by double-headed arrows. The remaining $\frac{(n-2)(n-1)}{2}$ are either connected by single-headed arrows or not connected at all. Therefore, the total number of arrows (i.e., the number of possible subjects for dissertations) can be no greater than $2(n-1+(n-2)(n-1)/2 = ((n-1)(n+2))/2$.

12. *Answer: Wrong.* In order to construct a counterexample, let us first find two differentiable functions $u(x)$ and $v(x)$, such that:

a) $u(x) < v(x)$ for any $x \in R$;
b) the derivative $u'(x)$ equals 0 only when $x=0$, the point $x=0$ is the minimum of function $u$;

c) the derivative $v'(x)$ never equals 0. For example, let's consider the following two functions: $u(x) = -\frac{1}{1+x^2}$, $v(x) = e^x$ Now, we can find the desired counterexample in the form:

$$f(x,y) = u(x) + (v(x) - u(x))(3y^2 - 2y^3).$$

When the value of x is fixed, this function is a cubic polynomial, which has two local extrema: a minimum which equals $u(x)$ at $y=0$, and a maximum which equals $v(x)$ at $y=1$. In order to check the properties given in the problem, we must find the critical points of the function $f(x,y)$. The condition $\frac{\partial f}{\partial y} = 0$ is equivalent to either $y=0$ or $y=1$. But $f(x,0) = u(x)$ and $f(x,1) = v(x)$, and therefore, the condition $\frac{\partial f}{\partial y} = 0$ is possible only in the first case and only at $x=0$. Thus, $f(x,y)$ has the unique critical point is the origin, which is a local minimum. This minimum is not global since $\lim_{y\to+\infty} f(x,y) = -\infty$ for any fixed $x$ ,which means that $\inf f(x,y) = -\infty$
Thus, the sought for counter example is:

$$f(x,y) = -\frac{1}{x^2+1} + \left(e^x + \frac{1}{x^2+1}\right)(3y^2 - 2y^3).$$

**Statistics:**

| Problem no | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
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| Num. of solutions | 256 | 296 | 226 | 29 | 65 | 32 | 139 | 57 | 119 | 79 | 1 | 24 |