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1989 Math. USSR Sb. 63 363

(http://iopscience.iop.org/0025-5734/63/2/A07)

Матем. Сборник
Том 135(177)(1988), Вып. 3

Math. USSR Sbornik
Vol. 63(1989), No. 2

# ON A SHIRSHOV BASIS OF RELATIVELY FREE ALGEBRAS OF COMPLEXITY $n$

UDC 519.48

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ABSTRACT. A Shirshov basis is a set of elements of an algebra $A$ over which $A$ has bounded height in the sense of Shirshov.

A description is given of Shirshov bases consisting of words for associative or alternative relatively free algebras over an arbitrary commutative associative ring $\Phi$ with unity. It is proved that the set of monomials of degree at most $m^2$ is a Shirshov basis in a Jordan PI-algebra of degree $m$. It is shown that under certain conditions on var($B$) (satisfied by alternative and Jordan PI-algebras), if each factor of $B$ with nilpotent projections of all elements of $M$ is nilpotent, then $M$ is a Shirshov basis of $B$ if $M$ generates $B$ as an algebra.

Bibliography: 12 titles.

Shirshov in [1] and [2] established the local boundedness of the height of a PI-algebra of degree $m$ over the set of words of degree at most $m$. Shirshov's theorem is directly related to Kurosh's problem for PI-algebras; in fact, they are equivalent. The precise assertion is contained in Theorems 3 and 4 of the present paper. Equivalence is established by using a "transfer" procedure, enabling us to gather, by means of a chain of so-called $f$-transformations, almost all symbols occurring in a set of long subwords of a word $c$ into $m-1$ groups, where $m$ is the degree of the identity $f$. The method works in the alternative and Jordan cases.

A well-known conjecture of Shestakov, which was proved by Ufnarovskiĭ [3], asserts that an associative PI-algebra of degree $m$ in which all words of degree at most $[m/2]$ are algebraic is locally finite; Ufnarovskiĭ also showed that $[m/2]$ can be replaced by the complexity. Note that Shestakov's conjecture is a consequence of Theorem 1 of the present paper; moreover the proof is constructive in nature.

In connection with Shirshov's theorem, several problems arise. Over which sets of words does the algebra $A$ have bounded height? For which classes of nonassociative algebras is the height theorem true? We are interested in algebras over an arbitrary ring.

L′vov [9] considered the case of algebras over an arbitrary ring; he established the boundedness of the height of an algebra $A$ over the set of words of degree at most $m-1$, and proved Shestakov's conjecture for $\deg A = 6$. The height theorem for $(-1,1)$ PI-algebras and alternative PI-algebras was proved by Pchelintsev [4], but the boundedness of the height was not established over a set of words.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 16A06, 17D05; Secondary 16A38.

In the present paper we describe the Shirshov bases consisting of words of a relatively free $\Phi$-algebra $A$, associative or alternative, where $\Phi$ is any commutative associative ring with unity and the variety $\operatorname{var}(A)$ is not necessarily homogeneous. We prove the existence of a Shirshov basis consisting of words for Jordan PI-algebras.

The author sincerely thanks S. V. Pchelintsev for posing these problems and for his active participation, and V. N. Latyshev for useful discussion and valuable comments.

## §1. Main results

We denote by $\#D$ the number of elements of the set $D$.

Throughout this paper, $A$ is a finitely generated (f.g.) associative PI-algebra over a commutative associative ring $\Phi$ with unity; for a set $M$ we let $\langle M\rangle$ denote the $\Phi$-module generated by $M$. We will say that an element $x \in A$ is linearly representable by a set $M$ if $x \in \langle M\rangle$, and that $A$ is linearly representable by $M$ if $A = \langle M\rangle$. We will say that $A$ has *essential height* $h$ over a set $Y \subset A$, and call $Y$ an *s-basis* of $A$, if there exists a finite subset $D \subset A$ such that $A$ is linearly representable by a set of elements of the form $t_1 \cdots t_N$, where $N \leq h$ and, for all $i$, either $t_i \in D$ or $t_i = b_i^{K_i}$, where $b_i \in Y$; if we can put $D = \varnothing$, then $A$ has height $h$ over $Y$, and $Y$ is called a *Shirshov basis* of $A$.

Note that if a set $Y$ generates $A$ as an algebra, then the elements of $D$ can be expressed in terms of $Y$; then we can put $D = \varnothing$, and if $Y$ is an $s$-basis, then $Y$ is a Shirshov basis, and conversely.

By the *complexity* $\operatorname{Pid}(\mathfrak{M})$ of a variety $\mathfrak{M}$ we mean that largest $n$ such that $\operatorname{Mat}_n(F)$ belongs to the variety $\overline{\mathfrak{M}}$ generated by the homogeneous components of the identities of $\mathfrak{M}$ for some simple factor $F$ of $\Phi$. For algebras over an infinite field this definition agrees with the usual one.

By the *degree* $\deg(A)$ of an algebra $A$ we mean the smallest $m$ such that $A$ satisfies an identity $f$ of the form

$$x_1 \cdots x_m - \sum_{\sigma \neq 1} K_\sigma x_{\sigma(1)} \cdots x_{\sigma(m)},$$

where $K_\sigma \in \Phi\ \forall \sigma \in S_m$.

The algebras of words that we consider do not contain the empty word $(\Lambda)$. If $c \neq 0$ is a homogeneous element, then $\|c\|$ is its degree of homogeneity; if $c$ is a word, then $|c|$ is its length; $|\Lambda| = \|\Lambda\| = 0$; if $c \neq \Lambda$, then we always have $|c| \geq 1$ and $\|c\| \geq 1$. For example, for letters $a, b, d$, if $\|a\| = 2$, $\|b\| = 3$, and $\|d\| = 4$, then

$$|a^3(bd)^2| = 7, \qquad \|a^3(bd)^2\| = 20.$$

THEOREM 1. *A finitely generated associative PI-algebra of complexity n has bounded height over the set of words of degree at most n; if $n = 0$, then* $\dim_\Phi(A) < \infty$.

The following corollary is a strengthening of Shestakov's conjecture:

COROLLARY. *Suppose $A$ is an f.g. associative algebra of degree $m$. Then $A$ has bounded height over the set of words of degree at most* $[m/2]$.

From now on, unless it is stated otherwise, we will consider $s$-bases and Shirshov bases consisting only of words of a relatively free algebra $A$. In this regard, an $s$-basis is a Shirshov basis if and only if it contains generators. Note that $M \cup \{c^l\}$ is an $s$-basis $\Leftrightarrow M \cup \{c\}$ is an $s$-basis.

A word of the form $c^l$, where $l > 1$, is called *cyclic*; two words $c_1$ and $c_2$ are called *cyclically conjugate* if

$$\exists d_1, \exists d_2 \colon \quad c_1 = d_1 d_2, \qquad c_2 = d_2 d_1.$$

Obviously, no $s$-basis that is minimal with respect to inclusion can contain two cyclically conjugate words. The problem of describing $s$-bases and Shirshov bases is equivalent to that of describing reduced (i.e. minimal with respect to inclusion and containing no cyclic words) Shirshov bases, since a reduced $s$-basis contains generators. We will denote by $Q_K^1$ the set of noncyclic words of length at most $K$. The relation of cyclic conjugacy partitions $Q_K^1$ into equivalence classes. A system of representatives will be called a *K-sample.*

THEOREM 2. *Suppose $A$ is a finitely generated, relatively free $\Phi$-algebra of complexity $n$. Then the set of reduced Shirshov bases of $A$ consisting of words is equal to the set of $n$-samples.*

Any Shirshov basis that is minimal with respect to inclusion can be obtained from a reduced basis by replacing the words of length greater than 1 by their powers. We denote by $M^{(K)}$ the ideal generated by the elements of the form $m^K$, where $m \in M$.

THEOREM 3. *Suppose $A$ is a finitely generated, graded, associative PI-algebra of complexity $n$, and $M \subset A$ is a finite subset of homogeneous elements. If $A/M^{(n)}$ is nilpotent, then $M$ is an s-basis of $A$. If, in addition, $M$ generates $A$, then $M$ is a Shirshov basis of $A$.*

MAIN LEMMA. *Under the hypothesis of Theorem 3, if $m = \deg(A)$ and $A/M^{(m)}$ is nilpotent, then $M$ is an s-basis of $A$.*

This assertion is weaker than Theorem 3.

REMARK. If we do not require homogeneity, then, as the example of an algebra with unity shows, the assertion of Theorem 3 is false. If the algebra $A$ satisfies a homogeneous identity of complexity $n$, then $A/M^{(l)}$ is nilpotent for all $l$. The example $A = \mathrm{Mat}_n(F)\otimes_F XF[X]$, where $F$ is a field, $M = \{m\}$, and $m$ is nilpotent of degree $n$, shows that the number $n$ in the hypothesis of the theorem cannot be diminished.

Suppose $B$, generally speaking, is an f.g. nonassociative $\Phi$-algebra. Then $L[B]$ is the algebra of its left multiplications. For $d \in B$ we let $L(d)$ be the operator of left multiplication by $d$, and we denote by $B_d$ the subalgebra of $B$ generated by $d$. Let $\mathrm{Mon}_l$ be the set of nonassociative monomials of degree at most $l$. For example:

$$\mathscr{R} = ((\cdot,\cdot),(\cdot,\cdot)) \in \mathrm{Mon}_4, \qquad \mathscr{R}(x,y,z,z) = (xy)z^2.$$

The algebra $B$ has *essential height $R$* over a set $M$ if there exists a finite set $D \subset B$ such that $B$ is linearly representable by the elements of the form $\mathscr{R}(t_1,\dots,t_N)$, where $\mathscr{R} \in \mathrm{Mon}_R$ and, for all $i$, either $t_i \in D$ or $t_i \in B_{m_i}$ for some $m_i \in M$. If we can put $D = \varnothing$, then $B$ has height $R$ over $M$; the connection between the height and the essential height is the same as in the associative case. This definition agrees with the usual definition of height for nonassociative algebras [4].

Let $M^{(K)}$ be the ideal of $B$ generated by $\bigcup_{m\in M} B_m^K$. We will say that the algebra $B$ has *L-length $K$* if $L[B]$ is linearly representable by the set of elements of the form $L(p_1)\cdots L(p_q)$, where $q \leq K$.

Suppose $\mathfrak{M}$ is a variety of algebras and $C_l$ is a relatively free $l$-generator algebra in $\mathfrak{M}$. The variety $\mathfrak{M}$ is called *nice* if there exist functions $\tau_{\mathfrak{M}}(l)$ and $\pi_{\mathfrak{M}}(l)$ such that, for all $l$,

(i) $C_l$ has $L$-length $\tau_{\mathfrak{M}}(l)$,
(ii) $L[C_l]$ has at most $\pi_{\mathfrak{M}}(l)$ generators, and
(iii) $L[C_l]$ is a PI-algebra.

THEOREM 4. *Suppose* var$(B)$ *is a nice variety, $B$ is an f.g. graded algebra, and $M$ is a finite set of homogeneous elements of $B$. If $\forall i\ B/M^{(i)}$ is nilpotent, then $M$ is an s-basis of $B$. If, in addition, $M$ generates $B$ as an algebra, then $M$ is a Shirshov basis of $B$.*

It was shown in [7] that condition (i) holds for Jordan algebras. It was proved in [5] that condition (i) holds for alternative algebras, and that condition (iii) holds for alternative and special Jordan PI-algebras; it was also shown there that $B/M^{(l)}$ is nilpotent for all $l$, where $B$ is an f.g. alternative algebra of degree $m$ and $M$ is the set of words of degree at most $m^2$. In [6] the validity of (iii) was established for Jordan PI-algebras, and in [8] the local finiteness of a Jordan PI-algebra of degree $m$ with algebraic words of degree at most $m^2$ was proved.

COROLLARY 1. *Suppose $B$ is an alternative or Jordan f.g. PI-algebra of degree $m$. Then $B$ has bounded height over the set of words of degree at most $m^2$.*

From the fact that any associative, alternative, or Jordan f.g. PI-algebra having no ideal with a nonnilpotent factor is simple we obtain

COROLLARY 2. *Suppose a set $M$ and algebra $B$ satisfy the assumptions of Theorem 4 and $B$ is an associative, alternative, or Jordan algebra. Then $M$ is an s-basis $\Leftrightarrow$ each simple factor of $B$ contains the nonnilpotent image of an element of $M$.*

Using the same argument as in the proof of the nonhomogeneous case of Theorem 1, we can prove

THEOREM 5. *Suppose $B$ is a relatively free f.g. alternative PI-algebra over $\Phi$. Then the set of Shirshov bases of $B$ consisting of words is equal to the set of Shirshov bases of the factor with respect to the associator ideal.*

This follows from the fact that in a Cayley-Dickson algebra there is always a nonnilpotent word in the generators of length at most 2, for any set of generators.

## §2. The auxiliary algebra $A_u^\Phi$

Throughout this paper, $E$ is an alphabet, $W(E)$ is the set of words of finite length over $E$, and $\Phi\langle E\rangle$ is a free associative $\Phi$-algebra over $E$. The notation $u_1 \sqsubset u_2$ signifies that the word $u_1$ is a subword of $u_2$. Otherwise $u_1 \urcorner \sqsubset u_2$. If $u \in W(E)$, then $u^\infty$ is a periodic word infinite in both directions: $\cdots uuu \cdots uuu \cdots$.

Let $\mathrm{Wd}(u) = \{v \in W(E) | v \sqsubset u^\infty\}$. Obviously $W(E)\backslash \mathrm{Wd}(u)$ is an ideal of the semigroup $W(E)$; it generates an ideal $I_u^\Phi$ of the algebra $\Phi\langle E\rangle$. Put $A_u^\Phi = \Phi\langle E\rangle / I_u^\Phi$. Suppose $|c| \geq n$. Let $(c)_n$ denote the leftmost subword of $c$ of length $n$, and $(c)^n$ the rightmost such subword. Obviously $c$ is representable in the form $c = (c)_n e$ and in the form $c = d(c)^n$. If $c = \tilde{c}b$, where $b$ is a letter, we put $\delta(c) = b\tilde{c}$. Obviously $\delta^{|c|}(c) = c$.

REMARK. $\delta(c)$ is cyclically conjugate to $c$. If $c = v^K$, then $\delta^{|v|}(c) = c$; if $c$ is a noncyclic word and $|c|$ does not divide $K$, then $\delta^K(c) \neq c$.

We will show that all words of length $l|u|$ in $\mathrm{Wd}(u)$ are cyclically conjugate. Indeed, suppose $v_1 \sqsubset u^\infty$, $v_2 \sqsubset u^\infty$, $|v_1| = |v_2| = l|u|$, the initial letters of $v_1$ and $v_2$ are adjacent in the word $u^\infty$, and the initial letter of $v_1$ is to the left. Since letters in $u^\infty$ that are a distance a multiple of $n$ apart are identical, it follows that $(v_1)_1 = (v_2)^1$; hence $v_2 = \delta(v_1)$. Therefore, if the distance between the initial letters of $v_1$ and $v_2$ is $K$ and the initial letter of $v_1$ is to the left, then $v_2 = \delta^K(v_1)$.

Everywhere below, $u$ is a noncyclic word, $n = |u|$, and in this case two subwords of $u^\infty$ of length $n$ are cyclically conjugate and are equal if and only if the distance between their initial letters is a multiple of $n$. Therefore if $c_1 \sqsubset u^\infty$, $c_2 \sqsubset u^\infty$, and $(c_1)_n = (c_2)_n$, it follows that the subwords $c_1$ and $c_2$ are at a distance a multiple of $n$ apart. This implies

PROPOSITION 1. *An initial subword of length $n$ uniquely determines a word of* $\mathrm{Wd}(u)$. *If $c_1, c_2 \in \mathrm{Wd}(u)$, $(c_1)_n = (c_2)_n$, and $|c_1| = |c_2|$, then $c_1 = c_2$. If $|c_1| > |c_2|$, then $c_1 = c_2 d$. If $|c| \geq n$, $|d_1| = |d_2|$, and $d_1 \neq d_2$, then $cd_1 \in I_u^\Phi$ or $cd_2 \in I_u^\Phi$.*

Suppose $v^2 = vv \sqsubset u^\infty$, $|v| \geq n$, $(v)_n = (v)_n$, and, in view of what was said above, the initial letters of the left and right $v$ are at a distance $|v|$ apart, a multiple of $n$. Then we have

PROPOSITION 2. *In the algebra $A_u^\Phi$, the set of nonnilpotent words is equal to the set of words that are cyclically conjugate to some power of $u$.*

PROPOSITION 3 (Combinatorial analogue of the assertion of simplicity of a matrix algebra). *Suppose $I \neq 0$ is a homogeneous ideal of $A_u^\Phi$. Then there exist $\lambda_0 \neq 0$, $\lambda_0 \in \Phi$, and $n \in N$ such that $\lambda(A_u^\Phi)^n = 0$.*

PROOF. Suppose $s = \lambda_0 c + \sum \lambda_i c_i \in I$, where $\lambda_i \in \Phi$, $|c_i| = |c|$, $c_i \neq c$, and $\lambda_0 \neq 0$. Each sufficiently long word in $\mathrm{Wd}(u)$ has the form $v_1 c v_2$, where $|v_1| \geq n$. By Proposition 1,

$$\lambda_0 v_1 c v_2 + v_1 \left(\sum \lambda_i c_i\right) v_2 = \lambda_0 v_1 c v_2 + \sum \lambda_i v_1 c_i v_2 \equiv \lambda_0 v_1 c v_2 \mod I_u^\Phi.$$

Therefore $\lambda_0 v_1 c v_2 \in I$. The proposition is proved.

Suppose $U$ is the $\Phi$-module generated by the words of length $n$ in the algebra $A_u^\Phi$. Put $e_1 = u$ and $e_i = \delta^{i-1}(u)$. The module $U$ is free, and $e_1, \dots, e_N$ is a basis.

THEOREM 6. $\mathrm{var}(A_u^\Phi) = \mathrm{var}(\Phi[x] \otimes_\Phi \mathrm{Mat}_n(\Phi))$.

PROOF. Since $\mathrm{var}(A_u^\Phi)$ is a homogeneous variety, it follows that to prove the theorem it suffices to construct mappings $h$ and $g$ such that $A_u^\Phi \xrightarrow{h} \mathrm{End}_\Phi(U) \to 0$ and $0 \to A_u^\Phi \xrightarrow{g} \mathrm{End}_{\Phi[x]}(U \otimes_\Phi \Phi[x])$.

Let us construct the mapping $h$. Suppose $d$ is a word, $d \in \mathrm{Wd}(u)$. Put $h(d)(e_i) = (\delta^i(u)d)^n$. If $|d| > n$, $(d)_n = e_i$, and $(d)^n = e_j$, then $h(d)(e_k) = \delta_{ik} e_j$, i.e. $h(d)$ is the matrix unit $E_{ji}$; hence $h$ is an epimorphism.

*Construction of $g$.* Suppose $d$ is a word in $\mathrm{Wd}(u)$ and $f \in U$. Put

$$g(d)(f \otimes 1) = h(d)(f) \otimes x^{|d|}.$$

Suppose $\varphi\colon U \otimes_\Phi \Phi[x] \to U$ and $\varphi(f \otimes P(x)) = f \otimes P(1)$. For any $d \in A_u^\Phi$ we have a commutative diagram:

$$\begin{tikzcd} U \arrow[r] & U \otimes 1 \arrow[r, "g(d)"] \arrow[dr, "h(d)"'] & U \otimes \Phi[x] \arrow[d, "\varphi"] \\ & & U \end{tikzcd}$$

In view of the homogeneity of $g$ and Proposition 3, $g$ is an embedding.

Theorem 6 was proved earlier in a weaker form by V. V. Borisenko.

DEDUCTION OF THEOREM 2 FROM THEOREM 1. It suffices to show that a reduced Shirshov basis contains a sample, i.e. for any noncyclic word $u$ such that $|u| \leq n$ a basis $Y$ contains a word cyclically conjugate to $u$.

Since $A$ is a relatively free algebra of complexity $n$, it follows from Theorem 6 that there exists a factor $C$ of $A$ such that $C \xrightarrow{\sim} A_u^{\Phi}$. Since the image of a Shirshov basis under a mapping onto an infinite-dimensional nonnilpotent algebra contains a nonnilpotent element, the desired assertion follows from Proposition 2.

## §3. Preparatory lemmas

Suppose $E = \{a_1, \dots, a_s\}$ is an alphabet and the order $a_1 \prec \cdots \prec a_s$ induces the lexicographic order on $W(E)$. Let

$$W^l(E) = \{c \in W(E) \mid |c| \geq l\}; \qquad \overline{W}^l = \{c \in W(E) \mid |c| = l\}.$$

There exists a unique isomorphism $\psi$ of the ordered sets $W^l(E)$ with the lexicographic order and the segment $[1; s^l]$ of the natural sequence. For $c \in W^l(E)$ we put $N_l(c) = \psi((c)_l)$. Everywhere below, $v$ is a fixed word of length $n$ in $W(E)$, $b \notin W(E)$, and $z$ denotes a fixed element of $W^{2mn}(E)$; we also assume that the word $(z)_{2mn}$ contains no subword of the form $q^m$, where $|q| \leq n$. Let $K = (m+1)n$ and $\mathscr{K} = 2mn$.

**Proposition 4.** *Suppose $0 < |q| \leq n$. Then $(qz)_K \neq (z)_K$.*

Indeed, if $(z)_K = (qz)_K$, then $(q^s z)_K = (q^{s+1} z)_K$, hence $(z)_K = (q^K)_K$, which is impossible by virtue of the choice of $z$.

**Lemma 1.** *Suppose $\mathscr{T} \sqsubset (vb)^\infty, b\urcorner \sqsubset v$, $\mathscr{T} = t_0 \cdots t_{p+1}$, and $\forall i\ |t_i| > 4n$, $\sigma \in S_p$, and $\mathscr{T}_\sigma = t_0 t_{\sigma(1)} \cdots t_{\sigma(p)} t_{p+1}$, where $t_0 = \bar{t}_0 b e_0$, $t_i = f_i b \bar{t}_i b e_i$ for $1 \leq i \leq p$, $t_{p+1} = f_{p+1} b \bar{t}_{p+1}$, $\mathscr{K} = 2mn$, $\mathscr{T}_\sigma\urcorner \sqsubset (vb)^\infty$, and, for any $i$, $b\urcorner \sqsubset e_i$ and $b\urcorner \sqsubset f_i$. Put $\sigma(0) = 0$ and $\sigma(p+1) = p+1$. Then there exists an $i$ such that $(e_{\sigma(i)} f_{\sigma(i+1)} z)_{\mathscr{K}} \prec (vz)_{\mathscr{K}}$.*

**Proof.** If $q_1 q_2 = v$, put $\overline{N}(q_1) = N_{\mathscr{K}}(q_2 z)$. If $v = q_1 q_2 = q_3 q_4$ and $q_2 \neq q_4$, then $q_1 q_4 \neq v$ and $\forall i\ |q_i| \leq n$.

It follows from Proposition 4 that $(q_1 q_4 z)_{\mathscr{K}} \neq (q_1 q_2 z)_{\mathscr{K}} = (vz)_{\mathscr{K}}$; hence $\overline{N}(q_1) - N_{\mathscr{K}}(q_4 z) \neq 0$, and if $(\overline{N}(q_1) - N_{\mathscr{K}}(q_4 z)) > 0$, then $(q_1 q_4 z)_{\mathscr{K}} \prec (q_1 q_2 z)_{\mathscr{K}} = (vz)_{\mathscr{K}}$.

Consider the sum

$$\sum_{i=0}^{p} (\overline{N}(e_{\sigma(i)}) - N_{\mathscr{K}}(f_{\sigma(i+1)} z)) = 0,$$

since $e_j f_{j+1} = v$ and $\overline{N}(e_j) = N_{\mathscr{K}}(f_{j+1} z)$. If this sum contains a positive term with index $i$, we have $(e_{\sigma(i)} f_{\sigma(i+1)} z)_{\mathscr{K}} \prec (vz)_{\mathscr{K}}$, as required.

Since the sum is equal to zero, it suffices to establish the presence of a nonzero term; but since $\mathscr{T}_\sigma\urcorner \sqsubset (vb)^\infty$, there is an $i$ such that $be_{\sigma(i)} f_{\sigma(i+1)} ab \neq bvb$. Then

$$\overline{N}(e_{\sigma(i)}) - N_{2mn}(f_{\sigma(i+1)} z) \neq 0.$$

The lemma is proved.

**Corollary.** *Suppose, under the hypothesis of Lemma 1, that $\tilde{\mathscr{T}_\sigma}$ is obtained from $\mathscr{T}_\sigma$ by the substitution $b \to z$. Then $\forall R$ $(2mn \leq R \leq |vz|)$ there exists $H \sqsubset \tilde{\mathscr{T}_\sigma}$ such that $|H| = R$ and $(H)_{2mn} \prec (vz)_{2mn}$.*

Indeed, since $|t_i| > 4n$, we can put $H = (e_{\sigma(i)} f_{\sigma(i+1)} zvz)_R$.

LEMMA 2. *Suppose $u$ is a noncyclic word, $|u| = n$, $f$ is a multilinear identity over a field $F$ of degree $p$, and* $\operatorname{Pid}(f) < n$. *Then* $\exists \mathscr{T} \sqsubset u^\infty$, $\mathscr{T} = t_0 t_1 \cdots t_p t_{p+1}$, $\forall i$ $4n \le |t_i| \le 10n$, *such that $\mathscr{T}$ is linearly representable* mod $T(f)$ *by words of the form* $\mathscr{T}_\sigma = t_0 t_{\sigma(1)} \cdots t_{\sigma(p)} t_{p+1}, \sigma \in S_p$, *such that* $\mathscr{T}_\sigma \urcorner \sqsubset u^\infty$.

PROOF. Suppose $\tilde{f}$ is an identity obtained from the identity $x_0 f(x_1, \dots, x_p) x_{p+1}$ by replacing the variables $x_i$ by monomials of degree $4p$, where all $4p(p+2)$ variables occurring in the monomials are distinct. Obviously $\operatorname{Pid}(\tilde{f}) = \operatorname{Pid}(f)$ and, by Theorem 6, $\tilde{f}$ is not satisfied in $A_u^F$. Consequently, there exists a set of words $\{t_0, t_1, \dots, t_{p+1}\}$, each of degree at least $4p$, such that $t_0 f(t_1, \dots, t_p) t_{p+1} \not\equiv 0 \bmod I_u^F$. We may assume that $\mathscr{T} = t_0 t_1 \cdots t_p t_{p+1} \sqsubset u^\infty$. Since $|t_0| > 4n$, it follows from Proposition 1 that $\forall \sigma \in S_p \mathscr{T}_\sigma \sqsubset u^\infty \Leftrightarrow \mathscr{T}_\sigma = \mathscr{T}$. Then there exists a set $\{\lambda_\sigma\}$ of elements of $F$ such that

$$\bmod T(f) \quad 0 \equiv \sum \lambda_\sigma \mathscr{T}_\sigma \not\equiv 0 \mod I_u^F,$$

or

$$\bmod T(f) \quad 0 \equiv \left( \sum_{\mathscr{T}_\sigma = \mathscr{T}} \lambda_\sigma \right) \mathscr{T} + \sum_{\mathscr{T}_\sigma \urcorner \sqsubset u^\infty} \lambda_\sigma \mathscr{T}_\sigma \not\equiv 0 \mod I_u^F.$$

Consequently,

$$\sum_{\mathscr{T}_\sigma = \mathscr{T}} \lambda_\sigma = \lambda \neq 0, \qquad \mathscr{T} \equiv \frac{-1}{\lambda} \sum_{\mathscr{T}_\sigma \urcorner \sqsubset u^\infty} \lambda_\sigma \mathscr{T}_\sigma \mod T(f).$$

The lemma follows from the fact that if $|t_i| > 10n$, we can eliminate from $t_i$ a subword equal to $u$, and for all $\mathscr{T}_\sigma$ the property of being a subword of $u^\infty$ or not being such a subword is preserved. The lemma is valid for all identities, not just multilinear ones.

From Lemma 2 and the corollary of Lemma 1 we obtain

PROPOSITION 5. $\forall k$ $2mn \le k \le |vz|$ *and for* $s > (p+2) \cdot 10n$, *the word* $(vz)^s$ *is linearly representable modulo $T(f)$ by words $c_j$, each of which contains a subword $H_j$ of length $k$ such that* $(H_j)_{2mn} \prec (vz)_{2mn}$.

Suppose $\alpha$ is a set of words. We will call it *distinguished* if the factor with respect to the ideal generated by $\alpha$ is nilpotent; we will denote its degree of nilpotency by $l(\alpha)$. If a set $\{u_i\}$ is distinguished and $v_i \sqsubset u_i$, then the set $\{v_i\}$ is distinguished and $l(\{u_i\}) \ge l(\{v_i\})$. The set $\alpha$ is distinguished if and only if each word of length at least $l(\alpha)$ is linearly representable by words of the form $d_1 c d_2$, where $c \in \alpha$.

LEMMA 3 (replication). *Suppose $f$ is a multilinear identity of the algebra $A$,* $\deg f = m$, *and $\{D_j\}$ is the set of words of degree at most $m$ (including the empty word) in the generators of $A$ and the elements $c_i$, where $\{c_i\}$ is a distinguished set. Then* $\forall R \in \mathbf{N}$ *the set* $\{(c_i D_j)^R\} = \alpha_R$ *is distinguished, and* $l(\alpha_R) < K(l(\{c_i\}), p, m, R)$ *for any field $F$.*

PROOF. Each word of length at least $l(\{c_i\})$ is linearly representable by words of the form $d_1 c_i d_2$. We use this fact and also the fact that each word having sufficiently high degree in $\alpha = \{c_i\}$ is representable by Shirshov's theorem in the necessary way. (We also use the fact that the $(R+1)$th power of a word contains the $R$th power of any cyclically conjugate word; hence the words $c_i$ appear at the beginning.) The lemma is proved.

## §4. Proof of Theorem 1

*Multilinear case.* Suppose $n = \mathrm{Pid}(A)$, $m = \deg A$, $\Phi$ is a field, $p = \deg f$, and $f$ is a multilinear identity of complexity $n$ satisfied in the algebra $A$. In view of the main lemma, it suffices to show that the set $Q_n^m$ of $m$th powers of the words of degree at most $n$ is distinguished. The following operations on sets of words preserve the property "of being distinguished":

1) Replacing a word by a subword; replacing a word by a set of words by which it is linearly representable.

2) "Replication"; replacing the set $\{c_i\}$ by the set $\{(c_iD_j)^R\}$, where $\{D_j\}$ is the set of words of degree at most $p$ in the $c_i$ and the generators of $A$.

3) If $R > 20p$, $|c_i| \geq 2mn$, and $c_i$ contains no subword in $Q_n^m$, then the word $(c_iD)^R$ can be replaced by the set of words of the form $\{H_y\}$, where $|H_y| = 2mn$ and $H_y \prec (c_i)_{2mn}$ (see Proposition 5).

The multilinear case follows from the fact that by means of operations 1)–3) we can make the set $Q_n^m$ from the set of all words of length $2mn$.

Deduction of Theorem 3. Suppose the set $M$ and algebra $A$ satisfy the conditions of Theorem 3. It follows from the nilpotency of $A/M^{(m)}$ and Lemma 3 that the set $\{(m_i^nD_j)^R\}$ is distinguished for each $R$. By Theorem 1, $(m_i^nD_j)^R \subset M^{(m)}$ for large $R$. We now use the main lemma.

Remark. From the proof of Theorem 1 we deduce the existence of a function $\rho(m,n,p,s)$, where $n = \mathrm{Pid}\,A = \mathrm{Pid}\,f$, $m = \deg A$, $p = \deg f$, and $s$ is the number of generators of the $F$-algebra $A$, such that all words of length $\rho(m,n,p,s)$ lie in $\mathrm{Id}(Q_n^m)$; $\rho$ does not depend on $F$.

From the above proof and estimates of Shirshov [2] it follows that $\rho(m,n,p,s)$ can be taken equal to

$$\exp(\exp(\exp(\exp(\exp(\exp(\exp(mnps))))))).$$

*General case.* Let $T(\mathrm{st}_n)$ be the ideal generated by the standard identity $\mathrm{st}_n$ of degree $n$, and $\hat{A}_s^F(g)$ a relatively free $s$-generator $F$-algebra in the variety generated by the identity $g$. Let $\mathrm{Alg}(A)$ denote the set of elements of $A$ that are algebraic over $F$, $I(A)$ the ideal generated by the homogeneous components of identities in $A$, $\overline{A} = A/I(A)$, $\pi$ the natural projection, $N_A$ the radical of $\overline{A}$, and $m = \deg A$.

Lemma 4. *Suppose $A$ is an algebra of degree $m$. Then for each $n$ there exists a finite set $Y_n = \{f_1,\dots,f_t\}$, $Y_n \subset T(\mathrm{st}_{2n})$, such that $A$ has bounded height over $Q_N^1 \cup \{f_1,\dots,f_t\}$.*

Proof. We may assume that $A$ is a graded relatively free algebra, $a_1,\dots,a_s$ its generators, and $\rho = \rho(2n,n,2n,s)$. We will show that all words of length $\rho$ in $\mathbf{Z}\langle a_1,\dots,a_s\rangle$ (and therefore also in $\Phi\otimes\mathbf{Z}\langle a_1,\dots,a_s\rangle$) lie in $T(\mathrm{st}_{2n}) + \mathrm{Id}(Q_n^m)$. Let $L$ be the module generated by the words of length $\rho$,

$$M = L \cap [T(\mathrm{st}_{2n}) + \mathrm{id}(Q_n^m)], \qquad N = L/M.$$

From the multilinear case of Theorem 1 and the right faithfulness of the tensor product it follows that $N\otimes\mathbf{Z}_q = 0$ for any prime $q$. Since $N$ is f.g., $N = 0$. Consequently, there exists $W = \{g_1,\dots,g_R\} \subset T(\mathrm{st}_{2n})$, $\#W < \infty$, such that the set $Q_m^n \cup W$ is distinguished. It remains to apply Lemma 3 and the main lemma.

It suffices to show that if $n = \mathrm{Pid}(A)$, then $T(\mathrm{st}_{2n}) \subset \mathrm{Alg}(A)$; this follows from Proposition 7.

PROPOSITION 6. *Suppose $F$ is a field, $\#F > \deg g$, and $F' \rhd F$. Then*

$$\operatorname{var}(\hat{A}_s^F(g) \otimes_F F') = \operatorname{var}(\hat{A}_s^{F'}(g)), \qquad \hat{A}_s^{F'}(g) \simeq \hat{A}_s^F(g) \otimes_F F'.$$

Indeed, $\hat{A}_s^F(g)$ satisfies all components of $g$ homogeneous in each variable and all of their linearizations (see [10], Chapter 1).

COROLLARY 1. *Suppose $x$ is a homogeneous component of $g$. Then each simple factor of $A$ in which the image of $x$ is nonnilpotent contains at most $(\deg g)^m = K$ elements.*

COROLLARY 2. *In this case, $x^{2K!} - x^{K!}$ lies in the radical of $A$; hence $x$ is an algebraic element and $I(A) \subset \operatorname{Alg}(A)$.*

PROPOSITION 7. *Suppose $A$ is a relatively free algebra. Then $\operatorname{Alg}(A) = \pi^{-1}(N_A)$ is an ideal; if $\Phi$ is Noetherian, then $\exists t \in \mathbf{N}$: $(\operatorname{Alg}(A))^t \subset I(A)$.*

PROOF. If $x \in \operatorname{Alg}(A)$, then $\pi(x)$ is an algebraic element of the graded algebra $\overline{A}$ and is nilpotent. The existence of the desired $t$ follows from a theorem of Braun (see [12]).

## §5. The "transfer" procedure. Proof of the main lemma

Consider this game: given a word $c$, the first player designates a set of words by which it is linearly representable, the second designates a word in this set, and the first wants to bring this word to a desired form; the possibility of doing this is equivalent to the linear representability of $c$ by words of this form.

Consider this game: there are $l$ groups of objects, with $K_1, \dots, K_l$ being the numbers of objects in these groups. The first player chooses $m$ groups and partitions each of them into a left part and a right part, and then the second player rearranges the right parts nonidentically, after which the left and right parts are reunited.

The aim of the first player is to try to see that at most $m - 1$ groups contain at least $m$ objects.

Suppose $K_{i_1}, \dots, K_{i_m} \geq m$. We select the groups with indices $i_1, \dots, i_m$ and put $\overline{K}_{i_q} = K_{i_q} - q$. Then $\forall \sigma \in S_m \backslash \{1\}$ the vector

$$(K_1, \dots, \overline{K}_{i_1} + \sigma(1), \dots, \overline{K}_{i_q} + \sigma(q), \dots, \overline{K}_{i_m} + \sigma(m), \dots, K_l)$$

is lexicographically greater than the vector $(K_1, \dots, K_l)$; in trying to increase the vector $(K_1, \dots, K_l)$ the first player arrives at the desired result.

An equivalent game: a word is printed on a tape, the first player cuts the word into $m+2$ pieces, and the second nonidentically rearranges the middle $m$. The aim of the first player is to achieve a "transfer" of almost all letters of the set of long subwords of $c$ into $m - 1$ subwords. It is easy to see that by replacing a subword of length $k$ with a word containing the $m$th power of a basic word and knowing how to "transfer" powers of a basic word together we can bring a word to the form required by the definition of essential height. Let us turn to a formal exposition (the definitions were given in §1).

Everywhere below, $E$ is an alphabet, possibly infinite, and $\Phi\langle E\rangle$ is a free graded associative algebra over $E$. For each $K$ we have $\dim T_K < \infty$, where $T_K = \{x \in \Phi\langle E\rangle \mid \|x\| \leq K\}$. Also, $\forall i = 1, \dots, p$, $B_i$ is a subsemigroup of $W(E)$ generated by letters, and $B_i \cap B_j = \varnothing$ for $i \neq j$. Put $A_i = B_i^m$, where $m = \deg f$, $f$ being a multilinear identity satisfied in the algebra $A$.

We call a representation $c = c_0\beta_0 \cdots c_R\beta_R c_{R+1}$ of a word $c$ *regular* if

a) $\forall j$, $\forall x \in A_j$ $x^{\urcorner} \sqsubset c_i$, and

b) the $\beta_i$ are maximal subwords of $c$ with respect to inclusion such that $|\beta_i| \geq m$ and $\beta_i \in A_{j(i)}$.

Since all of the $B_i$ are generated by letters and $B_i \cap B_j = \varnothing$ for $i \neq j$, a regular representation of $c$ exists and is unique; therefore the relation $\beta_i \in A_{j(i)}$ uniquely defines a function $j(i)$ connected with the word $c$. Put $k_i = |\beta_i|$; then obviously $k_i \geq m$ for $0 \leq i \leq R$. Put $N_i = \|\beta_i c_{i+1} \cdots c_R \beta_R c_{R+1}\|$ and $k_i = N_i = 0$ for $i > R$. Note that $N_0 > N_1 > \cdots > N_R > N_{R+1} = \cdots = 0$.

By the *indicator* $\overrightarrow{\mathrm{PK}}(c)$ of a word $c$ we mean the vector

$$\overrightarrow{\mathrm{PK}}(c) = (N_0, k_0, N_1, k_1, \dots, N_R, k_R, 0, \dots, 0, \dots)$$

in $\mathbf{N}^{\mathbf{N}}$. We order the set of indicators lexicographically. The following lemma is obvious.

LEMMA 5. *Suppose there is given a regular representation*

$$c = c_0 \beta_0 c_1 \beta_1 \cdots c_R \beta_R c_{R+1}$$

*of the word c. Then:*

a) *The last letter of $c_i$ and the first letter of $c_{i+1}$ do not belong to $B_{j(i)}$.*

b) *Suppose $\|c_i\| = \|c_i'\|$ and $c' = c_0 \beta_0 \cdots \beta_{i-1} c_i' \beta_i \cdots c_{R+1}$, where $c_i'$ contains a subword of $A_k$ for some $k$. Then $\overrightarrow{\mathrm{PK}}(c') \succ \overrightarrow{\mathrm{PK}}(c)$.*

c) *Suppose $\|c\| = \|c'\|$, the word $c'$ has the form $c' = c_0 \beta_0 \cdots c_i \beta_i' t$ for some $t$, and $|\beta_i'| > |\beta_i|$. Then $\overrightarrow{\mathrm{PK}}(c') \succ \overrightarrow{\mathrm{PK}}(c)$.*

For each homogeneous ideal $J \subset \Phi\langle E\rangle$ we denote by $\mathscr{L}_J$ the set of words not representable linearly mod $J$ by words with larger indicator.

PROPOSITION 8. *The $\mathscr{L}_J + J$ generate $\Phi\langle E\rangle$ as a $\Phi$-module.*

PROOF. Suppose $c \in \mathscr{L}_J$ and represent $c$ by a linear combination of words with larger indicator: $c = \sum \lambda_i y_i$, $\lambda_i \in \Phi$, $\|y_i\| = \|c\|$. If $y_i \notin \mathscr{L}_J$, carry out the same procedure with $y_i$. The finiteness of the number of words of given degree of homogeneity guarantees that the process terminates.

Everywhere below, $\hat{S}$ is the ideal of $\Phi\langle E\rangle$ generated by $\bigcup_1^p A_i$.

PROPOSITION 9. *Suppose that in $\Phi\langle E\rangle/(J + \hat{S})$ all homogeneous components of degree at least $l$ are zero, $c \in \mathscr{L}_J$, and $c = c_0 \beta_0 \cdots \beta_R c_{R+1}$ is a regular representation of $c$. Then $\forall i$ $\|c_i\| \leq l$.*

PROOF. If $\|c_i\| > l$, then $c_i$ is linearly representable by words $c_{ik}$, where $\|c_{ik}\| = \|c_i\|$ and $\overrightarrow{\mathrm{PK}}(c_{ik}) \succ (0, 0, \dots)$. We now use part b) of Lemma 5.

LEMMA 6. *Suppose there is given a regular representation $c = c_0 \beta_0 \cdots \beta_R c_{R+1}$ of the word $c$, and there exists a sequence $0 \leq i_1 < i_2 < \cdots < i_m \leq R$ such that $j(i_1) = j(i_2) = \cdots = j(i_m)$. Then $c \notin \mathscr{L}_{T(f)}$ and*

$$f = x_1 \cdots x_m + \sum_{\sigma \neq 1} \lambda_\sigma x_{\sigma(1)} \cdots x_{\sigma(m)}, \qquad \lambda_0 \in \Phi.$$

PROOF. Represent $\beta_{i_K}$ in the form $\delta_K \gamma_K$ so that $|\gamma_1| < \cdots < |\gamma_m|$ (this can be done since $\forall i$ $|\beta_i| \geq m$). Put

$$t_0 = c_0 \beta_0 \cdots c_{i_1} \delta_1, \qquad t_m = \gamma_m c_{i_{m+1}} \cdots c_{R+1},$$
$$t_K = \gamma_K c_{i_K+1} \cdots c_{i_{K+1}} \delta_{K+1}, \qquad 1 \leq K < m.$$

Then $c = t_0 t_1 \cdots t_m$.

To prove the lemma it suffices to show that $\overrightarrow{\mathrm{PK}}(c_\sigma) \succ \overrightarrow{\mathrm{PK}}(c)$, where $\sigma \neq 1$ and $c_\sigma = t_0 t_{\sigma(1)} \cdots t_{\sigma(m)}$. Let $K$ be the smallest number such that $\sigma(K) \neq K$; then $\sigma(K) > K$, $c_\sigma = t_0 \cdots t_{K-1} t_{\sigma(K)} \cdots t_{\sigma(m)}$, $\|c_\sigma\| = \|c\|$, and the word $c_\sigma$ has the form $c_\sigma = c_0\beta_0 \cdots c_{i_K}\delta_K\gamma_{\sigma(K)}\omega$ for some $\omega$, $|\gamma_{\sigma(K)}| > |\gamma_K|$. It remains to use part c) of Lemma 5.

From Lemma 6, Dirichlet's principle, and Proposition 9 we obtain

PROPOSITION 10. *Suppose $J \supset T(f)$, $c \in \mathscr{L}_J$, and all homogeneous components of $\Phi\langle E\rangle/(\hat{S} + J)$ of degree at least $l$ are zero. Then the regular representation $c = c_0\beta_0 \cdots \beta_R c_{R+1}$ satisfies the following conditions:*

a) $\forall i\ \|c_i\| \leq l$.

b) $R \leq (m-1)p$ *($p$ being the number of sets $A_i$).*

PROOF OF THE MAIN LEMMA. The alphabet $E$ will consist of letters $\overline{a}_i$ corresponding to the generators of the algebra $A$ and letters $\overline{m}_j$ corresponding to $m_j \in M$, $m_j \in M \subset A$, $\|\overline{a}_i\| = \|a_i\|$, and $\|\overline{m}_j\| = \|m_j\|$. Put $B_j = \{\overline{m}_j^k\}_{k \geq 1}$; then $M = \{\overline{m}_j\}_{j=1}^p$, $A_j = \{m_j^k\}_{k=m}^\infty$, and $\overline{S} = \overline{M}^{(m)}$. There is an exact sequence $0 \to J_A \to \Phi\langle E\rangle \xrightarrow{\pi} A \to 0$, where $\pi\colon \overline{a}_i \mapsto a_i$ and $\overline{m}_j \mapsto m_j$. The ideals $\hat{S}$ and $J_A$ satisfy the conditions of Proposition 10 for some $l$ if $A/M^{(m)}$ is nilpotent.

Proposition 10 means that the essential height of a word $c \in \mathscr{L}_{J_A}$ is at most $2(m-1)p$.

REMARK. In exactly the same way we can show that at most $m-1$ words $c_i$ can have length at least $m$ if $c \in \mathscr{L}_{J_A}$ (we "transfer" to the right).

## §6. The nonassociative case

Suppose $B$ is an algebra and $M = \{m_i\}_1^p$ a set of its elements satisfying the conditions of Theorem 4; $b_1, \dots, b_s$ are generators of $B$, $Q = \{q_i\}$ is a set of nonassociative monomials in $b_1, \dots, b_s, m_1, \dots, m_p$, the alphabet $E$ consists of symbols $\overline{L(q_i)}$ corresponding to the $q_i \in Q$, and $\|\overline{L(q_i)}\| = \|q_i\|$; $f$ is a multilinear identity in $L[B]$ of the form

$$f(x_1, \dots, x_m) = x_1 \cdots x_m + \sum_{\sigma \neq 1} \lambda_\sigma x_{\sigma(1)} \cdots x_{\sigma(m)}, \qquad \lambda_\sigma \in \Phi.$$

Let $B_j$ be the subsemigroup of $W(E)$ generated by $\overline{L(q_{\mathscr{S}})}$, where $q_{\mathscr{S}} \in B_{m_j}$, and let $A_j = B_j^m$ and $\hat{S} = \mathrm{id}(\bigcup_1^p A_j)$. There is an exact sequence

$$0 \to J_{L[B]} \to \Phi\langle E\rangle \xrightarrow{\pi} L[B] \to 0, \qquad \pi\colon \overline{L(q)} \to L(q).$$

LEMMA 7. *There exists $k \in \mathbf{N}$ such that all homogeneous components of $\Phi\langle E\rangle/(J_{L[B]} + \hat{S})$ of degree at least $k$ are zero.*

PROOF OF THEOREM 4. It follows from Lemma 7 and Propositions 9 and 10 that $\Phi\langle E\rangle$ is linearly representable mod $J_{L[B]}$ by elements of the form $c_0\beta_0 \cdots \beta_R c_{R+1}$, where $R \leq p(m-1)$, $\|c_i\| \leq k$, and $\beta_i \in A_{j(i)}$. Since the $L$-length of $B$ is bounded, there exists $h$ such that any element of $\pi(A_j)$ is linearly representable by elements of the form $L(q_1) \cdots L(q_{\mathscr{K}})$, where $\mathscr{K} \leq h$ and $q_i \in B_{m_j}$. Theorem 4 follows from the fact that each $x \in B$ is linearly representable by elements of the form $G \circ b_i$, where $G \in L[B]$, and from the definition of essential height.

PROOF OF LEMMA 7. Since $B/M^{(l)}$ is nilpotent for each $l$ and the $L$-length is bounded, it follows that for each $h$ all elements of sufficiently high degree of homogeneity lie in the ideal generated by the elements of the form

$$L(\tilde{N}(b_{i_1}, \dots, b_{i_p}, c, b_{i_{p+1}}, \dots, b_{i_k})),$$

where $\tilde{N} \in \mathrm{MoN}_{k+1}$ and $c \in B^h_{m_i}$. In view of condition (ii) for nice varieties, $L(\tilde{N}(b_{i_1}, \dots, b_{i_p}, c, \dots, b_{i_k}))$ lies in the ideal generated by the elements of the form $L(q_1) \cdots L(q_l)$, where $\tilde{N} \in \mathrm{MoN}_{l+1}$ and $l < \pi_{\mathrm{var}(B)}(p+s)$.

Using once again condition (ii) and Dirichlet's principle, we see that for sufficiently large $h$ we have that $L(\tilde{N}(b_{\mathscr{S}_1}, \dots, b_{\mathscr{S}_i}, c, \dots, b_{\mathscr{S}_l}))$, where $c \in B^h_{m_i}$, lies in the ideal generated by the elements of the form $L(q_1) \cdots L(q_l)$, where $q_j \in B_{m_i}$ $\forall j$. But $L(q_1) \cdots L(q_l) \in \hat{S}$.

It follows from the proof that the hypothesis of Theorem 4 can be weakened, it being sufficient to require only the nilpotency of $B/M^{(\gamma)}$, where

$$\gamma(\pi_{\mathrm{var}(B)}(p+s))^2 \deg(f).$$

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Received 6/OCT/86

## Bibliography


1. A. I. Shirshov, *On certain nonassociative nil rings and algebraic algebras*, Mat. Sb. **41(83)** (1957), 381–394; English transl. in Amer. Math. Soc. Transl. (2) **119** (1983).

2. ——, *On rings with polynomial identities*, Mat. Sb. **43(85)** (1957), 277–283; English transl. in Amer. Math. Soc. Transl. (2) **119** (1983).

3. V. A. Ufnarovskiĭ, *An independence theorem and its consequences*, Mat. Sb. **128(170)** (1985), 124–132; English transl. in Math. USSR Sb. **56** (1987).

4. S. V. Pchelintsev, *A theorem on height for alternative algebras*, Mat. Sb. **124(166)** (1984), 557–567; English transl. in Math. USSR Sb. **52** (1985).

5. I. P. Shestakov, *Finitely generated special Jordan algebras and alternative PI-algebras*, Mat. Sb. **122(164)** (1983), 31–40; English transl. in Math. USSR Sb. **50** (1985).

6. Yu. A. Medvedev, *Nilradicals of finitely generated Jordan PI-algebras*, Preprint No. 24, Inst. Mat. Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk, 1985. (Russian) R. Zh. Mat. **1986**, 6A398.

7. V. G. Skosyrskiĭ, *Jordan algebras with the minimality condition for two-sided ideals*, Preprint No. 21, Inst. Mat. Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk, 1981. (Russian) R. Zh. Mat. **1982**, 5A223.

8. E. I. Zel′manov, *Absolute zero-divisors and algebraic Jordan algebras*, Sibirsk. Mat. Zh. **23** (1982), no. 6, 100–116; English transl. in Siberian Math. J. **23** (1982).

9. I. V. L′vov, *On Shirshov's height theorem*, Fifth All-Union Sympos. Theory of Rings, Algebras, and Modules, Abstracts of Reports, Inst. Mat. Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk, 1982. (Russian)

10. K. A. Zhevlakov et al., *Rings that are nearly associative*, "Nauka", Moscow, 1978; English transl., Academic Press, 1982.

11. I. N. Herstein, *Noncommutative rings*, Math. Assoc. Amer. and Wiley, New York, 1968.

12. I. V. L′vov, *Braun's theorem on the radical of a finitely generated PI-algebra*, Preprint No. 63, Inst. Mat. Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk, 1984. (Russian) R. Zh. Mat. **1985**, 6A212.


Translated by G. A. KANDALL