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ON RINGS WHICH ARE ASYMPTOTICALLY CLOSE TO ASSOCIATIVE RINGS

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The subject of this work is extension of A. R. Kemer's results to a rather wide class of rings, close to associative rings, over a field of characteristic 0 (in particular, this class includes the varieties generated by finite-dimensional alternative and Jordan rings) For this case we prove finite-basedness of systems of identities (Specht property), representability of finitely generated relatively free algebras and rationality of their Hilbert series. For this purpose, we extend Razmyslov —Zubrilin's theory to Kemer polynomials. For a rather wide class of varieties we prove Shirshov theorem on height.

Ключевые слова и фразы: *PI*-algebra, representable algebra, universal algebra, non-associative algebra, alternative algebra, Jordan algebra, signature, polynomial identity, Hilbert series, Specht problem.

§1. Introduction

An identity in an algebra means a polynomial which vanishes on the whole algebra. For instance, a $(n-1)$-dimensional algebra satisfies *Capelli identity* C_n of order n:

$$C_n(\vec{x},\vec{y}) = \sum_{\sigma\in S_n} (-1)^\sigma y_0 x_{\sigma(1)} y_1 x_{\sigma(2)} y_2 \cdots y_{n-1} x_{\sigma(n)} y_n.$$

Various important classes of algebras (for example, associative, alternative, Lie, Jordan algebras) are axiomatizable by identities. The class of algebras which satisfy a given system of identities is a category which is called *a variety* (*variety*), and the free objects of this category are called *relatively free algebras*.

Various non-associative structures recently have obtained rather numerous applications in the major areas of mathematics. For instance, A. V. Yagzhev has elaborated the approach to Jacobian problem based on universal algebra. This reduced Jacobian problem to problems of *PI*-theory (this problem occurred to be equivalent to the question on weak nilpotency of any ternary Engel algebra over a field of characteristic 0). A. V. Yagzhev's approach is related to

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quantization theory and operads. It is not casual that I. P. Shestakov, one of leading experts in non-associative *PI*-theory, is actively working in polynomial automorphismas and quantum theory. Beatiful results of Ye. I. Zelmanov (in particular, solution of weakened Burnside problem) were preceded by his works in the theory of Jordan algebras. Numerous mathematicians specializing in *PI*-theory also did work in the theory of polynomial automorphisms. The monograph [54] is devoted to problems of these theories.

A rather evolved structure as well as combinatorial theory has been elaborated for associative algebras. First of all, this includes radical theory, prime algebras, Burnside-type problems, finite-basedness problems. Structure theory also is rather evolved for a certain class of rings close to associative ones, and analogues of Kemer's theorem have been obtained. At the same time, investigation of bounds for finite basedness (which has different nature in zero and in positive characteristics) and developing of the corresponding theory in a maximally general form seems to be a very important and actual problem. In the present paper, we consider only the case of algebras over a field of zero characteristic.

A point of an ordinary algebraic variety is a set of numbers satisfying some system of algebraic equations. A point of a variety of algebras is an algebra satisfying a set of identities. To a system of algebraic equations, there corresponds an ideal in the ring of commutative polynomials. What ideal corresponds to a set of identities?

Let $P(x_1,\dots,x_n)$ be an identity in an algebra A, $\{H_i(y_1,\dots,y_{q_i})\}$ an arbitrary set of polynomials, and $R(z_1,\dots,z_k)$ an arbitrary polynomial. Then the results of substituting $P(H_1,\dots,H_n)$ and multiplying by R (RP and PR) hold in A as well. These new polynomials are called *consequences* of the identity P. Moreover, a linear combination of identities is an identity again. An ideal generated by the values of a system of polynomials closed under substitution is called a *T-ideal*. To every T-ideal, there corresponds a T-ideal in a free algebra or, equivalently, *a completely characteristic* ideal in a free algebra, i. e., an ideal closed under all endomorphisms.

Similarly to the notion of T-ideal, the notion of *T-space* is introduced in a natural way; this is the space generated by values of a system of polynomials closed under substitutions. Some examples of T-spaces not being T-ideals are: the commutator subalgebra $[A,A]$; the set of values of central polynomials in the algebra of generic matrices. This notion is a new one in comparison with the clasical case.

In the classical case we have Hilbert theorem on basis which states that any ascending chain of ideals in the ring of commutative polynomials in several variables stabilizes. For identities in associative algebras we have Kemer's theorem (which answers the known question posed by W. Specht) stating that, in zero characteristic, an ascending chain of T-ideals stabilizes. In positive characteristic, the similar fact is valid for finite numbers of variables (the case of an infinite ground field is solved by A. R. Kemer in [29], and the general case recently by the author in [9]). For infinite numbers of variables, there exist some counterexamples constructed first by the author in [7; 8] and later by V. V. Shchigolev [50] and by A. V. Grishin [13].

The *PI*-theory produces a new point of view to non-commutative algebraic geometry. Consider an algebra endowed by a group of automorphisms or a semigroup of endomorphisms. In classical definitions, replace the notion of "ideal" by that of "invariant ideal". For instance, "primarity" means the absence of nonzero invariant ideals with zero product. Furthermore, Noetherian properties usually are provided by the transformation (semi)group, and algebraic operations become an additional structure. This approach makes it natural to work in multibased algebras of an arbitrary signature (and even in operads).

In many cases, a result obtained for the associative case produced possibilities for non-associative (in particular, alternative or Jordan) situation as well. Extension of results leads to a fundamentally new insight.

I. P. Shestakov has said that until A. R. Kemer's work [25] he was sceptical as regards super-generalizations. His opinion was changed by the "supertrick" which enables to reduce (in zero characteristic) investigation of identities in infinitely generated associative algebras to superidentities in finitely generated superalgebras. The supertrick has obtained a non-associative generalization but the main fact was that previous counterexamples solving open problems in the theory of alternative and Jordan algebras (in particular, examples of infinitely based varieties) obtained their explanation in I. P. Shestakov's work [47]. Non-associative theory often dealt with polynomials whose variables could be divided into several mutually anticommuting groups. (This property is usual for extremal polynomials in solvable alternative algebras, consisting of long associators.) In this case we can check non-vanishing of the corresponding series of polynomials working in a model which is the Grassmann hull of a finite-dimensional superalgebra [47] (cf. Kemer's work [27] which states that every proper variety of associative algebras over a field of zero characteristic is generated by the Grassmann hull of a finite-dimensional superalgebra).

I. P. Shestakov's approach was further developed by S. V. Pchelintsev and his students [1; 2; 38] which has resulted in constructing infinitely based varieties of commutative Moufang loops. (The author considers it possible that the works [1; 2] include the earliest correct proof of this fact, although they were preceded by publications of another author on the same topic[43].) In the sequel, the Grassmannian technique enabled to construct examples of infinitely based varieties for the associative case as well. It is of interest that systematical study of identities related to a Grassmann algebra was started by V. N. Latyshev [31; 32] already in 60ths.

Proofs of finite-basedness and recent solutions of open problems make it especially actual to clarify the nature of finite-basedness and its bounds. Moreover, the situation in zero and positive characteristic does differ both technically and conceptually. This paper treats the case of zero characteristic.

The main result of this paper is the following statement.

Theorem 1.1. (a) *Let $\mathfrak{M}$ be a convenient variety of algebras* (*see Definition* 1.1) *over a field of zero characteristic, such that all its subvarieties are representable. Then the Hilbert series H_Q of an arbitrary T-space Q in a relatively free algebra of $\mathfrak{M}$ is rational.*

(б) *A convenient variety of algebras over a field of zero characteristic is a locally Specht one, and its relatively free algebras are representable.*

Definition 1.1. A variety $\mathfrak{M}$ is called *structurizable* if every finite-dimensional algebra of $\mathfrak{M}$ decomposes into a sum of simple components and of the nilpotent radical.

A structurizable variety is called *convenient* if it is generated by some finite-dimensional algebra.

The varieties of alternative and Jordan algebras which are generated by a finite-dimensional algebra are structurizable. Nevertheless Theorem 1.1 is not a generalization of known results by A. V. Iltyakov [22], A. Ya. Vais and Ye. I. Zelmanov [12] on finite-basedness of varieties of alternative and Jordan algebras. We have to require that all identities of some finite-dimensional algebra hold.

We attempt to work (in particular, to extend A. R. Kemer's and Yu. P. Razmyslov's technique) in as general situation as possible. First we develop the technique of Kemer's polynomials (which are a convenient generalization of Capelli polynomials) in the most general situation. Next we consider a class of rings which are asymptotically close to associative ones. For this situation, we prove an analogue of Shirshov theorem on height. Passing to Specht-type problems, we impose additional conditions of structurizability and validity of all identitities of some finite-dimensional algebra. Closeness to associativity means possibility to extend information concerning an operator algebra $D(A)$ to the algebra A itself. A considerable part of work can be accomplished using the left multiplication algebra $L[A]$. We attempted to formulate criteria of asymptotical closeness to associative rings and to emphasize some basic concepts and constructions (see, in particular, Definition 1.1).

Studying finite-basedness problems, we follow Kemer's program. It consists in explicit description of an extremal ideal I which "forces" to determine the maximal possible number of radical specializations as well as the complete set of specializations related to each prime radical component. In the abstract non-associative situation, the program needs significant improvements. First of all, we have no rather evolved theory of supports which are finite-dimensional algebras, thus we have difficulties in using supports and the approach is more functional. On the same reason, instead of consideration of support models we present a direct, purely combinatorial proof for invariance of internal traces. By author's opinion, this is the base of this section.

If the ground field is of characteristic zero, the basic outline of argument can be extended rather well to the non-associative structurizable case. The only exclusion is the proof that a *PI*-algebra satisfies all identities of some finite-dimensional algebra. Hence in alternative and Jordan cases we go roundabout. Local representability has not been proved for this case yet, and we have no rather evolved theory concerning "supertrick". This makes the following problem important:

Problem. *When do in a non-associative PI-algebra satisfying the system* C_n, *all identities of some finite-dimensional algebra hold? In particular, how can we check validity of all identities of a finite-dimensional algebra for alternative and Jordan PI-algebras?*

Investigating Specht-type problems, we have to postulate validity of all identities of a finite-dimensional algebra. In the associative case this follows from Razmyslov — Kemer — Braun theorem on nilpotency of the radical (whose analogue can be proved for a rather wide class of rings close to associative ones) and from Levin's theorem [60] the extension of which is the main difficulty here. Levin's theorem is closely related to matrix representation of bimodules. Respectively, its analogues for non-associative algebras have to be related to investigation of bimodules. In this aspect, I. P. Shestakov's study of bimodules over Jordan prime algebras seems rather promising. Problems of finite-basedness are in some sense problems of describing interaction between prime components by means of the radical, and bimodules are elementary cells of this interaction.

§2. Preliminaries

We use the following notation. In this section, $\mathrm{T}(f)$ denotes *the* T*-space generated by a polynomial* f (and not the T-ideal). The symbol A usually denotes an algebra, and $a_1, \ldots, a_s$ denote its generators. All rings and algebras, if the contrary is not specified, are assumed to be finitely-generated. The formula $f\big|_A = 0$ means that the polynomial f is an identity of the algebra A, and the formula $f\big|_A \neq 0$ means the contrary. The ideal generated by the set $\mathcal{M}$ is denoted by $\mathrm{id}(\mathcal{M})$. The set of variables $x_1, \ldots, x_n$ sometimes will be treated as a *multivariable* and denoted by $\vec{x}$, thus using expressions like $P(\vec{x}, y)$, $\mathbb{K}[\vec{x}]$, $\mathbb{K}\langle\vec{x}\rangle$ etc. Even for rings without unit, we use notation like $y(1+z)$ (for the element $y + yz$). The symbol E_{ij} denotes a matrix unit: this operator maps the ith basis vector to the jth one, and all others to zero.

The basis rank of a variety $\mathfrak{M}$ is the minimal s such that $\mathfrak{M}$ is generated by its s-generated algebras. The basis rank of the variety of all associative algebras equals 2; that of the variety generated by the algebra of generic matrices also equals 2; the basis rank of a Grassmann algebra or of the variety defined by $\big[x,[y,z]\big]=0$ equals infinity.

A. R. Kemer has established that the basis rank of a variety of associative *PI*-algebras equals infinity if and only if this variety contains an infinitely generated Grassmann algebra [24]. This formed a step in proving representability of varieties of associative algebras over a field of zero characteristic by the Grassmann hull of a finite-dimensional superalgebra.

The growth function $V_A(n)$ of an algebra A is defined as the dimension of the space spanned by the words of length $\le n$; generating function $\sum V_A(n)t^n$ is called the *Hilbert series* of A. Sometimes we will consider *the complete Hilbert series*

$$H_A(t_1,\dots,t_s)=\sum V_A(n_1,\dots,n_s)t_1^{n_1}\cdots t_s^{n_s}$$

where $V_A(n_1,\dots,n_s)$ is the dimension of the space generated by the words containing $\le n_i$ occurrences of the letter a_i for each i.

The growth function depends on the choice of the set of generators. If we define the following equivalence relation on the set of functions:

$$f\equiv g \iff \exists c\in\mathbb{N}:\forall n\, f(cn)\ge g(n) \text{ и } g(cn)\ge f(n)$$

then the equivalence class $V_A(n)$ is now an invariant of the algebra itself. *The Gelfand – Kirillov dimension* of an algebra A is the limit

$$\operatorname{GKdim}(A)=\lim_{n\to\infty}\ln V_A(n)/\ln n$$

if this limits exists; GKdim is an invariant of the algebra itself.

By $G_{\mathfrak{M}}(n)$ we denote the dimension of the vector space generated by the words of length n and containing each of $x_1,\dots,x_n$ once, in the relatively free n-generated algebra of $\mathfrak{M}$. If $\mathfrak{M}$ is a variety of associative algebras then $G_{\mathfrak{M}}(n)=n!$

Definition 2.1. Let $Y=\{u_i\}$ be a set of words. *The height of the set of words W relative to Y* is the least h such that every word $w\in W$ is representable in the form $u_{i_1}^{k_1}u_{i_2}^{k_2}\cdots u_{i_r}^{k_r}$ where $r\le h$. *An algebra A is of height h over Y* if A is linearly representable by a set of words, having height h over Y. Furthermore, Y is called *the Shirshov basis of A.*

We say that Y is an *s-basis of an algebra A* if there exist a number $\mathrm{H}_{\mathrm{Ess}}$ and a finite set $D(Y)$ such that A as a vector space is generated by elements of the form $t_1\cdots t_N$ where $N\le 2\,\mathrm{H}_{\mathrm{Ess}}+1$, and for each i either $t_i\in D(Y)$ or $t_i=y_i^{k_i}$, $y_i\in Y$. Here the number of factors $t_i\notin D(Y)$ does not exceed $\mathrm{H}_{\mathrm{Ess}}$.

The essential height of A over Y is the least number $\mathrm{H}_{\mathrm{Ess}}(A)$ having these properties. For varieties with associative powers the definition is similar.

An algebra is *Noetherian-type* if it is an R-algebra and at the same time a Noetherian module over a Noetherian associative-commutative ring R. An algebra is called *representable* if it embeds into a Noetherian-type algebra. Then the ring R is called *the representaion ring*, and we may assume that R contains unit.

Definition 2.2. We call an ideal I *representable* if the corresponding quotient algebra is representable. An ideal of identities is *representable* if it is a T-ideal of a representable algebra or, equivalently, a relatively free algebra of the corresponding variety is representable.

The least integer n such that the algebra A embeds into an algebra which is a module of rank n over its center is called *the representation order*. If an ideal I is representable then *the representation order of I* equals the representation order of A/I.

An algebra is called a *PI_n-algebra* if it belongs to the variety generated by representable algebras of order n.

Lemma 2.1. *The set of representable ideals of a fixed order satisfies ACC.*

As a matter of fact, a sufficient level of invariance of an ideal relative to a sufficiently rich semigroup of endomorphisms results in representability. The most invariant are T-ideals; hence we have Specht property in the associative case and in "good"structurizable varieties with a sufficiently rich endomorphism semigroup.

Lemma 2.2 (On intersection of representable ideals)**.** *The intersection of a finite number of representable ideals is representable.*

Доказательство. Let $\{I_\alpha\}$ be representable ideals of an algebra A. Then the kernel of the homomorphism $A \to \bigoplus_\alpha A/I_\alpha$ equals $\bigcap_\alpha I_\alpha$, and the direct sum of representable algebras is representable. □

Definition 2.3. A representable algebra is called *irreducible* if it does not contain a finite set of representable nonzero ideals with zero intersection. An algebra is called *irreducible of order n* if it does not contain a finite set of representable nonzero ideals of order n with zero intersection.

Note that the decomposition into simple components is, in general, not unique.

Statement 2.1. (a) *Every Noetherian-type algebra embeds into a direct sum of finite number of its irreducible quotients by ideals stable under multiplication by elements of the representation ring.*

(b) *For any $m \geq n$, any representable algebra of order n embeds into the direct sum of its irreducible quotients of order m.*

We say that an ideal I *contains no obstacle for representation of an ideal* J if there exists a representation ρ of the whole algebra in a Noetherian-type algebra such that $\ker(\rho) \cap I = I \cap J$.

Statement 2.2. *Suppose an ideal I contains no obstacle for representability of J. Then if $A/(I+J)$ is representable then A/J is representable as well.*

Доказательство. Let ρ be a representation of A/J such that its restriction to I has kernel equal to $I \cap J$. It suffices to consider the direct sum of this representation and of faithful representation of $A/(I+J)$. □

Замечание. It is not clear whether representability of the meet $I \cap J$, of the sum $I+J$, and of the ideal I itself implies representability of J. Perhaps no. It seems probable that an embedding into a Noetherian-type algebra may expand the ideal J so that some new elements in the meet with I may arise. It would be worth while to construct corresponding examples.

Extensions and the choice of free elements. Let A be a finite-dimensional algebra. If the ground field is infinite then we can construct a relatively free s-generated algebra $\widetilde{A}$ which generates $\mathrm{Var}(A)$ and embeds into the extension of A by a polynomial ring (and thus is representable). Namely, take a basis for A as a vector space, multiply each element of the basis by a free variable and sum up. We get a generator for the algebra $\widetilde{A}$. To get the set of generators for $\widetilde{A}$, take disjoint sets of variables.

Statement 2.3. (a) *The algebra $\widetilde{A}$ is relatively free, representable, and* $\mathrm{Var}(\widetilde{A}) = \mathrm{Var}(A)$.

(b) *The extension of $\widetilde{A}$ by the ring R generated by the values of trace operators of* (*forms*) *is a Noetherian R-module.*

Доказательство. Assertion (a) is already proven. Assertion (b) follows from Proposition 2.5 to be proved below. □

Let M be a Noetherian module over a Noetherian ring S. Then the finite sum $\oplus_i \otimes^{k_i} M$ is Noetherian as well. Let A be a representable algebra embedded into a Noetherian-type algebra $\hat{A}$, and $\sqcup = \{t_i\}$ be a finite set of letters. Consider the set of words $\mathcal{U}_k$ containing $\leq k$ occurrences of letters from $\sqcup$. Furthermore consider various specializations of elements of A in $\mathcal{U}_k$.

Statement 2.4. *Let A be a representable algebra. Then there exists a finite set $\mathcal{M}$ of elements from A such that for any polynomial $F(\vec{y}, \vec{\Lambda})$ of homogeneity degree $\leq k$ in variables from a finite set $\vec{\Lambda}$, constant vanishing of $F(\vec{y}, \vec{\Lambda})$ is equivalent to vanishing of $F(\vec{y}, \vec{\xi})$ for every $\vec{\xi} \in \mathcal{M}^k$.*

The above statement may reformulated in terms of extensions by *restrictedly free elements of order k*. A set of n elements $\mathcal{M} = \{m_i\} \subset A$ is called a *free set of order k* if for any polynomial F of degree $\leq k$ in variables from $\{t_i\} = \Lambda$

we have the following: if the result of substitution $m_i \to t_i$ vanishes in A then the result of every substitution $s_i \to t_i$ for every $s_i \in A$ vanishes as well.

Consider an extension $A_{Var(A)}\langle\Lambda\rangle$ and various specializations $t_i \to m_i$; $i = 1, \dots, |\Lambda|$, each of which corresponds to a homomorphism of this extension onto A. Since the set $\mathcal{M}$ is finite, we get a homomorphism of the extension $A_{Var(A)}\langle\Lambda\rangle$ into the direct sum $\oplus^k A$ where k is the number of all these specializations. Elements of A map to constants, and images of elements from Λ will be free elements of order k in the image. We obtain *the free extension of order k by $\mathcal{M}$, which is a free set of order k.*

Now we need some universal construction.

Definition 2.4. Suppose B is an associative algebra, $\{\beta_i\}$ are its generators, $\{b_i\}_{i\in I}$ is the set of its elements, $\{\delta_{ij}\}_{i\in I;\, j=1,\dots,m}$ is a set of independent commuting variables. *The canonical algebraic representation of order m* is the algebra

$$\widehat{B}^{(m)} = B[\delta_{ij}]/\operatorname{id}\left(\{b_i^m + \delta_{i1}b_i^{m-1} + \cdots + \delta_{im}\}_{i\in I}\right).$$

If the index i runs over the set corresponding to words of degree $\leq s$ in the generators of the algebra B then the resulting object will be called *the canonical algebraic representation of length s and of order m* and will be denoted by $\widehat{B}^{(m,s)}$.

Definition 2.5. Suppose $b \in B$. Extend the algebra B by free commuting constants δ_i, $i = 1, \dots, n-1$, and consuder the quotient by the ideal $\operatorname{id}(b^m + \delta_1 b^{m-1} + \cdots + \delta_m)$. The algebra B admits a natural map to this algebra, and the kernel of this map is called *the obstacle for algebraicity of order m for the element b*. The kernel of the canonical algebraic representation of order m is called *an obstacle for algebraicity of order m*. The definition of *the obstacle for algebraicity of order m of the system of elements* $\{b_i\}$ is similar.

Suppose a system of forms $\delta_i(b)$ satisfies the Hamilton —Cayley identity. This means that the identity

$$a^n + \delta_1(a)a^{n-1} + \cdots + \delta_n(a) \equiv 0$$

holds in the algebra. Then the *canonical Hamilton — Cayley representation of order n* is constructed in a natural way, and the corresponding kernel is called *the obstacle for the Hamilton —Cayley identity of order m.*

Informally speaking, if all elements of B (resp. the words of length $\leq s$) are “forcedly” turned into algebraic elements of degree m then we get the canonical algebraic representation of order m (resp. of length s).

Similarly we define the ideal J_k, *the obstacle for representability by matrices of order k over a Noetherian ring.*

Local finiteness of algebraic algebras implies

Lemma 2.3 (on canonical representation [59]). (a) *A canonical algebraic representation is a Noetherian module over values of the trace operator. The same is true for the representation of length* s *if* s *does not exceed* $\leq m$ *or* PIdeg(B).

(b) *If* B *is representable by matrices of order* m *then the natural maps* $B \to \widehat{B}^{(m)}$ *and* $B \to \widehat{B}^{(m,s)}$ *are embeddings.*

(c) *If* B *is representable, that is, embeds into a Noetherian-type algebra then the canonical algebraic representation of some order is an embedding.*

Замечание. The canonical algebraic representation may be defined in the non-associative case as well. Then assertion (b) of the above lemma remains valid, and assertion (a) holds for the so-called *Kurosh varieties* (see Definition 3.1).∎

Definition 2.6. Suppose B is a finite-dimensional algebra, $\{\vec{e}_i\}_{i=1}^n$ is its basis, $\{x_{kl}\}_{k=1}^{s}{}_{l=1}^{n}$ are independent variables. *The* s*-generated algebra of generic elements from* B is the algebra generated by $a_i \sum_{l=1}^n x_{il}\vec{e}_l$.

The above definition immediately extends to multibased algebras of an arbitrary signature. The algebra of generic elements is relatively free and generates a homogeneous variety.

Now we will describe the procedure of *linearization.* Suppose a polynomial P has degree n in a variable x. Substitute $\sum_{i=1}^n x_i$ for x and take the sum of terms multilinear in all x_i. The resulting polynomial Q is called *the complete linearization* of P. *A partial linearization* is the sum of terms having a given inhomogeneity degree in the variables x_i. In characteristic zero which is treated in the present paper, an identity is equivalent to all its linearizations. In particular, all identities are equivalent to multilinear ones.

The main idea which is due to Yu. P. Razmyslov is as follows. If we extend a representable algebra by the values of traces then it becomes a Noetherian-type algebra.

Suppose $f(\vec{x}, \vec{y})$ is a polynomial, multilinear and skew symmetrical in $\vec{x} = (x_1, \dots, x_n)$, V is a vector space generated by x_i, and $A \in \mathrm{End}(V)$ is an operator. Then

$$f(A \cdot \vec{x}, \vec{y}) = f(Ax_1, \dots, Ax_n, \vec{y}) = \det(A) f(\vec{x}, \vec{y}).$$

Now we obtain a linearization. Set $A = E + ta$. We have a decomposition in powers of t:

$$f(A \cdot \vec{x}, \vec{y}) = \left(\sum_{k=0}^{n} \Phi_k(a) t^k\right) f(\vec{x}, \vec{y})$$

where $\Phi_k(a)$ is a *form of order* k over the operator a. It equals the trace of the operator $\bigwedge^k(a)$ which acts on the vector space $\bigwedge^k(V)$ or, equivalently, the sum of principal minors of order k in the matrix of the operator a. In particular, $\Phi_1(a) = \mathrm{Tr}(a)$, $\Phi_n(a) = \det(a)$, $\Phi_0(a) = 1$.

Theorem (Yu. P. Razmyslov). (a) *The algebra of generic matrices of size n satisfies the Capelli identity of order n^2+1 and does not satisfy the Capelli identity of order n^2.*

(b) *The following equations hold where x_i are alternated and y_i are "layers":*

$$n\,\mathrm{Tr}(Z)C(x_1,\dots,x_{n^2};\,y_1,\dots,y_{n^2}) = \sum_{i=1}^{n^2} C(x_1,\dots,x_{n^2};\,y_1,\dots,y_{n^2})\Big|_{x_i=Zx_i}, \quad (1)$$

$$\det(Z)C(x_1,\dots,x_{n^2};\,y_1,\dots,y_{n^2}) = C(Zx_1,\dots,Zx_{n^2};\,y_1,\dots,y_{n^2}) = C(x_1,\dots,x_{n^2};\,y_1,\dots,y_{n^2})\Big|_{x_i=Zx_i\forall i}. \quad (2)$$

The above theorem and a corollary from Shirshov theorem on height (if all words of length not exceeding the degree of the algebra are algebraic then the algebra is finite dimensional) immediately imply

Statement 2.5. (a) *Suppose Y is the set of words having length $\le 2n$, in the generators of the algebra of generic matrices M_n, and Z is the following set of traces:*

$$Z = \big\{\,\mathrm{Tr}(y_i^{k_j}) \;\big|\; y_i \in Y,\ 0 < k_j \le \deg(A)\big\}.$$

Then the extension $M_n[Z]$ of M_n is integer over $\mathbb{K}[Z]$ and is a Noetherian module.

(b) *The algebra of generic matrices with trace (or with forms) is a Noetherian module over values of the trace operator of (taking the form). In turn, the values of these operators generate a Noetherian commutative ring.*

(c) *A similar statement is valid for any representable algebra A. An extension $\hat{A}$ of a representable algebra A by the value of the trace operator on elements of the above type is a Noetherian module over a commutative ring. (Traces of the (form) are determined by the representation.)*

The procedure of swap.

Statement 2.6 [52]**.** *Consider the following game. Given n piles of some objects. The first player may choose any m piles and divide each of them into right and left part. The second player interchanges right parts non-identically. Then the first player can guarantee that all piles except $m-1$ ones contain $\le m-1$ objects each.*

Доказательство. Order the piles and consider the vector whose ith coordinate is the number of objects in the ith pile. Order such vectors lexicographically. We will show that if the first player cannot increase the vector corresponding to the present piles then the distribution of objects is as required.

Suppose there are m piles; $k_1,\dots,k_m$ are the corresponding numbers of objects. Suppose $k_i \ge m$ for any i. Set $k_i = k_i' + q_i$, $q_i = i$, $k_i' = k_i - i$. Since $k_i \ge m$, we have $k_i' \ge 0$. It remains to apply Proposition 2.7. □

Statement 2.7 [52]**.** *Suppose* $k_i \geq m$*. Put* $k_i = k'_i + q_i$*. Suppose* $k'_i \geq 0$ *and* $q_j > q_i$ *for* $j > i$*. Then for any non-identity permutation* $\sigma \in S_m$ *the vector* $\vec{k}_\sigma = (k'_1 + q_{\sigma(1)}, \ldots, k'_m + q_{\sigma(m)})$ *is lexicographically smaller than* $\vec{k} = (k'_1 + q_1, \ldots, k'_m + q_m)$*.*

Доказательство. If $\sigma(1) \neq 1$ then $\sigma(1) > 1$ and $k'_1 + q_{\sigma(1)} > k'_1 + q_1$. In this case $\vec{k}_\sigma \succ \vec{k}$. If $\sigma(1) = 1$ then we get inductive descent from m to $m-1$.□

Proposition 2.6 implies

Lemma 2.4 (on swap)**.** *Let* A *be a PI-algebra satisfying a multilinear identity* f *of degree* m*. Let a word* W *be of the form*

$$W = c_0 v_1 c_1 \cdots v_m c_{m+1}$$

where c_i *are letters not occurring in the words* v_j*. Then* W *can be represented modulo* $T(f)$ *as a linear combination of words having the form*

$$W' = c_{i_0} v'_1 c_{i_1} \cdots v'_m c_{i_{m+1}}$$

where c_i *do not occur in the words* v'_j *and not more than* $m-1$ *words* v'_i *have length exceeding* $m-1$*.*

The sense of the lemma is that the identity enables to collect almost all symbols from "piles" v_i into $m-1$ piles v'_i.

We play for the first player when we represent the word W as a product $W_0 \cdots W_{m+1}$ "cutting" the words v_i. Then the identity turns $W_0 \cdots W_{m+1}$ into a sum of words where W_i are permuted non-identically. The second player chooses the most "unconvenient" term.

If all v_i are powers of the same element then we obtain a gathering procedure.Suppose $M \subset A$. Let $M^{(k)}$ denote the ideal generated by kth powers of elements from M. The swap lemma implies

Statement 2.8 [52]. *Suppose* A *is a finitely generated graded associative PI-algebra,* $M \subset A$ *is a finite set of homogeneous elements which generates* A *as an algebra. Suppose the quotient* $A/M^{(m)}$ *is nilpotent of degree* r*. Then* A *is generated as a vector space by elements of the form*

$$v_0 m_0^{k_0} v_1 m_1^{k_1} \cdots m_{s-1}^{k_{s-1}} v_s$$

where for each i *we have* $|v_i| < r$*,* $k_i \geq m$ *and furthermore, not more than* $m-1$ *of the words* v_i *have length* $\geq m$*,* $m_i \in M$*, and there are no* m *equal elements among* m_i*.*

In other words, A has bounded *essential height* over M (see Definition 2.1).

If an algebra satisfies a rarefied identity then there exist k and coefficients α_σ such that for any polynomial $F(x_1, \ldots, x_k, y_1, \ldots, y_r)$ multilinear in x_i the following equation holds:

$$\sum_\sigma \alpha_\sigma F(c_1 v_{\sigma(1)} d_1, \ldots, c_k v_{\sigma(k)} d_k, y_1, \ldots, y_r) = 0. \tag{3}$$

Similarly to the swap lemma, we can prove using this fact

Lemma 2.5 (on rarefied swap [52]). *Let A be a PI-algebra such that for all F multilinear in the variables x_i the equation* (3) *holds. Replace x_i by v_i. Then $F(v_1, \ldots, v_m, \vec{y})$ is linearly representable by elements of the form $F(v'_1, \ldots, v'_m, \vec{y})$ where not more than $k-1$ words v'_i have length greater than $k-1$.*

Замечание. The equation $\sum \alpha_\sigma F(x_{\sigma(1)}, \ldots, x_{\sigma(m)}, \vec{y}) = 0$ for any F is the definition of a rarefied identity for the non-associative case (and moreover for algebraic systems of arbitrary arity).

§3. Capelli polynomials and Kemer polynomials

This section is devoted to one of main tools used in this paper, that is, to polynomials which are multilinear and skew symmetric in several groups of variables. *The Capelli polynomial C_n of order n* is the polynomial of the form

$$C_n = \sum_{\sigma \in S_n} (-1)^\sigma x_{\sigma(1)} y_1 x_{\sigma(2)} \cdots y_{n-1} x_{\sigma(n)}.$$

Here y_i are called *layers*.

In the non-associative case (including algebras of an arbitrary signature Ω) the term *the system of Capelli polynomials C_n of order n* denotes a set of polynomials which are multilinear and skew symmetric in some set of n variables $\{x_i\}$. If in an algebra B each Capelli polynomial of order n vanishes then we say that B satisfies *the system of Capelli identities*. The system C_n holds in algebras of dimension less than n. For instance, the algebra of matrices of order n satisfies C_{n^2+1} (but does not satisfy C_{n^2}).

3.1. *Kemer diagrams.* To each Young diagram D we may attach a collection of disjoint sets of variables $\{\Lambda_i\}$ corresponding to its columns. The number of elements in Λ_i equals the length of the corresponding column.By $S(D)$ we will denote the T-ideal generated by polynomials, multilinear and skew symmetric in each Λ_i.

Now we define a non-determinate operator $'$ on the set of diagrams. If all columns of D are distinct then the diagram D' is obtained from D by adding two unit columns, otherwise we take two maximal coinciding columns. Let m be their length. Replace one of them by a column of length $m-k$, and the other one by a column of length $m + k$. Thus we obtain m distinct diagrams of the form D' (for $k = 1, \dots, m$). We do not include zero columns.

It is not difficult to verify

Lemma 3.1. *Every diagram of the form $D^{(s)}$ for $s \geq \frac{n(n+1)(2n+1)}{12}$ includes a column of length $\geq n$.*

Доказательство. The lemma may be reformulated as follows. Suppose $s \geq n(n+1)(2n+1)/12$. Consider the following operation over a set of numbers. We add two units if all numbers are distinct. And if m, m is the pair of maximal coinciding numbers in the set then they are replaced by the numbers $m-h, m+h$ where $1 \leq h \leq m$. Repeated s times, this operation results in appearing a number not less than n.

First of all, it is clear that the numbers cannot remain bounded. In fact, consider maximal numbers to which the operation was applied infinitely many times. If it was applied to a number n more than n times then we get two equal numbers exceeding n. Hence there is no maximal number to which the operation was applied infinitely many times.

Thus a number $\geq n$ will appear, and the only question is on the number of operations. Consider a process having maximal number of steps before a number $\geq n$ appears. Clearly, at the next to last step there are two numbers equal to $n - 1$ (otherwise we can avoid appearance of a number $\geq n$ at the next step, and the process will not be the longest one.)

Similarly, at the preceding step we must have one copy of $n - 1$ and two copies of $n-2$. Using induction, we ensure easily that at the kth step from the end in the longest process we must have single copies of $n - (k - 1), \dots, n - 1$ and two copies of $n - k$. At the $(n - 2)$th step from the end we get the set $2, 2, 3, 4, \dots, n-1$, and at the $(n-1)$th step from the end we have $1, 1, 2, \dots, n-1$ and perhaps one unit more.

It is not difficult to estimate the number of operations which precede the appearance of this set. Obviously, the operation $'$ increases the sum of squares of the numbers in the set not less by 2, and this number does not exceed $\left(2 + \sum_{k=1}^{n-1} k^2\right)/2 = \frac{n(n-1)(2n-1)}{6} + 1$. The total number of operations differs from this number by n. Thus we obtain the required estimate. □

Замечание. The same estimate holds for the maximal number of operations necessary to obtain the number n if we must have $k = 1$ when replacing n, n by $n-k, n+k$. A corresponding problem was suggested by the author at the 27th international mathematical Tournament of towns.

Define $b(\mathfrak{M})$ as the greatest natural b which satisfies the following condition.

There exist diagrams D consisting of arbitrarily many cells, such that their columns are of length $\geq b$ and all polynomials from $S(D)$ are not identities of *the variety* $\mathfrak{M}$.

For an algebra C, set $b(C) = b\big(\operatorname{Var}(C)\big)$. If C is nilpotent then $b(C) = 0$, and if C generates the variety of all associative algebras then $b(C) = \infty$.

Definitions. Suppose $\mathfrak{M}$ is a variety of algebras of signature Ω, D is a Young diagram such that all columns are of length $\geq b = b(\mathfrak{M})$. If furthermore $S(D)\big|_{\mathfrak{M}} \neq 0$ then such a diagram will be called *curious*. We call a diagram D *interesting* if there exist arbitrarily large curious diagrams including D. We call a column in a curious diagram D *large* if its length exceeds b, and *small* if it equals b. The set of variables corresponding to a small (resp. large) column is *small* (resp. *large*). The set of large columns forms *the head* $H(D)$ *of the diagram* D. A diagram is called *extremal* if it is interesting and moreover for $H(D') \supset H(D)$ and $D' \supset D$ all polynomials from $S(D')$ are identities of $\mathfrak{M}$; $k(H)$ denotes the minimal number of small columns in an extremal diagram with head H (if there is no such diagram then $k(H) = \infty$). *A good diagram with head* H is an extremal diagram having not less than $k+1$ small columns. An extremal diagram is called *a Kemer diagram* if all large columns are of length $b+1$. Then d denotes their number, and k is the minimal number of small columns in the Kemer diagram. So a minimal Kemer diagram is described by the parameters b, d and k.Thus for the variety $\mathfrak{M}$ we define the values $b(\mathfrak{M})$, $d(\mathfrak{M})$ and $k(\mathfrak{M})$. The types of varieties or triples (b, d, k) are ordered as follows: $(b_1, d_1, k_1) \prec (b_2, d_2, k_2)$ if any of three conditions holds:

- $b_1 < b_2$;
- $b_1 = b_2, d_1 < d_2$;
- $b_1 = b_2, d_1 = d_2, k_1 > k_2$.

Intervals between variables which correspond to the diagram are called *layers*. If no confusion can occur, we will use the term "layer" when we consider the values of the corresponding polynomials (if substitutions of variables from the set $\bigcup \Lambda_i$ are fixed).

In the sequel, if the contrary is not specified, *a Kemer diagram* means *a good Kemer diagram*.

Extension by generic elements. Suppose B is an algebra from a variety $\mathfrak{M}$, $X = \{x_i\}_{i\in I}$ is a set of variables. The following lemma is an analogue of the lemma from [18].

Lemma 3.2. *For any lemma* $B \in \mathfrak{M}$ *and arbitrary set of variables* X *there exists an algebra* $B_{\mathfrak{M}}\langle X\rangle \in \mathfrak{M}$ *which is generated by* B *and* X *and has the following properties:*

- *any map* $X \to B_{\mathfrak{M}}\langle X\rangle$ *extends uniquely to an endomorphism* $B_{\mathfrak{M}}\langle X\rangle \in \mathfrak{M}$*;*
- X *generates a free algebra of* $\mathfrak{M}$*;*

- *the algebra $B_{\mathfrak{M}}\langle X\rangle \in \mathfrak{M}$ is the universal object with the properties from the above items.*

Similarly, there exists a universal algebra $B^{alt}_{\mathfrak{M}}\langle X\rangle$ in the class of all extensions of B by the set of $|X|$ absolutely anticommuting elements from X, as well as the algebra $B_{\mathfrak{M}}(D)$ corresponding to the diagram D. Here to the columns of the diagram there correspond absolutely anticommuting sets of variables.

If $\mathfrak{M} = \mathrm{Var}(B)$ then we omit the index $\mathfrak{M}$ in notation for algebras like $B_{\mathfrak{M}}\langle X\rangle$ and write $B\langle X\rangle$.

If $\mathfrak{M}'$ is a subvariety in $\mathfrak{M}$ then the algebra $B_{\mathfrak{M}}\langle X\rangle$ maps onto $B_{\mathfrak{M}'}\langle X\rangle$ in a natural way. Furthermore, to a morphism of algebras $B^1 \to B^2$ there corresponds naturally a morphism $B^1_{\mathfrak{M}}\langle X\rangle \to B^2_{\mathfrak{M}}\langle X\rangle$, and this functor is faithful.

A set Λ is called *absolutely commuting* if $xcy = ycx$ for any $c \in A$ and any $x, y \in \Lambda$. If $xcy = -ycx$ for any $c \in A$ and any $x, y \in \Lambda$ then the set Λ is *absolutely anticommuting.*

3.2. *Razmyslov — Zubrilin theory for Kemer polynomials.* This subsection is devoted to the technique presented in [18; 19], which stem from the paper [41] by Yu. P. Razmyslov whose student K. A. Zubrilin was. For the associative case, this technique is presented in [59]. It extends easily to the non-associative case (and moreover to algebras of arbitrary signature) using the appropriate definition of Kemer polynomials.

Suppose a polynomial $F(\vec{y}, x_1, \dots, x_n)$ is multilinear and skew symmetric in variables x_i, $a \in A$. Define the operators of internal forms δ^k_a by

$$\delta^k_a(F) = \sum_{i_1<\cdots<i_k} F(\vec{y}, x_1, \dots, x_n)\Big|_{x_{i_1}=ax_{i_1},\dots,x_{i_k}=ax_{i_k}}; \quad \delta^0_a(F) = F. \qquad (4)$$

The polynomial $\delta^k_a(F)$ is the homogeneous of degree k in a component in the result of substitution $F\big|_{(a+1)x_i \to x_i;\, i=1,\dots,n}$.

It is easily verified that

$$\delta^k_a(C_n) = \sum_{i_1<\cdots<i_k} \sum_{\sigma\in S_n} (-1)^\sigma x_{\sigma(1)} y_1 \cdots x_{\sigma(i_1)} a y_{i_1} \cdots x_{\sigma(i_k)} a y_{i_k} \cdots y_{n-1} x_{\sigma(n)}.$$

Hence $\delta^k_a(F)$ also is skew symmetric in the set of variables $\{x_i\}_{i=1}^n$.

Put $\mathrm{Tr}(a) = \delta^1_a$. Clearly $\mathrm{Tr}(a+b) = \mathrm{Tr}(a) + \mathrm{Tr}(b)$.

The operators δ^k_a are defined only for records of elements, so the result of their application may depend, in general, on the representation of an element from A as a polynomial F and on the choice of $\{x_i\}$. If F is multilinear and skew symmetric in several sets of variables then considering δ_k we will indicate the specific set.

We shall use the following technical statement.

Lemma 3.3 (on absorption of a variable)**.** *Suppose an algebra satisfies the system of Capelli identities of order $n+1$; the polynomial F is multilinear and skew symmetric in $x_1, \dots, x_n$ and moreover linear in the variable z. Then the following equation holds:*

$$F(z, x_1, \dots, x_n, \vec{y}) = \sum_{i=1}^{n} F(z, x_1, \dots, x_n, \vec{y})\Big|_{z=x_i;\, x_i=z}. \tag{5}$$

Доказательство. The difference between right and left sides of the equation is a polynomial belonging to $\mathrm{T}(C_{n+1})$ because it is multilinear and skew symmetric in the set of variables $\{z, x_1, \dots, x_n\}$. □

Now we formulate the basic lemma from [18] which is an analogue of Hamilton — Cayley theorem for operators having an internal definition.

Lemma 3.4. *Suppose a polynomial $F(y, \vec{z}, x_1, \dots, x_n)$ is multilinear and skew symmetric in the variables x_i, and $a \in D(A)$ is an element of an operator algebra (for instance, an operator of multiplication by $\bar{a}$). Then modulo C_{n+1} we have the equation ("Hamilton — Cayley theorem")*

$$F\big(a^n(y), \vec{z}, x_1, \dots, x_n\big) = \sum_{k=1}^{n} (-1)^k \delta_a^k\Big(F\big(a^{n-k}(y), \vec{z}, x_1, \dots, x_n\big)\Big). \tag{6}$$

Доказательство. Write down equation (6) in the form

$$\sum_{k=0}^{n} (-1)^k \delta_a^k\Big(F\big(a^{n-k}(y), \vec{z}, x_1, \dots, x_n\big)\Big) = 0. \tag{7}$$

Suppose $i_1 < \dots < i_k$, $I = \{i_1 < \dots < i_k\}$. Consider the term

$$t_I = F\big(a^{n-k}(y), \vec{z}, x_1, \dots, x_n\big)\big|_{x_{i_1}=ax_{i_1}, \dots, x_{i_k}=ax_{i_k}}.$$

For $n-k > 0$ represent $a^{n-k}y$ in the form $a^{n-k-1}ay$ and put $y' = ay$. For $n = k$ represent $a^{n-k}y$ in the form y and put $y' = y$. Now suppose $x_i' = x_i$ for $i \notin I$ and $x_i' = ax_i$ for $i \in I$. Apply identity (5) from lemma 3.3 to t_I. We get

$$\begin{aligned} t_I = &-\sum_{j \notin I} F\big(a^{n-k-1}(x_j), \vec{z}, x_1, \dots, ay, \dots, x_n\big)\Big|_{x_{i_1}=ax_{i_1}, \dots, x_{i_k}=ax_{i_k}} \\ &-\sum_{j \in I} F\big(a^{n-k}(x_j), \vec{z}, x_1, \dots, ay, \dots, x_n\big)\Big|_{x_{i_1}=ax_{i_1}, \dots, x_{i_k}=ax_{i_k}} \end{aligned}$$

for $I \neq \{1, \dots, n\}$ and

$$t_{\{1,\dots,n\}} = \sum_{j=1}^{n} F\big(x_j, \vec{z}, ax_1, \dots, ay, \dots, ax_n\big).$$

To complete the proof of the lemma, note that in the expression for $\sum_I (-1)^{|I|} t_I$ the terms of the sums

$$\sum_{j\notin I} F\big(a^{n-|I|-1}(x_j), \vec{z}, x_1, \ldots, ay, \ldots, x_n\big)\Big|_{x_{i_1}=ax_{i_1},\ldots,x_{i_{|I|}}=ax_{i_{|I|}}}$$

for $|I| = k < n$ cancel with the terms of the sums

$$\sum_{j\in I} F\big(a^{n-|I|}(x_j), \vec{z}, x_1, \ldots, ay, \ldots, x_n\big)\Big|_{x_{i_1}=ax_{i_1},\ldots,x_{i_{|I|}}=ax_{i_{|I|}}}$$

for $|I| = k+1$, hence $\sum_I (-1)^{|I|} t_I = 0$. □

Extend the original algebra A by coefficients λ_i. Consider the ideal I_a generated by $a^{n+1} - \sum_i \lambda_i a^{n+1-i}$.

Let $\Lambda_{\mathfrak{M}}(A, X)$ be the subspace in the algebra $A_{\mathfrak{M}}\langle x_1, \ldots, x_n\rangle$ consisting of polynomials which are multilinear and skew symmetric in the set of variables $X = \{x_1, \ldots, x_n\}$. Put $A(n; a) = A\big[\{\lambda_i\}\big]/I_a$.

To λ_i, attach the operators $\delta_i(a)$. Lemma 3.4 immediately implies

Statement 3.1. *Suppose $A \in \mathfrak{M}$, and the variety $\mathfrak{M}$ satisfies the system C_{n+1}. Then we have the natural embedding*

$$\Lambda_{\mathfrak{M}}(A, X) \to \Lambda_{\mathfrak{M}}\big(A(n; a), X\big).$$

Let K_a be the kernel of the map $A \to A^n(a) = A\big[\{\lambda_i\}\big]/I_a$.

Corollary 3.1. *If $h \in K_a$, $A = A'_{\mathfrak{M}}\langle y\rangle$, y is a variable not from X, $F(y, X) \in \Lambda_{\mathfrak{M}}\big(A(n), X\big)$, then $F(y, X)\big|_{h\to y} = 0$.*

Let $\mathcal{S} \subset A$ be a set of elements from the algebra A. Let us define the algebras $A(n, \mathcal{S})$, $A(n, \mathcal{S})_{\mathfrak{M}}\langle X\rangle$ and the space $\Lambda_{\mathfrak{M}}\big(A(n; \mathcal{S}), X\big)$ in a natural way. Let $A(n)$ be the minimal universal object obtained by forced declaring all elements of A being algebraic of degree n. It is the injective limit of extensions $A_0 = A$, $A_1 = A_0(n, A_0), \ldots, A_{k+1} = A_k(n, A_k)$. Let K_n be the kernel of the natural map $A \to A(n)$.

Note that it suffices to show that $F_h = 0$ if h belongs to the obstacle for the canonical algebraic representation of order n for any previously fixed finite subset $\{a_1, \ldots, a_s\} \subseteq A$, because the joint of these obstacles is an obstacle for the canonical algebraic representation of order n for the whole algebra. Hence using induction, from Proposition 3.1 we obtain

Corollary 3.2. (a) *Suppose $A \in \mathfrak{M}$, and the variety $\mathfrak{M}$ satisfies the system C_{n+1}. Then we have the natural embedding*

$$\Lambda_{\mathfrak{M}}(A, X) \to \Lambda_{\mathfrak{M}}\big(A(n), X\big).$$

(b) *If $h \in K$, $A = A'_{\mathfrak{M}}\langle y\rangle$, y is a variable not from X, $F(y, X) \in \Lambda_{\mathfrak{M}}\big(A(n), X\big)$, then $F(y, X)\big|_{h\to y} = 0$.*

The above corollary implies an important technical statement.

Lemma 3.5. *Suppose a polynomial F is multilinear and skew symmetric in some set of n variables Λ and is linear in a variable x_0 not belonging to Λ. Furthermore let v belong to the obstacle Obstr_n for algebraicity of order n. Then $F_v = F\big|_{v\to x_0} \equiv 0$ modulo C_{n+1}.*

The subspace $\mathrm{T}(g)$ skew symmetric in sets of variables from Λ_i is a Noetherian module over operators $\mathrm{Tr}(a)$ and $\delta_k(a)$.

Corollary 3.3. (a) $C_n \,\mathrm{Obstr}_n \subseteq C_{n+1}$.

(b) $\mathrm{Obstr}_1 \cdots \mathrm{Obstr}_n \subseteq C_{n+1}$.

(c) $(\mathrm{Obstr}_n)^n \subseteq C_{n+1}$. *In particular, if an algebra A satisfies C_{n+1} then* $(\mathrm{Obstr}_n)^n = 0$.

(d) *If a finitely generated algebra satisfies a system of Capelli identities then the radical is nilpotent.*

Доказательство. Assertion (a) is reformulation of the above lemma. Assertion (b) is deduced from (a) by obvious induction. Assertion (c) follows from (b) and the inclusion $\mathrm{Obstr}_k \subseteq \mathrm{Obstr}_n$ for $k \le n$. Thus if the algebra satisfies C_{n+1} then it includes an ideal Obstr_n of nilpotency degree n with representable quotient. But in a representable finitely generated algebra the radical is nilpotent. Assertion (d) is proved. □

Consider two disjoint sets X and Y having m elements each, and the symmetric group S_{2m} acting on $X \cup Y$. In the group algebra $\mathbb{Z}S_{2m}$, define elements $T(Z)$, $Z \subseteq X$, as follows:

$$\begin{aligned} T(Z) &= \sum_{\sigma(Z)\subseteq Y} (-1)^\sigma \cdot \sigma, \quad Z \ne \varnothing, \\ T(\varnothing) &= \sum_{\sigma\in S_{2m}} (-1)^\sigma \cdot \sigma. \end{aligned} \tag{8}$$

Statement 3.2 [18]**.** *The following equation holds:*

$$\sum_{Z\subseteq X} (-1)^{|Z|} T(Z) = \sum_{\sigma(X)=X} (-1)^\sigma \cdot \sigma. \tag{9}$$

The left-hand sum is taken over all subsets Z of X, including X itself and $\varnothing$.

Замечание. Suppose $X = \{1, \dots, m\}$ and $Y = \{m+1, \dots, 2m\}$. Consider an action of the group algebra $\mathbb{Z}S_{2m}$ on the vector space over $\mathbb{Z}$ of multilinear polynomials of degree $2m$ in $2m$ variables $x_1, \dots, x_{2m}$: namely,

$$\sigma \cdot x_1, \dots, x_{2m} = x_{\sigma(1)} \cdots x_{\sigma(2m)}$$

where $\sigma \in S_{2m}$. Then $T(Z) \cdot x_1 \cdots x_{2m}$ is a polynomial, skew symmetric in variables with numbers from Z and in variables with numbers from $X \backslash Z \cup Y$. If $Z \ne X$ then $|X \backslash Z \cup Y| \ge m+1$, and if $|Z| = m-k$ then $|X \backslash Z \cup Y| \ge m+k$.

The next lemma follows from Proposition 3.2 and the above remark.

Lemma 3.6 [18]. *Let $f(x_1,\dots,x_{2n})$ be a polynomial, multilinear and skew symmetric in the sets of variables $x_1,\dots,x_n$ and $x_{n+1},\dots,x_{2n}$ and maybe depending on other variables. Then*

$$f(x_1,\dots,x_n,x_{n+1},\dots,x_{2n}) - f(x_{n+1},\dots,x_{2n},x_1,\dots,x_n) \equiv 0 \mod I$$

where I is the sum of T-ideals generated by polynomials which correspond to diagrams D having a column of length $n-k$ and another column of length $n+k$, $k=1,\dots,n$.

Corollary 3.4 (on transfer [18]). (a) *Suppose an algebra A satisfies the system of Capelli identities of order $n+1$, and let a polynomial F be multilinear and skew symmetric in variables $\{x_i\}_{i=1}^n$ and $\{z_i\}_{i=1}^n$. Then the value of $\delta_a^k(F)$ does not depend on the choice of the group $\{x_i\}$ or $\{z_i\}$, and the operators δ_a^k and δ_b^s commute.*

(b) *Furthermore in this case* $\mathrm{Tr}(ab)=\mathrm{Tr}(ba)$.

Lemma 3.7 [18]. *Suppose a polynomial F is multilinear and skew symmetric in each of two disjoint sets of n variables Λ_i, $i=1,2$, and moreover is linear in the variable x_0 not belonging to Λ. Suppose v belongs to the obstacle for Hamilton — Cayley identity of order n. Then $F\big|_{x_0\to v} \equiv 0$ modulo $\sum_{k=1}^n (C_{n+k}, C_{n-k})$.*

The subspace $\mathrm{T}(g)$, skew symmetric in sets of variables from Λ_i is a Noetherian module over the operators δ_k.

Here (C_r, C_s) denotes the T-ideal generated by polynomials, multilinear and skew symmetric in two groups consisting of r and s variables.

Замечание. Since the calculations in the proof of Lemma 3.5 (resp. Lemma 3.7) involve variables from a single set Λ_i (resp. from only two sets), the proof of these lemmas extends to the case of extremal diagrams (that is, of several sets of variables).

Statement 3.3 (on transfer). *Suppose $|\Lambda_1|=|\Lambda_2|=n=b(A)$, a polynomial F corresponds to the Young diagram $D=D_0\cup D_1$ where D_0 is an extremal Young diagram, and the diagram D_1 consists of two columns of length b corresponding to the sets Λ_1 and Λ_2.*

Then all statements similar to (a) *and* (b) *in Corollary* 3.4 *as well as Lemma* 3.7 *and Lemma* 3.4 *hold.*

Operators $\delta_k(x)$ are forms defined internally.

The statement below is a complete analogue of Lemma 3.7 for extremal polynomials.

Statement 3.4. *Suppose a Young diagram D includes a pair of columns of the same length m, a polynomial $f\in S(D)$ is linear, besides the variables corresponding to columns of D, also in a variable x_0, and an element $v\in A$*

belongs to the obstacle for Hamilton — Cayley identity of order m, $g = f\big|_{v\to x_0}$*. Then* g *belongs to the sum of ideals of the form* $S(D_i)$ *where the diagram* D_i *is obtained from* D *by replacing some pair of columns of length* n *with a pair of columns of length* $n+k$ *and* $n-k$ *respectively* $(k>0)$*. In particular, if* m *is the maximal length of coinciding columns then* g *belongs to the sum* $\bigoplus_{D_i=D'} S(D_i)$.

Note that summation extends over all possible results obtained by applying the operator $'$.

Using Lemma 3.1 and induction, we obtain the following

Theorem 3.1. *Let an algebra* A *satisfy the system of Capelli identities* C_n*. Then the operator obstacle for Hamilton — Cayley identity of order* n *has nilpotency index not exceeding* $\frac{n(n+1)(2n+1)}{12}$.

Note that the quotient by this obstacle is a representable algebra, and for representable algebras Razmyslov — Kemer — Braun theorem is obvious. Thus we in fact obtain a direct combinatorial proof of Theorem 3.1.

Since in an associative algebra with trace all matrix identities follow from Hamilton — Cayley identity [17; 40; 61], we have

Corollary 3.5. *In the conditions of Theorem* 3.1*, suppose that* A *is an associative algebra. Then the verbal ideal generated by identities of matrices of size* n *has nilpotency index not greater than* $\frac{n(n+1)(2n+1)}{12}$.

3.3. *Representable spaces.* A variety $\mathfrak{M}$ is called *Kurosh variety* if any algebraic algebra from $\mathfrak{M}$ is locally finite. An algebra is called *algebraic* if any 1-generated subalgebra is finite-dimensional.

Definition 3.1. A space $S \subseteq A$ is called *representable* if it has zero meet with the obstacle for representability of some order. We call a space $S \subset B$ *representable in an algebra* B if there exists a morphism B to a Noetherian-type algebra such that its restriction to S is an embedding.

An equivalent definition: a space S is *representable* if the restriction to S of some homomorphism of A to a Noetherian-type algebra is an embedding.

For a diagram D of a variety $\mathfrak{M} \supseteq \mathrm{Var}(A)$ and for an algebra A, define the subspace $S_{D,\mathfrak{M}}(A) \subseteq A_{\mathfrak{M}}\langle\{\Lambda_i\}\rangle$ as the space of polynomials, multilinear and skew symmetric in variables from each set Λ_i corresponding to the diagram D. If the algebra in question is clear, we simply write $S_{D,\mathfrak{M}}(A)$.

In view of the remark after Lemma 3.7, the following statement is valid.

Statement 3.5. *Let* D *be an extremal diagram for a Kurosh variety* $\mathfrak{M} \supseteq \mathrm{Var}(A)$, $S = S_{D,\mathfrak{M}}$*. Then if* D *includes a small column then the space* S *is representable.*

Замечание. The Kurosh property is necessary because almost all multiplications may be subsequently applied to variables which correspond to D, and then powers of elements from A not appear. To avoid restrictions of such kind, we need to consider a pair consisting of the algebra A and the algebra of operators acting on A.

To a diagram D, attach a system of sets of variables $\xi_{ij} \in \Lambda_i$ where Λ_i is a set of strongly anticommuting variables, which corresponds to the jth column of the diagram D.

Suppose $\mathfrak{M}$ is a variety of algebras, D is a Kemer diagram. Consider the algebra $K_0(A, D) = A *_{\mathfrak{M}} \big\langle \{\xi_{ij}\} \big\rangle / Q$ where the ideal Q is generated by non-associative monomials M_α containing two occurrences of any variable ξ_{ij}, and by elements of the form $M(\xi_{ij_1}, \xi_{ij_2}) + M(\xi_{ij_2}, \xi_{ij_1})$. The algebra $K_0(A, D)$ is an extension of A in $\mathfrak{M}$ by a system of strongly anticommuting variables, which is defined by the diagram D. Usually $\mathfrak{M} = \mathrm{Var}(A)$, and this is assumed if the index $\mathfrak{M}$ is omitted. In any case, the inclusion $\mathfrak{M} \supseteq \mathrm{Var}(A)$ is necessary for correctness of constructions. This is always assumed.

Let J_{b+1} be the ideal in $K_0(A, D)$ generated by values of polynomials of the form $f(u_1, \dots, u_{b+1}, \vec{y}\,)$ where u_k are monomials in generators of A, and f is multilinear and skew symmetric in u_i. Put $K_1(A, D) = K_0(A, D)/J_{b+1}$. Let $K(A, D)$ be the space (the ideal) in $K_1(A, D)$ generated by monomials, multilinear in all variables of the form ξ_{ij}. Extremality of D implies a natural isomorphy between the corresponding spaces in the algebras $K_1(A, D)$ and $K_0(A, D)$.

Statement 3.6 (on representable spaces)**.** *Let D be an extremal diagram. Then the following is true.*

(a) $\mathrm{Var}(A) = \mathrm{Var}\big(K_0(A, D)\big) = \mathrm{Var}\big(K_1(A, D)\big)$.

(б) *The spaces $S_1(A, D)$ and $S_0(A, D)$ (sometimes denoted by $S(A, D)$) in the algebras $K_1(A, D)$ and $K_0(A, D)$ respectively, generated by monomials multilinear in all variables of the form ξ_{ij} are isomorphic in a natural way.*

(в) *If $\mathfrak{M}$ is a Kurosh variety then the above spaces are representable.*

If the ground field is of characteristic zero then $K(A, D)$ is isomorphic to the subspace $A_{\mathfrak{M}}\langle X\rangle$, the space of polynomials with coefficients in A, multilinear and skew symmetric in the corresponding sets of variables.

In the space $K(A, D)$ multiplication by internal traces (forms) is defined in a natural way, and if $\mathfrak{M}$ is a Kurosh variety then this action determines the structure of a Noetherian module on $K(A, D)$.

Now we formulate an additional useful statement concerning the above constructions.

Statement 3.7. *The correspondence $A \to K_{\mathfrak{M}}(A, D)$ is a covariant functor* (*the index $\mathfrak{M}$ specifies the variety in question*)*. The space $K_{\mathfrak{M}}(A, D)$ always is representable and moreover isomorphic to $K_{\mathfrak{M}}(A', D)$ where A' is a quotient* *of A by some representable ideal. The T-ideal $H(D)$ of the algebra $K_{1\mathfrak{M}}(A,D)$ lies in the space $K_{\mathfrak{M}}(A,D)$.*

We will prove a somewhat different statement concerning representable spaces.

Statement 3.8. *Suppose D is a rectangle of size*

$$\big(k(\mathfrak{M}) + d(\mathfrak{M}) + 1\big) \times b(\mathfrak{M}),$$

and $\mathfrak{M}$ is a Kurosh elastic variety (that is, a variety with associative powers). Then the space $S = S_{D,\mathfrak{M}}$ is representable.

Доказательство. It suffices to show that S is disjoint with the obstacle to algebraic representation of some order (which depends on $\mathfrak{M}$ only) for an arbitrary element $r \in B$. Pass to the algebra $B\big[\delta_i(r)\big]$ which we will consider as an algebra over the associative-commutative ring $\mathbb{F}\big[\delta_i(r)\big]$, choose a new element $r' \in B$, and so on. Thus we obtain that we can pass from B to its operator canonical algebraic representation of some order, and the latter algebra is a finite-dimensional module over an associative-commutative ring.

Suppose $b = b(A)$ and ψ_i are operators of the form $x_i \to r(x_i)$. It suffices to show that the operator r is algebraic over the operators ψ_i (and the algebraicity order depends on $\mathfrak{M}$ only). Then we may interprete the coefficients in the algebraicity relation using operators ψ_i and argue as in the proof of Lemmas 3.5 and 3.7.

Thus the statement will be proved if we will show that for some m every polynomial $F(\vec{x}, \vec{y}, r^m)$, multilinear and skew symmetric in variables from the sets Λ_i satisfies

$$F(\vec{x}, \vec{y}, r^k) = \sum_{i=1}^{k} \Psi_i F(\vec{x}, \vec{y}, r^{m-i})$$

where the coefficients Ψ_i are polynomials in the operators ψ_i and do not depend on F.

Let $\delta_{ik}(r)$ be the operator of an internal form of order k defined by the set of variables Λ_j. If a polynomial G is of the form $G\big(\vec{y}, \bigcup_i \Lambda_i, r^b, d\big)$ then the difference

$$G\bigg(\vec{y}, \bigcup_i \Lambda_i, r^b, d\bigg) - \sum_{k=1}^{b} \delta_{jk} \cdot G\bigg(\vec{y}, \bigcup_i \Lambda_i, r^{b-k}, d\bigg)$$

is representable as a linear combination of polynomials obtained by choosing $b+1$ subwords inside some polynomial $\overline{G}$ and by alternating these subwords; here the subwords containing any variable from Λ_j are not alternated for $j \neq i$, and furthermore only those occurrences of r are involved which are contained in the chosen power r^b.

Suppose $m = b\cdot(d+1)$. Then r^m divides into $d+1$ parts of length b, and the difference

$$G\Big(\vec{y},\bigcup_i \Lambda_i, r^m\Big) - \sum_{k_1=1}^{b}\sum_{k_2=1}^{b}\cdots\sum_{k_{d+1}=1}^{b}\delta_{1k_1}(r)\delta_{2k_2}(r)\cdots\delta_{d+1,k_{d+1}}(r)G\Big(\vec{y},\bigcup_i \Lambda_i, r^{m-\sum_i k_i}\Big)$$

belongs to the T-ideal which corresponds to the diagram including $d+1$ columns of length $b+1$ and $k(A)$ columns of length b. But in $\mathfrak{M}$ any ideal with these properties is zero. □

Note also the following useful

Statement 3.9. *A finite set of elements which are values of polynomials from representable spaces described in this subsection generates a Noetherian A-module (left, right, bimodule).*

Доказательство. It suffices to prove the proposition for a polynomial $f \in S_{D,\mathrm{Var}(A)}$. The latter follows from the fact that any increasing chain of representable ideals stabilizes. □

3.4. *Thinning of alternators.* Here we consider one of the basic tools which enables to guarantee that alternated constructions have bounded degree. We will use Kemer polynomials.

We start from the associative case. Let a pair (A, H) be given where A is an algebra with a fixed set of generators, and $H \in \mathbb{N}$ is a positive integer.

Suppose D is a diagram, $f \in S(D) \cap A_{\mathfrak{M}}\langle \bigcup_i \Lambda_i\rangle$ is a polynomial, g is obtained from f by substituting the words $\{v_j\} \in A$ instead of variables from $\bigcup_i \Lambda_i$. If all these words have length $\leq H$ then we call the polynomial g (and the sets Λ_i) *thin* relative to D. If at least one v_j has length $> H$ then g is *thick*.

Замечание. In fact, the notions <thick> and <thin> relate not to g itself but to its recording. (We permit this inaccuracy when the specific recording is clear and so any confusion is impossible.)

If H is not fixed, we may speak of *h-thin* and *h-thick* polynomials for each $h \in \mathbb{N}$.

If each Capelli polynomial of thickness h vanishes then we say that the algebra satisfies *the system of Capelli identities of thickness h*. If $h = H$ then the algebra *satisfies the system of thin Capelli identities*. Similar is the definition of somewhat more general notions of validity of *a system of rarefied identities of thickness h*, *a thin system of rarefied identities*, *a thick set Λ_i* and *a set Λ_i of thickness h* as well as of *thickness of a variable* considered as a set of a single variable.

To generalize these notions to an algebra of an arbitrary signature Ω, we have to modify the notion of the *word length*. Here *a word* is an arbitrary monomial in generators. To the layout of operations, there corresponds a tree such that the generators correspond to its end vertices (except the root), and the monomial itself corresponds to the root. This tree is called *the tree of the monomial*. *A branch* is a part of a path without self-intersections, which leads from the root to an end vertex. The length of a monomial v is the maximal possible length of a branch in its tree, it will be denoted by $l(v)$. The value $l(v)$ is the maximal length of a chain of submonomials mutually comparable by inclusion; $br(v)$ denotes the total number of branches of length $l(v)$. The parameter $CH(v)$ is the vector $\big(l(v), br(v)\big)$. Its values are ordered lexicographically (first by the first coordinate, then by the second one).

The following lemma due to K. A. Zubrilin [19] concerns the structure of trees of monomials for algebras with rarefied identities. Its proof is easily obtained using the swap procedure.

Lemma 3.8 (on a tree)**.** *Suppose an algebra A satisfies a system of rarefied identities of order m. We call a branch long if its length is $\geq m$.*

Then any monomial is linearly representable by monomials such that the corresponding tree has not more than $m-1$ disjoint long branches.

It is easy to see that all notions related to thickness are immediately extended to the general case. In the associative case, there is no need to consider the parameter CH: it suffices to use lengths of monomials only. This reduces the technical aspect of proofs, so the reader may originally have in mind just the associative case.

Note that the degree of a monomial v does not exceed $q^{l(v)}$ where q is the maximal arity of an operation in the signature Ω.

The swap procedure enables to prove directly

Statement 3.10. *Suppose an algebra B satisfies a system of rarefied identities of order h and of thickness h, and $g \in S(D)$. Then g is linearly representable by polynomials $g' \in S(D)$ such that for each of them not more than h sets of variables Λ_i are of thickness greater than h. Moreover the total number of variables (in all sets) of thickness greater than h does not exceed $h-1$.*

Corollary 3.6. *If an algebra B satisfies the system of Capelli identities of order m and of thickness m then it satisfies the Capelli identity of order $2m-1$ as well.*

We consider polynomials from $A_{\mathfrak{M}}\big\langle \bigcup_i \Lambda_i \big\rangle$ together with a family of correspondences between monomials and variables from $\bigcup_i \Lambda_i$, which determines a substitution resulting in g. This allows to speak of *thick* or *thin* variables from $\bigcup_i \Lambda_i$ as well as of variables having *thickness h*.

Definition 3.2. $S^{(m)}_{D,m_1,m_2,k}$ is a space generated by polynomials which correspond to a diagramm $D \cup E$ having the following properties:

- the diagram E consists of $m_1 + m_2$ columns of length k;
- not more than m_1 sets of variables corresponding to these columns have length greater than m.

Clearly $S^{(m)}_{D,m_1,m_2,k}$ is an ideal.

We also will need the following corollary from Proposition 3.10 and Lemma 2.5 on rarefied swap.

Corollary 3.7 (Thinning of small sets). *Suppose an algebra satisfies a system of rarefied identities of order m and of thickness m, $m' = m_1 + m_2 \geq m$. Then we have the equality*

$$S^{(m)}_{D,m',0,k} = S^{(m)}_{D,m,m'-m,k}$$

and the inclusion

$$S^{(m)}_{D,m',0,k} \subseteq S^{(m)}_{D,0,m'-m,k}.$$

Now we turn to the procedure of thinning large sets in Kemer polynomials. This procedure is based on the following relation. Let a polynomial f be multilinear and skew symmetric in the set of variables $\{y_i\}_{i=1}^m$ and retains these properties after adding the variable x. Consider

$$f' = f - \sum_{i=1}^{m} f\Big|_{x \to y_i;\, y_i \to x}. \tag{10}$$

The polynomial f' is multilinear and skew symmetric in the set of variables $\{x\} \cup \{y_i\}_{i=1}^m$.

Замечание. Similarly, starting from a polynomial, multilinear and skew symmetric in each of the sets of variables $\{y_i\}_{i=1}^{m_1}$ and $\{x_j\}_{j=1}^{m_2}$ we can construct a polynomial f', multilinear and skew symmetric in the joint set $f' = \sum_{\sigma \in S} (-1)^\sigma \sigma \cdot f$ where S is a system of representatives of cosets of $S_{m_1+m_2}$ relative to $S_{m_1} \times S_{m_2}$.

Lemma 3.9. *Suppose $m = |\Lambda|$, a polynomial $f \in A_{\mathfrak{M}}\langle \Lambda \cup \{x_0\} \rangle$ is multilinear and skew symmetric in the set of variables Λ and is linear in the variable x_0. Suppose the polynomial $g \in \mathrm{T}(f)$ is obtained by replacing x_0 by a monomial v_0 of length k, $k > m+1$, and by replacing the variables from Λ by monomials of length $\leq m$.*

Then g is linearly representable by

1. *the values of substitutions to f such that $CH(v_0') < CH(v)$;*
2. *the polynomials which correspond to substitutions of thickness $m+1$ of elements from A to a polynomial from $A_{\mathfrak{M}}\langle \Lambda \cup \{x_0\} \rangle$, multilinear and skew symmetric in the set of variables $\Lambda \cup x_0$, and of arbitrary elements to another set Λ' where $|\Lambda'| = m$.*

Доказательство. Choose a submonomial of thickness $m+1$ in the monomial v_0, denote it by x'_0 and use the expression (10). It remains to observe that the terms corresponding to

$$f' = f - \sum_{i=1}^{m} f\Big|_{x\to y_i;\, y_i\to x}$$

satisfy condition (2), and those corresponding to the terms of the sum

$$\sum_{i=1}^{m} f\Big|_{x\to y_i;\, y_i\to x},$$

satisfy condition (1) of the present lemma. Let u be a submonomial in the monomial v and u belongs to some chain of submonomials which increases by inclusion and has maximal length. Then replacing u by a monomial of smaller length we decrease the parameter of the original monomial. □

Consider a pair (A, H). Suppose $H \geq k$. Similarly to the above, we can prove the following

Statement 3.11. *Suppose a polynomial f belongs $A_{\mathfrak{M}}\big\langle \bigcup_{i=0}^{1} \Lambda_i\big\rangle$, $|\Lambda_0| = k$, $|\Lambda_1| = k+1$. Suppose the set Λ_1 contains just s thick variables, and all variables in the set $|\Lambda_0|$ are thin. Then the following holds.*

- (a) *If $s > 1$ then the polynomial g corresponding to f is linearly representable by values of polynomials g_μ which correspond to not more than one thick variable in the set Λ_1.*
- (b) *Suppose $s = 1$. Then the polynomial g corresponding to f is linearly representable by values of polynomials g_μ such that parameters (CH) of all but one its variables in the set Λ_1 are the same and a single parameter is strictly smaller.*

(All polynomials g_μ correspond to f for some verbal substitution $x_j \to v_j$ where $x_j \in \bigcup_{i=0}^{1} \Lambda_i$.)

Доказательство. Assertion (a) uses a construction related to formula (10), and assertion (b) follows from (a) applied to the thick variable in Λ_1. □

Замечание. Similarly, if $|\Lambda_0| = |\Lambda_1| - s$ then in view of the remark preceding Lemma 3.9 we may guarantee that not more than s variables from Λ_1 have thickness exceeding m. But for $s > 1$ no analogue of assertion (b) has been proved. This is one of the reasons for using Kemer diagrams and not arbitrary extremal diagrams.

Thus we can make a large set thinner at the price of «thickening» a small set. Combine this process with the swap procedure which enables to thin almost all small sets. Summing up the results of Proposition 3.11 and of Corollary 3.7 we obtain

Statement 3.12. *Suppose $\mathfrak{M}$ is a variety of algebras with parameters (b, d, k) satisfying a rarefied identity of degree m, and S_{m_1,m_2} is the space of values of Kemer polynomials with $m_1 + m_2$ small columns, such that all variables corresponding to large sets have thickness $\le m + 1$ and not more than m_1 variables from small sets have thickness $> m$ (the number of large sets for a Kemer polynomial always equals d). Then the following holds:*

(a) *If $m_1 = m-1$ and $m_1+m_2 \ge k+1$ then the space S_{m_1,m_2} coincides with the space generated by all values of Kemer polynomials which have the diagram D consisting of d large and $m_1 + m_2$ small columns.*

(b) *Suppose c is the number of small columns in the Kemer diagram D and the diagram E consists of m columns of length b. Then we have the inclusion $S(D \cup E) \subset S(0, c)$.*

The sense of this proposition (and of this subsection) is in the inclusion: the space of values of polynomials corresponding to the Kemer diagram $D \cup E$ is contained in the space of thin values of Kemer polynomials corresponding to a somewhat smaller diagram D.

Corollary 3.8. *Suppose A is a PI-algebra with parameter (b, d) satisfying a system of rarefied identities of order m, A' is the quotient by the ideal generated by polynomials of thickness* $\max(m, b+1)$ *which correspond to some Kemer diagram for the variety* $\mathrm{Var}(A)$. *Then the pair (b, d) for* $\mathrm{Var}(A')$ *is strictly smaller than the pair (b, d) for* $\mathrm{Var}(A)$.

Замечание 1. The thinning technique is used for extremal polynomials corresponding to diagrams which contain $\ge m$ small columns and with a large column of length q contain a column of length $q - 1$.

Замечание 2. For associative and structurizable algebras, the possibility for thinning follows also from the fact that an extremal ideal is a Noetherian module over traces.

3.5. *Rings with operators.* Suppose B is an algebra with signature Ω, and $D(B)$ is an operator algebra for B. To each element of $D(B)$ there corresponds a monomial from $B_{\mathrm{Var}(B)}\langle X\rangle$, linear in the variable x. To multiplication of operators $D_1 * D_2$ there corresponds a substitution of $D_1 \to x$ to D_2, and to action of an operator on an element v there corresponds a substitution $v \to x$.

Thus we obtain a pair (B, D) where D is an operator algebra. We want to investigate identities of this pair. We may define in a natural way the variety $\mathfrak{M}$ of double-based algebras (multiplication operators must belong to the operator algebra D) as well as extensions $B\langle \vec{x}, \vec{y}\,\rangle$ where $\vec{x}$ denotes free variables of the algebra, and $\vec{y}$ denotes free operator variables. Similarly we define the algebra $B_{\mathfrak{M}}\langle \vec{x}, \vec{y}\,\rangle$ for any variety $\mathfrak{M}$ of double-based algebras (to every variety of ordinary algebras there corresponds, in a natural way, a variety of double-based algebras.) Note that investigation of alternative and Jordan algebras involves using of multibased systems (alternative systems, Jordan triple systems).

We will investigate varieties such that the corresponding operator algebra is a *PI*-algebra. This is a natural class of algebras. The necessity of restrictions of such kind is evident from the following example.

Пример. Suppose $\mathfrak{M}$ is a variety of algebras determined by the identity $x(yz) = 0$, B is a free 2-generated algebra of $\mathfrak{M}$ and $\mathfrak{M}' = \mathrm{Var}(B)$. It is easy to see that the basis $\mathfrak{M}$ consists of monomials of the form

$$(((\cdots((x_{i_1}x_{i_2})x_{i_3})\,x_{i_4})\cdots)\,x_{i_k})\,,$$

and substitution of monomials of length greater than 1 to all positions except the first one results in zero. Hence if x_{i_1} is fixed then substitutions of the form $x_{i_1} \to x_{i_1}x_\alpha$ realize the action of left multiplication in the operator algebra. It is easily seen as well that the variety $\mathfrak{M}'$ satisfies a system of rarefied identities, namely, the Capelli identities of order 4. At the same time, the growth in a free algebra of $\mathfrak{M}'$ is exponential, $\mathrm{GKdim}(B) = \infty$, and the variety $\mathfrak{M}'$ itself is neither locally Specht nor locally representable.

Similarly to the case of ordinary algebras, we may consider polynomials, multilinear and skew symmetric in sets of operator variables (these polynomials may also include ordinary variables). Corresponding diagrams will be called *operator diagrams*, and the space of polynomials, which corresponds to an operator diagram D will be denoted $S_2(D)$; to a pair $\mathcal{D} = (D_1, D_2)$ consisting of an ordinary and an operator diagram, there corresponds in a natural way the space $S(\mathcal{D}) = S(D_1, D_2)$. Let D_1 be an extremal diagram for an algebra B. An operator diagram D_2 *is compatible with the diagram* D_1 if the algebra $B_{\mathfrak{M}}\langle\,\vec{x},\vec{y}\,\rangle$ contains a nonzero polynomial from $S(D_1, D_2)$, and D_2 *admits* D_1 if D_2 is compatible with any extremal diagram $D' \supset D_1$. Fix an extremal diagram D_1 and consider the set of operator diagrams which admit it. Similarly to the case of ordinary algebras, define operator diagrams *extremal relative to* D_1, their *heads* and *tails* (the remaining parts of the diagrams). We may define, in a natural way, *a pair of extremal diagrams* $\mathcal{D} = (D_1, D_2)$. (The diagram D_2 has to be extremal in the class of diagrams which admit arbitrary large extremal diagrams containing D_1.)

Замечание. In the definition of an extremal diagram D_2 we require non-vanishing of a polynomial which is not necessarily purely operator polynomial. Moreover it is quite possible that the space $S(D_2)$ in the operator algebra D is zero.

We call a family of diagrams *regular* if together with any diagram it contains all its subdiagrams.

Lemma 3.10. *Every descending by inclusion chain of regular families stabilizes.*

Let D_1 be a Kemer diagram for an algebra B. When D_1 increases, the set of diagrams compatible with D_1 decreases and at some moment stabilizes. Now the symbol k will denote the minimal number of small columns necessary for stabilization. *A good diagram* will mean a Kemer diagram with $\geq k+2$ small columns. Considering diagrams compatible with a good Kemer diagram D_1 we define *the operator Kemer diagram* D_2 corresponding to D_1 as well as parameters b_2, d_2 and k_2. The pair (D_1, D_2) is called *the pair of Kemer diagrams.*

Note that operator alternators and alternators of elements from the algebra are disjoint. All statements related to thinning of alternators and to representable spaces extend immediately to this case. We formulate only the eventual result.

Statement 3.13. *Let $\mathfrak{M}$ be a variety of double-based algebras with parameters (b, d, k) satisfying a rarefied identity of degree m and an operator rarefied identity of the same degree. Let $S_{m_1,m_1';\,m_2,m_2',n}$ be the space of Kemer polynomials which have m_1+m_1' and m_2+m_2' small columns in the first and in the second diagram respectively and satisfy the following conditions.*

1. *For each type, the variables which correspond to large sets are of thickness not exceeding $m+1$ (this means that we substitute to these variables from Λ monomials of thickness not greater than $m+1$).*
2. *Not more than m_1 variables in small sets are of thickness greater than m.*
3. *All variables from A contained in the alternated sets are of thickness not exceeding n.*

Then the following holds.

(a) *If $m_1 = m_2 = m-1$, $m_1+m_1' \geq k+1$, $m_2+m_2' \geq k+1$ then the space $S_{m_1,m_1';\,m_2,m_2'}$ coincides with the space generated by all values of Kemer polynomials having the pair of diagrams (D_1, D_2) where D_i consists of d_i large and m_i+m_i' small columns.*

(b) *Suppose c_i is the number of small columns in D_i, and E_i consists of m columns of length b_i. Then we have the inclusion*

$$S(D_1 \cup E_1, D_2 \cup E_2) \subset S_{m_1,m_1';\,m_2,m_2',m} \subset S_{0,m_1';\,0,m_2',m}.$$

Since the operator algebra D is a PI-algebra and every finitely generated subalgebra of it is a Kurosh algebra, the lemma on tree implies

Statement 3.14. *Let $\mathcal{D} = (D_1, D_2)$ be an extremal pair of diagrams for a variety $\mathfrak{M} \supseteq \mathrm{Var}(A)$ of double-based algebras with operators, $S = S_{\mathcal{D},\mathfrak{M}}$. Then if each D_i includes a small column then the space S is representable.*

§4. Height theorem for the non-associative case

Definition 4.1. An algebra A has *bounded L-length* if for some k the algebra $L[A]$ of its left multiplications is linearly representable by a set of elements of the form $L(p_1)\cdots L(p_q)$ where $q<k$ and $L(x)$ is the operator of the left multiplication by x. A variety $\mathfrak{M}$ is called *not bad* if the algebra of left multiplications of any finitely generated algebra from this variety

(1) is finitely generated,

(2) is of bounded L-length,

(3) and moreover the algebra A is *elastic*, that is, any 1-generated algebra from $\mathfrak{M}$ is associative or $\mathfrak{M}$ is a *variety with associative powers*.

The above conditions mean that the associative algebra $L[A]$ is close to the original algebra A and enable to extend to A statements related to $L[A]$.

Definition 4.2. A class of rings $\mathfrak{E}$ is called *a Kaplansky class* if for any $R\in\mathfrak{E}$ the following holds:

(1) if R is prime and $\operatorname{Nil}(R)=0$ then R has nontrivial center;

(2) the quotient $R/\operatorname{Nil}(R)\in\mathfrak{E}$ is a subdirect product of simple rings;

(3) $R\otimes\mathbb{F}[\lambda]\in\mathfrak{E}$;

(4) $\operatorname{Nil}(R)=0\Rightarrow\operatorname{Jac}\big(R\otimes\mathbb{F}[\lambda]\big)=0$.

The condition (4) is equivalent to the following two assertions:

- if R is prime and $\operatorname{Nil}(R)=0$ then localization by the center $Z(R)$ is simple;
- if $\operatorname{Nil}(R)=0$ then $R\otimes\mathbb{F}[\lambda]$ is a subdirect product of prime algebras. The intersection of any ideal with $Z(R)$ is nontrivial.

Замечание. Any variety of non-nilpotent Lie algebras cannot be not bad because l-length is unbounded. It does not belong to a Kaplansky class as well.

A variety $\mathfrak{M}$ is called *good* if it is not bad, that is, satisfies (1)–(3) from the definition of a not bad variety, and moreover

(4) the algebra of left multiplications of any finitely generated algebra from this variety is a PI-algebra.

An algebra is called *representable* if it embeds into a finite-dimensional algebra over an associative-commutative ring.

Statement 4.1. (a) *In the category of n-dimensional representations there exists a universal object.*

(b) *An algebra C is representable if and only if there exist a family of ideals $\{\mathcal{J}_i\}_{i\in I}$ and a number $n\in\mathbb{N}$ such that*

(1) $\bigcap_{i\in I}\mathcal{J}_i=0$;

(2) $\dim C/\mathcal{J}_i\le n$.

Доказательство. Assertion (b) is an immediate consequence of assertion (a) which is well-known. Now we describe the construction of universal n-dimensional over the center representation. Let $\{\bar{e}_i\}_{i=1}^n$ be the basis vectors. The multiplication is determined by structure coefficients C_{ij}^k: $\bar{e}_i\bar{e}_j = C_{ij}^k\bar{e}_k$. To every ith generator of the algebra C, attach the element $\sum_j \lambda_{ij}\bar{e}_i$. Furthermore impose relations on the coefficients C_{ij}^k and λ_{ij}, which follow from the relations of the algebra C. These coefficients are determined from the quotient of the ring of commutative polynomials by the above relations. □

Замечание. Note that the class of n-dimensional algebras over a field, even in the associative case, may contain no universal object.

Now we introduce the following notation:

Mon_r is the set of non-associative monomials of degree r;

$M^{(k)}$ is the ideal generated by kth powers of elements from M.

If J is an ideal in C then I_J is the ideal in $L[C]$ generated by operators of muliplication by elements of J. Suppose $D \subseteq C$; $I_{D,s}$ denotes the ideal in $L[C]$ generated by operators of left multiplication by elements of the form $W(t_1,\dots,t_k)$ where $W \in \mathrm{Mon}_k$, $k \le s$ and there exists i such that $t_i \in D$. If I is an ideal in $L[C]$ then define the ideal $\mathcal{J}_I$ as the set

$$\bigl\{x \in C : \forall d \in \mathrm{id}(x) \;\; L(d) \in I\bigr\}.$$

Let $g(r)$ denote the number of generators in the left multiplication algebra for $\mathfrak{M}_r$, the free r-generated algebra of $\mathfrak{M}$. Let $l(r)$ denote the l-length of $\mathfrak{M}_r$. Since the algebra of left multiplications is finitely generated, there exists a function denoted by $G_{\mathfrak{M}}(r)$ $\bigl($or simply $G(r)\bigr)$, such that for $s > G(r)$ we have the inclusion $I_{D,s} \ni I_{\mathrm{id}(D)}$ in an r-generated algebra C from $\mathfrak{M}$.

The fact that algebras of left multiplications are finitely generated implies

Statement 4.2. *There exists a function $h(r)$ such that $L[\mathfrak{M}_r]$ is generated by operators of left multiplication by monomials of degree not exceeding $h(r)$ in the generators of the algebra $\mathfrak{M}_r$.*

Statement 4.3. *If I is an ideal in $L[C]$ of codimension k then the codimension of the ideal $\mathcal{J}_I$ does not exceed $k \cdot N$ where N is the number of non-associative monomials of degree not greater than $G(r{+}1)$, and r is the number of generators of C.*

Доказательство. The fact that the algebra of left multiplications is finitely generated implies that $x \in \mathcal{J}_I$ if $L(d) \in I$ for any non-associative monomial d of degree not greater than $G(r+1)$, including x. Hence the codimension of $\mathcal{J}_I$ does not exceed the product of the codimension of I by the number of these monomials. □

Theorem 4.1. (a) *Let $\mathfrak{M}$ be a not bad variety, $C \in \mathfrak{M}$ be a finitely generated algebra. Then representability of C is equivalent to representability of $L[C]$.*

(b) *If a variety $\mathfrak{M}$ is good and $C \in \mathfrak{M}$ is a semiprime algebra then C is representable.*

(c) *If a variety $\mathfrak{M}$ is good and $C \in \mathfrak{M}$ is a simple algebra then C is finite-dimensional.*

(d) *If a variety $\mathfrak{M}$ is good and an algebra $C \in \mathfrak{M}$ has no ideals with non-nilpotent quotients then C is simple*

(e) *If a variety $\mathfrak{M}$ is good and any simple algebra from $\mathfrak{M}$ has no basis consisting of nilpotent elements then for $\mathfrak{M}$ Kurosh problem has positive solution. Moreover, if C is homogeneous and finitely generated then there exists $M \subset C$ such that the algebra $C/M^{(k)}$ is nilpotent for every k.*

(f) *The lattice of prime ideals in a variety satisfying a system of Capelli identities satisfies both ACC and DCC.*

Доказательство. Assertion (a) follows from Propositions 4.1–4.3. Assertions (c) and (d) follow from (b). Assertion (e) follows from (a)–(d), and assertion (f) from the rank theorem. Now we prove assertion (b). In the algebra $L[C]$ there exists a sequence of ideals $\{I_\alpha\}$ such that its intersection lies in the radical $R\big(L[C]\big)$ and any quotient $L[C]/I_\alpha$ embeds into the algebra of $m\times m$ matrices for some m the same for all I_α. The corresponding sequence of ideals $\mathcal{J}_{I_\alpha}$ in C is such that any quotient $C/\mathcal{J}_{I_\alpha}$ embeds into an algebra of dimension over the center, not exceeding some m'. If $x \in \cap \mathcal{J}_{I_\alpha}$ then $L(x) \in R\big(L[C]\big)$ and $x \in R(C)$. Primarity of C implies $R(C) = 0$. It remains to apply Proposition 4.1. □

Suppose $q(n)$ is the nilpotency degree of $C/M^{(n)}$, s is the number of generators of the algebra C. Denote by $I(M,n)$ the ideal in $L[A]$ generated by elements of the form $L(t_1)\cdots L(t_n)$ for which there exists $m \in M$ such that $t_i = m^{k_i}$ for any i (the element $m \in M$ is the same for all t_i). The definition of not bad variety implies

Statement 4.4. *The operator $L(x^q)$ generates for some q a proper ideal in $L[B]$ if and only if x^t for some t generates a proper ideal in B.*

The next proposition establishes a connection between nilpotency of nil-algebras and local finiteness condition.

Statement 4.5. *Let all nil-algebras in a good variety $\mathfrak{M}$ be nilpotent. Then there exist functions $F_{\mathfrak{M}}(n,k)$ and $H_{\mathfrak{M}}(n,k)$ such that any k-generated algebra from $\mathfrak{M}$ having algebraic of order n elements which are sums of monomials of length $\leq H_{\mathfrak{M}}(n,k)$, is of dimension not exceeding $F_{\mathfrak{M}}(n,k)$.*

The next statement is related to combinatorics in not bad varieties. It also enables to pass from an algebra to its algebra of left multiplications, and conversely.

Statement 4.6. (a) *The quotient* $L[A]/I_{M^{(k)}}$ *is nilpotent of degree* $\le q(k)\cdot l(s+1)$.

(b) *We have the inclusion* $\mathrm{Id}(D) \supseteq I_{D,r}$, *and if* $r \ge h(s+1)$ *then for all* D *the equality* $\mathrm{Id}(D) = I_{D,r}$ *holds.*

(c) *For* k *sufficiently large* $\big($*for* $k > g(r+1)\big)$ *we have the inclusion* $I_{M^{(k)},s} \subseteq I(M,n)$ $\big(k > K(|M|, n, s)\big)$.

(d) *For* $n > k\cdot l(2)$ *we have the inverse inclusion* $I(M,n) \subseteq I_{M^{(k)},s}$.

Доказательство. (a) An element of $L[A]^t$ belongs to the ideal generated by the operators of left multiplication by elements of $L\big[A^{t/l(s)}\big]$. If $t/l(s) \ge q(k)$ then $A^{t/l(s)} \supseteq M^{(k)}$.

(b) The inclusion $\mathrm{Id}(D) \supseteq I_{D,r}$ is obvious. The converse inclusion for $r \ge h(s+1)$ follows from Proposition 4.2.

(c) Suppose $z \in \mathrm{Id}(m^k)$. Denote the element m^k by a new letter m'. Then $L[z]$ lies in the ideal generated by m' and by monomials of degree not exceeding $G(s+1)$ in the generators of A. Let z' be such a monomial. Now replace m' by m^k to obtain a monomial w (in generators of the algebra and in the letter m).

The operator $L(w)$ is linearly representable by products of the form $\prod L(w_i)$ where the degree of each w_i does not exceed $h(s+1)$ (the alphabet consists of s generators of the algebra and of m). We say that a monomial w_i is *of the first kind* if it contains a letter distinct from m, and *of the second kind* otherwise. The number of monomials of the first kind does not exceed $G(s+1)-1$, and taken together they contain not more than $\big(h(s+1)-1\big)\big(G(s+1)-1\big)$ occurrences of m. Thus we have not less than $k - \big(h(s+1)-1\big)\big(G(s+1)-1\big)$ occurrences of m in monomials of the second kind.

An operator of multiplication by a monomial of the second kind is linearly representable by a product of monomials of the second kind of degree not exceeding $h(2)$ (a monomial of the second kind includes the letter m and also the letter corresponding to the element of the algebra to which the operators are apllied). Hence we may assume that the number of operators of the second kind is not less than

$$\frac{k - \big(h(s+1)-1\big)\big(G(s+1)-1\big)}{h(2)}.$$

Since the number of monomials of the first kind is not greater than $G(s+1)-1$, there is an interval consisting of not less than

$$\frac{k - \big(h(s+1)-1\big)\big(G(s+1)-1\big)}{h(2)\big(G(s+1)-1\big)}$$

operators of multiplication by monomials of the second kind, disposed succesively. ∎
Thus for

$$k > n\cdot h(2)\cdot\big(G(s+1)-1\big) + \big(h(s+1)-1\big)\big(G(s+1)-1\big)$$

we have the required inclusion $I_{M^{(k)},s} \subseteq I(M,n)$.

(d) Observe that in transformation of expressions of the form

$$x \cdot L[w_1] \cdots L(w_n)$$

where w_i are powers of m, two symbols are involved: m and x. Hence assertion (d) follows from the fact that the L-length of the algebra generated by elements x and m is bounded by $l(2)$. □

Let a number n exceed the degree of an identity valid in $L[A]$. Apply proposition 4.6 and the swap lemma to obtain the following

Lemma 4.1. *Suppose C is a finitely generated graded PI-algebra from a good variety, $M \subset C$ is a finite set of homogeneous elements, such that for any k the quotient algebra $C/M^{(k)}$ is nilpotent. Then there exist a number H and a finite set $D(M)$ such that $L[C]$ is linearly representable by elements of the form $t_1 t_2 \cdots t_k$ where $k < H$, and either $t_i \in D$ or there exists $m_i \in M$ such that $t_i = L(x_{i1})L(x_{i2}) \cdots L(x_{ij})$ where (for a fixed i) all $x_{i\alpha}$ are powers of the same m_i.*

Boundedness of the L-length implies

Statement 4.7. *Let all x_α be powers of the same element m. Then the product $L(x_1)L(x_2) \cdots L(x_j)$ is linearly representable by elements of the form $L(y_1)L(y_2) \cdots L(y_\lambda)$ where $\lambda \leq l(1)$ and all y_α are powers of m.*

Definition 4.3. An algebra C has *essentail height* not greater than H over a set M which is called *an s-basis* of C if there exist a finite set $D(M)$ and a number N such that C is lineraly representable by elements of the form $Q(t_1, \ldots, t_l)$ where $Q \in \mathrm{Mon}_l$, $l \leq N$, and for any i either $t_i \in D$ or there exist $m_i \in M$ и $k_i \in \mathbb{N}$ such that $t_i = m_i^{k_i}$, and the number of $t_i \notin D$ does not exceed N. The minimal of these H is called *the essential height.* If for some H we may assume $D = \varnothing$ then M is a Shirshov basis of C. This is equivalent to the following condition: M generates C as an algebra.

Note that in the associative case we may set $N = 2H + 1$.

The theorem below follows from Lemma 4.1 and Proposition 4.7.

Theorem 4.2. *Suppose C is a finitely generated graded PI-algebra from a good variety, $M \subset C$ is a finite set of homogeneous elements. Then if the algebra $C/M^{(k)}$ is nilpotent for any k then C has bounded height over M. Furthermore if M generates C as an algebra then M is a Shirshov basis for C.*

Замечание. Theorem 4.2 holds even without the associativity condition for 1-generated algebras. In this case, the condition $\exists m_i \in M : t_i = m_i^{k_i}$ in the definition of essential height must be replaced by the condition of representability of t_i in the form of a monomial in a single element of M.

Corollary 4.1. *Suppose C is a finitely generated graded PI-algebra from a good variety, $M \subset C$ is a finite set of homogeneous elements. The set M is an s-basis if and only if every simple quotient of C contains a non-nilpotent image of an element from M.*

Using the fact that simple algebras in good varieties are finite-dimensional, we obtain

Corollary 4.2. *Let $\mathfrak{M}$ be a good variety such that any simple algebra from $\mathfrak{M}$ has no basis consisting of nilpotent elements. Let C be a homogeneous finitely generated algebra from $\mathfrak{M}$. Then C has bounded height over some finite set M.*

In a number of works, asymptotical closeness of certain algebras to associative ones is proven. In fact, what is proved is the property of a variety to be «good». In [44] and in [46] it is shown that l-length of finitely generated Jordan and, respectively, alternative algebras is bounded. In [46] it is proved that the algebra of left multiplications of an alternative or special Jordan finitely generated *PI*-algebra is a *PI*-algebra[1]. In the same work it is shown that for a finitely generated alternative *PI*-algebra of degree m, the condition of the theorem holds if for M we take the set of words of degree not exceeding m^2. In [34] condition (4) is proved for finitely generated Jordan *PI*-algebras, in [16] it is established that a Jordan *PI*-algebra of degree m such that all words of degree $\leq m^2$ are algebraic, is locally finite. Thus we have

Corollary 4.3. (a) *Suppose A is a finitely generated graded associative (alternative, Jordan) PI-algebra, $M \subset A$ is a finite set of homogeneous elements which generates A as an algebra, $M^{(k)}$ is an ideal generated by kth powers of elements from M. Then if for any k the quotient algebra $A/M^{(k)}$ is nilpotent then A has bounded essential height over M.*

(b) *Let B be an alternative or Jordan finitely generated PI-algebra of degree m. Then B has bounded height over the set of words of degree $\leq m^2$.*

Statement 4.8. *Let B be the Cayley — Dickson algebra over an arbitrary field. Then some word of length not exceeding 2 in the generators of B is non-nilpotent.*

This proposition, Theorem 4.2 and Theorem 4.1 imply

Theorem 4.3. *Suppose B is a relatively free alternative algebra, M is a some set of (non-associative) words in its generators. Then M is a Shirshov basis (an s-basis) for B if and only if M is a Shirshov basis (an s-basis) for the quotient of B by the associator ideal.*

[1] a Jordan algebra is called *special* if it embeds into A^+ for some associative algebra A, and is called a *PI-algebra* if it satisfies an identity not valid in the free special algebra. An alternative *PI-algebra* is an alternative algebra with an identity not valid in the free associative algebra

Combinatorial-asymptotic notions and results can be extended to good varieties. *The complexity* of a variety $\mathfrak{M}$ is defined as the class of simple algebras belonging to $\mathfrak{M}$.

Theorem 4.4. *Let $\mathfrak{M}$ be a good variety. Then the following holds.*

(a) *The essential height of a representable algebra from $\mathfrak{M}$, if it exists, is equal to the Gelfand — Kirillov dimension of the algebra as well as to the essential height and the Gelfand — Kirillov dimension of its algebra of left multiplications, and the dimension in question is finite.*

(b) *Any finitely generated algebra from $\mathfrak{M}$ satisfies the strong algebraicity identity as well as the natural analogue of the Capelli identity. Hence its algebra of right multiplications also is PI.*

(c) *The radical of a finitely generated algebra from $\mathfrak{M}$ is nilpotent* (*an analogue of Braun theorem*).

(d) *If every simple algebra from $\mathfrak{M}$ has center then a complete analogue of the theorey of Razmyslov polynomials is valid. In particular, a localization of a prime algebra by the center is finite-dimensional over the center, and a prime algebra embeds into an algebra finite-dimensional over the center. The Gelfand — Kirillov dimension equals the transcendence degree of the center.*

(e) *Homogeneous components of identities form an algebraic ideal which satisfies the identity $x^{n-1} - x^{n_2} = 0$.*

Moreover, as was established above, finite-dimensional algebras from good Kurosh varieties satisfy the height theorem.

Here we do not present the proof of Theorem 4.4 as well as the definitions of some involved notions since the situation is quite similar to the associative case. All argument concerning swap immediately extends to good varieties.

The notion of a *good variety* generalizes to (multibased) algebras of arbitrary signature. An operator D is called *elementary* if there exists a monomial M of degree $n+1$ and constants $c_1, \ldots, c_n \in B$ such that $D(x) \equiv M(x, c_1, \ldots, c_1)$. An operator algebra for a finitely generated algebra has *bounded length* if it is generated as a vector space by products of elementary operators in a bounded number.

Definition 4.4. A variety $\mathfrak{M}$ is called *not bad* if the following conditions hold:

(1) for a finitely generated algebra from $\mathfrak{M}$, the operator algebra also is finitely generated;

(2) an operator algebra for a finitely generated algebra has bounded length.

A not bad variety is called *good* if in addition to (1), (2) it has the following property:

(3) the operator algebra for a finitely generated algebra from $\mathfrak{M}$ is a *PI*-algebra.

Замечание. For an algebra A (of arbitrary signature) satisfying C_n there exists the maximal locally nilpotent ideal [20] but the «Bair chain» stops at the nth step. This means that the algebra includes the maximal locally solvable ideal $B_0(A)$, the quotient $A/B_0(A)$ includes the maximal locally solvable ideal $B_1(A)$, similarly the quotient $A/B_1(A)$ includes $B_2(A)$ and so on. Furthermore we have $B_n(A) = 0$ (see [21]). For good varieties, the Bair radical coincides with the nilradical (and with the Jacobson radical), and $B_1(A) = 0$.

4.1. *On Burnside-type problems.* It is of interest of describe Shirshov bases for Lie and Jordan cases. This problem reduces to the case of varieties generated by simple algebras.

It is known that a simple Jordan algebra is either an algebra of a quadratic form or a non-special algebra $\mathbb{HC}_3$ or an algebra of matrices with operation $A \circ B = AB + BA$ or the algebra of symmetric matrices with operation $\circ$. In the first case the generators may be nilpotent but then all words of length 2 are non-nilpotent and the Shirshov basis must consist of words having length 1 and 2. In the latter case the set of monomials is a Shirshov basis if for any regular word u of length not exceeding the matrix size n there exists a monomial in this set such that after removing parentheses the leading coefficient is u. Hence to improve the estimates in Ye. I. Zelmanov's result [16], it suffices to calculate in $\mathbb{HC}_3$. In any case the bound for the degree of words does not exceed $\max(m/2, \mathrm{const})$. Note that since various bracketings are possible, the above condition for the set of Jordan (Lie) monomials is sufficient but not necessary. To all appearance, it may be weakened. In this context the question arises on description of monomials which are Shirshov bases. Theorem4.3 provides that it suffices to check only Kurosh condition. Seemingly the latter has some connection with tensor ranks of expressions. It would be of great importance to clarify this connection.

It is of interest to compute the lattice of ideals of identities in prime algebras for *PI*-rings close to associative ones and to obtain theorems on finiteness of the lattice for as general situation as possible.

It is known that all simple *PI* (λ, δ)-algebras are associative. So the question arises:

Is the variety of PI (λ, δ)*-algebras good*?

It is easy to see that a finitely generated Engel — Lie algebra generates a not bad variety. Is it possible to prove directly that it is good (i.e., that the algebra of multiplications is *PI*)? For this, it suffices to estimate from above the growth order of codimensions as $o(n)!$. Then we shall obtain another proof of nilpotency of these algebras (a well-known result by Ye. I. Zelmanov).

Also the following questions arise:

Is any algebra which is good from the left also good from the right? Is this valid in zero characteristic? Is the class of not bad (good) varieties in the finitely generated case closed under tensor product? Is this true in the general case?

Since a tensor product of associative *PI*-algebras is also *PI*, the left multiplication algebra in a tensor product of good varieties is *PI*. Moreover it is obvious that the tensor product of algebras A and B with bounded L-length has bounded L-length as well; and if the algebras $L[A]$ and $L[B]$ are finitely generated then $L(A \otimes B)$ is finitely generated as well. However *is it true for a free object of the variety* $\mathrm{Var}(A \otimes B)$?

When solving the Kurosh problem, we imposed the condition of absence of a nil-basis in a simple algebra. It does hold in associative, Jordan and alternative cases. On the other hand, a finite Lie algebra with a nil-basis generates a good variety. It is of interest to obtain any general criteria of absence of a nil-basis. Is it possible *in Corollary* 4.2 *to replace this condition by a weaker condition of absence of simple nil-algebras*?

§5. Finite-basedness problems

Dealing with a wide class of algebras, it is difficult to perform specific calculations, so we will use only abstract tools.

Definition 5.1. Define *the complexity of a variety* as the set of prime algebras belonging to it. Each prime algebra is considered up to the variety generated by it. Order complexities of algebras as follows: $A_i \prec A_j$ means that $\mathrm{Var}(A_i) \subset \mathrm{Var}(A_j)$.

The complexity type of a variety $\mathfrak{M}$ is the family of sets of prime algebras subordinate to $\mathfrak{M}$. A set of prime algebras $\{A_i\}$ is called *subordinate to the variety* $\mathfrak{M}$ if $\mathfrak{M}$ contains an algebra including $\oplus_i A_i$ and having a nonzero non-associative monomial which includes an occurrence of some element from A_i for each i.

In contrast to the associative case, for arbitrary $\mathfrak{M}$ the set of complexities of prime algebras is, in general, only partially ordered. However Martindale's theory provides that this set satisfies both ACC and DCC if some rarefied identity holds.

Similarly to the associative case, we deal with the sets of algebras $\{A_i\}$ (some of them may be identical) connected by means of the radical. Two *complexity types* T_1 and T_2 for these *sets of algebras* will be ordered as follows. If a prime algebra A_i is contained both in T_1 and T_2, decrease the multiplicity of its occurrence by 1 to obtain the sets of algebras T_1' and T_2' which have to be in the same relation as T_1 and T_2. Thus it suffices to order the sets without common elements. In this case we have $T_i \prec T_j$ if and only if each element of

T_i is smaller (relative to $\prec$) than some element of T_j. The obtained relation of the sets of algebras is a relation of partial strict ordering. Replacing any of A_i by any set of strictly smaller algebras decreases the complexity type.

In this section, the ground field has zero characteristic. Any finite-dimensional algebra, if the contrary is not specified, is assumed to be irreducible, that is, it does not include two nonzero ideals with zero meet. All varieties are assumed to be structurizable.

The main aim of this section is to prove Theorem 1.1.

5.1. *The Martindale centroid, the rank theorem and first Kemer lemma.* We start with considering prime components. For use in the sequel, we have to introduce two new essential restrictions on the variety in question, namely, to pass to *structurizable* and *convenient* varieties (see definition 1.1).

Let $\mathfrak{M}$ be a prime *PI*-algebra of arbitrary signature Ω. Then its Martindale centralizator is a finite-dimensional algebra over some field, of dimension equal to the maximal degree of a Capelli polynomial non-vanishing on this algebra.

Let R be a prime algebra of signature Ω. Recall the construction of the *Martindale centroid.* It is defined as the injective limit of equivalence classes of morphisms

$$\varinjlim \bigl\{ \mathrm{Hom}_R(I,R) \bigm| 0 \neq I \lhd R \bigr\}.$$

Equivalence of pairs (ψ_1, I_1) and (ψ_2, I_2) means coincidence of restrictions of ψ_1 and ψ_2 to the meet $I_1 \cap I_2$. The sum of morphisms ψ_i and ψ_j is defined in a natural way, and the product is the composition $\psi_i \circ \psi_j$. So we obtain the structure of a commutative ring $C(R)$.

The central closure $Q(R)$ is the set of formal sums $\sum \psi_i r_i$ where $r_i \in R$, with the natural equivalence relation and naturally defined operations.

Theorem 5.1 [42]**.** *Suppose $C(R)$ is a commutative ring with unit, $Q(R) = C(R)R$, any nonzero D-submodule in $Q(R)$ intersects R in a nonzero ideal, and for any D-homomorphism χ of a nonzero D-submodule J from $Q(R)$ to $Q(R)$ there exists an element $c \in C(R)$ such that $cj = \chi(j)$ for any $j \in J$. If J is a large D-submodule in $Q(R)$ then the element c is uniquely determined by the homomorphism χ.*

A submodule M of a module P is called *large in* P if any nonzero submodule in P has nonzero meet with M.

Another definition of the Martindale centroid may be given using the injective envelope (see [42]). *The central closure* of a prime algebra A is constructed as the injective envelope of A as a $D(A)$-module; here the ring of $D(A)$-endomorphisms of this module occurs to be a field such that $D(A)$-endomorphisms of A embed into it. This field is called the *Martindale centroid.*

Suppose a polynomial $F(\vec{y}, x_1, \ldots, x_n)$ is multilinear and skew symmetric in variables x_i. Substitute $e_i \to x_i$ and $\vec{y}' \to \vec{y}$ where $e_i, y'_k \in A$. Now for each

i, fix $\vec{y}$ and all e_j except e_i and put $\varphi_i(e_i) = F(\vec{y}, e_1, \ldots, e_n)$ and $\varphi_i(e_j) = 0$ for $j \neq i$ (no confusion because of skew symmetricity of F).

We have obtained a non-associative analogue of matrix units. Instead of basic units E_{ij} we have e_i. Instead of the position $*$ between facings $E_{ij} * E_{kl}$ for substituting the matrix unit E_{jk} (substituting other matrix units results in zero), there are positions in the polynomials φ_i.

Nonvalidity of the system C_n means «essential linear independence». Let R be a prime algebra such that C_{n+1} does hold but C_n does not. Then R embeds into a central simple algebra R' of dimension n over a field, generating the same variety (see, for example, [62] or [42]). The precise formulation is as follows.

Theorem 5.2 (on rank [42])**.** *Let V be a subspace in a prime algebra A of signature Ω. If* $\operatorname{rank}(A, V) < \infty$ *then for the central closure $Q(A)$ we have*

$$\dim_{c(A)} C(A)V = \operatorname{rank}(A, V) - 1.$$

The rank $\operatorname{rank}(A, V)$ of a vector space V relative to an algebra A is the least positive integer k such that V satisfies all Capelli identities of order k. We say that V *satisfies all Capelli identities of order* k if any polynomial $F(x_1, \ldots, x_k, \vec{y})$, multilinear and skew symmetric in a set of variables $\{x_1, \ldots, x_k\}$ vanishes on A after replacing x_i by elements of V skew symmetric in a set of k variables.

The proof of Theorem 5.2*.* We present this proof because we will use similar argument in the sequel. Suppose $a = C_n(v_1, \ldots, v_n, \vec{z}) \neq 0$, $v \in A$, $G(x, y, \vec{t})$ is bilinear in x and y. Put

$$a_i(u) = C_n(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_n, \vec{z}).$$

Using Lemma 3.3 on absorption of a variable $\big($equation (5)$\big)$ we get

$$g\big(a_i(u), a, \vec{t}\big) = \sum_{j=1}^{n} g\big(a_i(v_j), a_j(u), \vec{t}\big).$$

If $u = v_k$ and also $k \neq i$ then $a_i(u) = 0$ by skew symmetricity of the Capelli polynomial. If $k = i$ then $a_i(u) = a_i(v_i) = a$. Hence the preceding equation implies the identity

$$g\big(a_i(u), a, \vec{t}\big) \equiv g\big(a, a_i(u), \vec{t}\big). \tag{11}$$

Since the algebra A is simple for any $g, a \in A\backslash\{0\}$, $n \in \mathbb{N}$, there exist $\alpha \in \operatorname{Id}(a)^n$ and a nonzero multilinear polynomial G such that $G(g, \alpha, \vec{t}) \neq 0$. Together with the equation (11) which enables to transfer occurrences this easily implies that the operator $c_i(a, u)$ replacing a by $a_i(u)$ is well defined and does not depend on a specific recording. Moreover this operator belongs to the field of $D(A)$-endomorphisms of the central extension of A.

To prove the rank theorem, it suffices to establish the equality

$$b = u - \sum_{i=1}^{n} c_i(a,u)v_i = 0.$$

However for any polynomial H bilinear in the first two arguments we have

$$H(b,a,\vec{t}) = H(u,a) - \sum_{i=1}^{n} H(v_i, c_i(a,u)a, \vec{t}) = H(u,a) - \sum_{i=1}^{n} H(v_i, a_i(u), \vec{t}) = 0.$$

The last equality holds by virtue of the identity (5) from Lemma 3.3 (taking u for the variable z). The rank theorem is proven. □

The properties of the Martindale centroid and of the central closure imply

Statement 5.1. *Let $A \in \mathfrak{M}$ be a prime algebra which satisfies the system of Capelli identities of order $(n+1)$ but does not satisfy the system of Capelli identities of order n. Then A embeds into an algebra B, finite-dimensional over an associative-commutative ring $\mathbb{K}$, so that for any $a \in D(A)$ there exists a $\lambda(a) \in \mathbb{K}$ such that for any polynomial $F(\vec{y}, x_1, \ldots, x_n)$, multilinear and skew symmetric in $x_1, \ldots, x_n$, we have*

$$\sum_{i=1}^{n} F(\vec{y}, x_1, \ldots, x_n)\Big|_{a(x_i)\to x_i} = \lambda(a) \cdot F(\vec{y}, x_1, \ldots, x_n).$$

Here the ring $\mathbb{K}$ is generated by these $\lambda(a)$ and is Noetherian.

Доказательство. In view of properties of the Martindale central closure it suffices to ensure that if a polynomial $F_{\vec{y}}(\vec{x}, \vec{z})$ is multilinear and skew symmetric in the sets of variables $x_1, \ldots, x_n$ and $z_1, \ldots, z_n$ then the operation

$$F_{\vec{y}}(\vec{x}, \vec{z}) \to \sum_{i=1}^{n} F_{\vec{y}}(\vec{z}, x_1, \ldots, x_n)\Big|_{a(x_i)\to x_i}$$

determines a morphism of $D(A)$-modules (from $D(A)$ to the operator algebra) generated by the values of F on the T-ideal of A, generated by C_n. The latter follows from Lemma 3.6. □

Corollary 5.1. *The lattice of T-prime ideals in a finitely generated algebra of arbitrary signature, satisfying a system of Capelli identities satisfies ACC and DCC.*

In some cases it suffices to consider only operators of multiplication by elements of A.

Let $\{e_i\}$ be a basis of the ring R' considered as a vector space. Then the rank theorem implies the existence of a polynomial $F(x_1, \ldots, x_n, \vec{y})$, multilinear and skew symmetric in the set of variables $\{x_1, \ldots, x_n\}$ and such that $\Psi = F(e_1, \ldots, e_n, \vec{t}) \neq 0$. Since R' is simple, any element $x \in R'$ belongs to the kth power of the ideal J generated by Ψ, and there exists a linear combination of monomials containing k occurences of Ψ.

Let now R be a representable algebra from a structurizable good variety. It can be embedded into a finite-dimensional algebra R' such that $\mathrm{Var}(R') = \mathrm{Var}(R)$. Moreover R' decomposes into the sum of its prime components and the radical. Here every prime component R'_i is of dimension n_i equal to the maximal degree of the Capelli identity not valid in R'_i. Then to R'_i there corresponds some Ψ_i.

Consider an arbitrary polynomial G and its specializations in R' (the variables are replaced by elements of prime components and of the radical). Suppose a specialization H of a polynomial G does not vanish and to a variable y_i there corresponds an element x_i of R'_i. For any $k \in \mathbb{N}$, such a specialization is linearly representable by specializations of polynomials of $\mathrm{T}(G)$ obtained by substitutions of the form $y_i \to M_j$ where M_j is some non-associative monomial and an element Ψ_i replaces k variables in which M_j is multilinear. Since the specialization H is nonzero, all these specializations of the new polynomial from $\mathrm{T}(G)$ cannot vanish.

Furthermore if for any i the specialization H contains an element from R'_i then the basis elements $\{e_{ij}\}_{j=1}^{n_i}{}_{i=1}^{s}$ occiring in Ψ_i can be joined into a single set and alternated. Since replacing e_{ij_2} by e_{ij_1} for $j_1 \neq j_2$ results in zero, the result of alternation coincides with the original specialization and is not zero (the argument is similar to that of A. R. Kemer [27]).

Since we have substituted k terms Ψ_i into M_j, we can perform alternating relative to k sets. (In the sequel, we will take k sufficiently large.)

Summing up the above, we obtain

Lemma 5.1 (on refinement)**.** *Let a polynomial g take a nonzero value $\bar{g}$ for some specialization involving all prime components R'_i. Then for any arbitrarily large $q \in \mathbb{N}$ there exists a consequence of $\mathrm{T}(g)$ which is obtained from g by replacing variables by nonassociative monomials and alternating in q sets of $b(A)$ variables, and has the same value.*

Thus for convenient structurizable algebras of arbitrary signature we have

Lemma 5.2 (Kemer's first lemma)**.** *Let $\mathfrak{M}$ be a structurizable convenient variety of arbitrary signature, $A \in \mathfrak{M}$, $A^{(i)}$ is the quotient of A by the ideal generated by the ith prime compoment.*

Then either $\mathrm{Var}(A) = \mathrm{Var}(\bigcup A^{(i)})$ or $b(A) = \sum \dim(A^{(i)})$.

If Γ is a T-ideal in A then either $\Gamma \bigcap_i A^{(i)} = 0$ or $b(A, \Gamma) = \sum \dim(A^{(i)})$.

Замечание 1. Similar argument is used in the proof of the rank theorem.

Замечание 2. The proof of the non-associative analogue of second Kemer lemma differs fundamentally from the associative case. The point is that in the non-associative case the interaction between prime components and the radical may have different properties. In particular, any power of a composition of elements from operator algebras $D(A^{(i)})$ with distinct i can have nonzero action on the radical (for instance, in the case of Jordan algebras). Hence we cannot apply argument from [27] which uses moving a mixed element into a monomial such that all other elements of it have semisimple specializations only.

Since an explicit «chasing» the radical as in A. R. Kemer's work is rather difficult here, we use a more abstract and complicated method. We will show that internal traces in the structurizable case are defined invariantly and do not depend on the way of recording. This is the main content of this section.

5.2. *Structurizable varieties. second Kemer lemma.* The aim of this subsection is construction of internal traces in an extremal ideal and proof of their invariance. Also we present here the main technical results of the section, in particular, a non-associative analogue of second Kemer lemma.

Let A be a finite-dimensional algebra with generators $a_1, \dots, a_s$ from a structurizable variety $\mathfrak{M}$, $\overline{A} = A/J(A)$. Put

$$A(q) = \overline{A} *_{\mathfrak{M}} \mathbb{F}\langle \theta_1, \dots, \theta_s \rangle / \Theta^q$$

where $\Theta = \mathrm{Id}(\theta_1, \dots, \theta_s)$. This is the «free extension of the semisimple part of $\overline{A}$ by the radical with the nilpotency degree q». The construction is similar to the associative case.

Suppose D is a Kemer diagram,

$$\Lambda = \bigcup_{i=1}^{k+2} \Lambda_0^i \cup \bigcup_{j=1}^{d} \Lambda_j$$

is the corresponding set of variables, and

$$k \geq \max\bigl(k(A), c(A)\bigr), \quad d = d(A), \quad |\Lambda_0^i| = b(A),$$

$|\Lambda^j| = b(A) + 1$. To each polynomial f multilinear in variables from Λ there corresponds a polynomial $S_\Lambda(f)$ from $S(D)$ obtained by alternating in the sets Λ_0^i and Λ^j. Moreover for some f the polynomial $f' = S_\Lambda(f)$ does not vanish.

We call the T-ideal consisting of polynomials of the form $S_\Lambda(f)$ *extremal.*

Suppose $g \in S(D)$. Then g can be represented in the form $\sum S_\Lambda \breve{g}_i\big|_{v_i \to x_i}$ where v_i are some monomials which replace variables. We will use *recordings of* g, or expressions of the form

$$g = \sum S_{\Lambda^{(i)}} g_i$$

where alternated are the words corresponding to variables from Λ.

For any recording of this form and for any element $a \in A$ define the expression

$$\delta(a)(g) = \sum_i \sum_k S_{\Lambda^{(i)}} g_i \Big|_{av^1_{ik} \to v^1_{ik}}$$

where $\{v^1_{ik}\}$ is the set of words corresponding to variables from the set Λ^1_0 for the ith term.

Results of Subsection 3.2 imply that the operators $\delta(u_i)$ commute.Moreover the result of applying the operator $\delta(a)$ does not depend on the choice of the small subset Λ^i_0 in Λ which is used to determine (by substitutions) the operator $\delta(a)$ (see Lemma 3.6 and Proposition 3.3).

Our aim is to establish the invariance of these operators, that is, their independence of recordings of an element $g \in S(D)$ from the extremal ideal to which the operators of the form $\delta(a)$ are applied. For this, it suffices to check that a recording of the zero element turns into a recording of the zero element.

In fact, suppose

$$\sum S_{\Lambda^{(i)}} g_i = g = h = \sum S_{\Lambda^{(j)}} h_j.$$

Then

$$\sum S_{\Lambda^{(i)}} g_i - \sum S_{\Lambda^{(j)}} h_j$$

is a recording of the zero element $g-h$, and if $\delta(a)(g) \neq \delta(a)(h)$ then $\delta(a)(g-h) \neq 0$. Thus the recording of the zero element has turned into a recording of the zero element.

We shall require some auxiliary statements and constructions.

A recording $S_{\Lambda^{(i)}} g_i$ will be called *a letter recording* if all variables from the set Λ are replaced by words of length 1, that is, all alternations involve single letters.

Semisimplicity of the group algebra for the symmetric group implies

Statement 5.2 (on letter alternators)**.** (a) *Let f be a sum of multilinear polynomials of the form $S_\Lambda(f_i)$ where all alternations are in single letters. Then f can be represented in the form*

$$f = \Big(\sum_j \alpha_j S_{\Lambda_j}\Big)(h)$$

where $\alpha_j \in \mathbb{K}$ and S_{Λ_j} is the operator of alternating in sets of variables corresponding to columns of the diagram D.

(b) *Moreover $S_{\Lambda_j}(f) \neq 0$ for some j.*

Доказательство. It suffices to observe that in the group algebra, the elements

$$\frac{1}{(b+1)!^d b!^{k+2}} S_{\Lambda_j}$$

are idempotents and any linear combination of these does not vanish when multiplied by an element suitably chosen among them. □

Замечание 1. The above statement is a fact from tensor algebra and holds for any algebra of arbitrary signature over a field of zero characteristic.

Замечание 2. Semisimplicity of the group algebra results in the following fact: if the alternation procedure can be performed once then it can be repeated unboundedly. Lemma 5.3 on letter thinning may be considered as a far-reaching generalization of these considerations. Perhaps this approach enables to establish PI_n-properties for good structurizable Specht varieties, that is, to deduce from the Specht property local representability by constructing T-ideals such that after any T-space operation we can return to T-ideals represented in a convenient form.

Proposition 5.2 and results of Subsection 3.3 imply

Corollary 5.2. *The space of polynomials which are letter alternators corresponding to a Kemer diagram D is representable.*

The following result is rather important:

Lemma 5.3 (on letter thinning)**.** *Suppose A is a finite-dimensional algebra from a structurizable good variety and $g = \sum_i S_{\Lambda^{(i)}} g_i$. Then any value of g in A is linearly representable by values of letter alternators from $\mathrm{T}(g)$ corresponding to a Kemer diagram D.*

Замечание. We emphasize that this statement relates to T-spaces and moreover that we substitute elements from a structurizable representable algebra, that is, we substitute semisimple and radical components separately.

Доказательство. In view of Proposition 5.2 it suffices to verify our assertion for a single term $S_{\Lambda^{(i)}} g_i$. We may assume that $g = S_{\Lambda^{(i)}} g_i$ and g is multilinear.

We argue as in the proof of Lemma 5.2. Consider an arbitrary specialization of variables such that $g \neq 0$. To this specialization, attach the following system of substitutions $M_i \to x_i$.

- To a specialization of a variable x_i to a radical element there corresponds the identity substitution $x_i \to x_i$.
- To a specialization of a variable x_i to a prime component $A^{(i)}$ of A there corresponds a substitution of the form $M_i \to x_i$ where M_i is a non-associative monomial containing a sufficient number of occurrences of a monomial Ψ_i, multilinear and skew symmetric in some set of n_i variables, $n_i = \mathrm{rank}(A^{(i)})$.

- All variables in the monomials Ψ_i are distinct.

We will call a variable *passive* if it occurs in a (non-associative) word v_i which replaces a variable contained in a large set corresponding to Λ; otherwise the variable is *active*. Substitutions to small sets of active variables create multiplications by traces and the structure of a Noetherian module.

Since the specialization is fixed, to each of the original variables there corresponds either a radical specialization or a prime component A, and thus to each Ψ_i there corresponds some component $A^{(i)}$.

Note that for any specialization with nonzero result there is a small set of variables (see definitions 1) which contains elements from each prime component R_i'. Hence we can find a sufficient number of sets of active variables Δ_j such that

- each of these sets Δ_j is a joint of some sets of n_i variables, relative to which Ψ_{ij} is skew symmetric;
- for each Δ_i the set of prime components of A which correspond to sets from Ψ_{ij} coincides with the set of prime components of A taken once;
- thus any set Δ_j consists of $b(A)$ variables.

Now we alternate relative to all sets Δ_j.

Thus we have produced a sufficient supply of letter alternators corresponding to small sets of variables and consisting of active variables. The value of the resulting polynomial equals the originally fixed specialization for the original polynomial g by virtue of construction of Ψ_i and results of Subsection 5.1. (If a variable specialized in $A^{(i)}$ gets to the nest corresponding to $A^{(k)}$ where $k \neq i$, the result is zero, hence alternation does not change the value of the polynomial for the given specialization.)

The procedure of turning large sets of variables into letter sets is similar to the procedure of thinning, used in the proof of Lemma 3.9 and Proposition 3.11. Let a polynomial f be multilinear and skew symmetric in a set of variables $\{y_i\}_{i=1}^{b(A)}$ and this is preserved after adding a variable x, and let

$$f' = f - \sum_{i=1}^{b(A)} f\Big|_{x\to y_i;\, y_i\to x}. \tag{12}$$

Then the polynomial f' is multilinear and skew symmetric relative to the set $\{x\} \cup \{y_i\}_{i=1}^{b(A)}$.

Consider a system of small sets Δ_j (see definitions in Subsection 3.1).

Any of these Δ_j is a letter alternator, and the variables belonging to it do not appear in large sets.

For any radical specialization appearing in a large set, add the corresponding variable to some of small sets and use relation (12). Now the original value of the polynomial G will be represented as the sum of a value of a letter alternator (which we have to construct) and a value of a polynomial from $\mathrm{T}(G)$ such that large sets involve a smaller number of its specializations.

It suffices to observe that if words corresponding to large set do not include radical specializations then the polynomial vanishes because the dimension of the semisimple part equals $b(A)$ and is less than the number of elements in the large set relative to which we alternate. □

Замечание. Instead of applying proposition 5.2, we may construct for each $S_{\Lambda^{(i)}}g_i$ a sufficient number of sets of active variables Δ_j, disjoint for distinct i, and deal with each item separately. (Is a variable passive or active, this depends on the choice of the item g_i, this is the matter of fact.) The fact that dealing with the previous items will multiply the subsequent ones is not dangerous because «multiplied» items are equivalent for the sequel. The reason is that the sets Δ_j are disjoint, and the procedure of alternation changes positions of variables inside these sets (these variables will not be used at subsequent steps of «splitting-out») as well as positions of variables having radical specialization, whose number is bounded by $c(A)-1$. The number of variables necessary for constructing the required number of sets of active variables is estimated by $2r\big(b(A)+1\big)\big(d(A)+2\big)$.

Suppose now

$$h=\sum_{i=1}^{r} S_{\Lambda^{(i)}}(h_i)=0$$

and

$$h_a=\delta(a)(h)=\sum_{i=1}^{r}\sum_{k} S_{\Lambda^{(i)}}h_i\Big|_{av_{ik}\to v_{ik}}=\sum_{i=1}^{r}\delta(a)S_{\Lambda^{(i)}}(h_i)\neq 0.$$

So h_a is an obstacle for correctness of the definition of the internal trace (for the element a). The idea of the proof is to construct a nonzero element $h'_a\in\mathrm{T}(h_a)$ (without any substitution into a) having a sufficient number (2) of active small and $d(A)$ large letter alternators. Then the operator $\delta(a)$ may be «transferred» to these small alternators not changing the result, and then applying of this operator amounts to some substitutions using these variables, that is, to a T-space operation. This results in $h'_a\in\mathrm{T}(h')$ and $h'\in\mathrm{T}(h)=\mathrm{T}(0)=0$. A contradiction with $h'_a\neq 0$.

Now apply the lemma on letter thinning to each $\delta(a)S_{\Lambda^{(i)}}(g_i)$ occurring in this sum. Then we obtain a nonzero polynomial $\check{h}\in\mathrm{T}(h)$ representable in the form

$$\check{h}_a=\sum_{i=1}^{r}\sum_{j}\delta(a)S_{\Lambda^{(ij)}}(h_{ij})$$

where all alternators are letter ones and for each h_{ij} the number of letters occurring in large alternators equals $(b+1)d$.

Here the corresponding polynomial h vanishes:

$$h = \sum_{i=1}^{r} \sum_{j} S_{\Lambda^{(ij)}}(h_{ij}) = 0.$$

The number of h_{ij} can be estimated by examining the proof of the lemma on letter thinning and does not exceed c^d. Call a variable *passive* if it occurs in some large alternator, and *active* otherwise. The number of passive variables is easily estimated by $r(b+1)dc^d$. In this way, we can provide any number of small alternators consisting of active variables.

Furthermore we assume additionally (for the sake of «transfers») that *for any $S_\Lambda(h_{ij})$ there exists a small set of alternated active variable (letters) $\{z_k\}$ and a small set of alternated monomials $\{v_k\}$ such that no z_α occurs in any v_β.*

In all h_{ij} we may fix the same set of active variables which form a small alternator, and define using them the operator $\delta_1(a)$. Since

$$h = \sum_{i=1}^{r} \sum_{j} S_{\Lambda^{(ij)}}(h_{ij}) = 0$$

and the range of $\delta_1(a)$ is included in the T-space and adjusted for all terms, we have $\delta_1(a)(h) = 0$. On the other hand, by virtue of argument concerning «transfer» (see Proposition 3.3 and Corollary 3.4), the difference $\delta_1(a)(h_{ij}) - \delta(a)(h_{ij})$ lies in the ideal $H(D_1)$ where the diagram D_1 is obtained from D by extending the small column to the large one. Hence $\delta_1(a)(h_{ij}) - \delta(a)(h_{ij}) = 0$ and consequently $h_a = \delta_1(h) = 0$.

We have proved vanishing of the obstacle for correctness of the definition for the internal trace. Thus we have established the basic

Lemma 5.4 (on internal traces)**.** *For convenient varieties, the operators of the internal trace are defined correctly, that is, independently of a recording of the element.*

Similarly, the «transfer» procedure enables to prove

Statement 5.3. *Let the radical component* Rad(a) *of an element a be zero. Then $\delta(a)$ amounts to multiplication by the trace of the corresponding operator.*

Доказательство. The assertion is obvious if all specializations of the small set of variables involved in the definition of $\delta(a)$ are semisimple. On the other hand, by Proposition 3.3 the substitutions forming $\delta(a)$ can be «transferred» from one set of variables to another. Lemma on refinement enables to provide that the number of small sets exceeds $c(A)$ and for any specialization of variables all specializations in some of these small sets are semisimple. □

Proposition 5.3 immediately (without use of lemma on letter thinning) implies

Corollary 5.3. *If $R(a) = 0$ then $h_a = 0$.*

The main lemma 5.4 implies

Corollary 5.4. *If $\mathfrak{M}$ is a convenient variety such that all its varieties are representable then the Hilbert series of any relatively free algebra from $\mathfrak{M}$ is rational.*

Доказательство. The T-идеал $J = \mathrm{T}(S_\Lambda)$ is a Noetherian module over internal traces. Its Hilbert series H_J is rational, and $H_A = H_J + H_{A/J}$. The algebra A/J is relatively free, and moreover either $b(A/J) < b(A)$ or $b(A/J) = b(A)$ and $d(A/J) < d(A)$. The decreasing induction completes the proof. □

Thus to prove rationality of Hilbert series it suffices to show local representability.

Definition 5.2. An ideal $J \subseteq S_\Lambda$ is called *closed* if it is closed under operators of internal traces $\delta(a)$. Let J be an arbitrary ideal. Then J^0 is a maximal closed ideal included in J.

Замечание. The notion of closedness is naturally defined also in the case $b(A, J) = b(A)$ for ideals lying in the spaces H_r. The corresponding constructions are presented in Subsection 5.2.1.

If J is a T-ideal then J^0 is a T-ideal as well. Moreover

$$d(A/J^0, J) < d(A) = d(A, J).$$

Suppose $I = \bigcap_{i=1}^{s} R_i \cap R(A)^{c(A)-1}$.
Proposition on letter alternators implies

Statement 5.4. *If $J \subset I$ and J is a nonzero T-ideal then $J^0 \neq 0$.*

Lemma on refinement and proposition on letter alternators imply

Statement 5.5. (a) *Suppose $g \in \mathrm{T}\left(S_\Lambda(f)\right)$ and $g\big|_A \neq 0$. Then there exists $h \in \mathrm{T}(g)$ such that $S_\Lambda(h) \neq 0$.*

(b) *Let Q be the T-subspace $\bigcup_f S_\Lambda(f)$. Then there exists a closed T-ideal $\Gamma \subseteq \bigcup_f S_\Lambda(f)$ such that $Q^0 = Q \cap \Gamma$ is a closed T-space.*

Recall that in this section $\mathrm{T}(f)$ denotes the T-space generated by f. Hence our argument retains for T-spaces, and we may reinforce the result of Corollary5.4.

Theorem 5.3. *Suppose $\mathfrak{M}$ is a convenient variety such that all its subvarieties are representable. Then the Hilbert series H_Q of an arbitrary T-space Q in a relatively free algebra from $\mathfrak{M}$ is rational.*

Доказательство. Let Q be a T-space in a relatively free finitely generated algebra $B \in \mathfrak{M}$. We have to show rationality of H_Q. We may assume that $\mathfrak{M} = \mathrm{Var}(B)$. Let A be a finite-dimensional algebra such that $\mathrm{Var}(A) = \mathfrak{M}$. By induction argument, we may assume that A includes no two nonzero ideals with zero meet.

Suppose Λ is a set of variables, corresponding to a Kemer diagram D, $I = \bigcup_g \mathrm{T}\big(S_\Lambda(g)\big)$, $Q' = Q \cap I$. It suffices to prove rationality of $H_{Q'}$, so we may assume that $Q = Q'$. It remains to apply assertion (b) of Proposition 5.5 since the Hilbert series H_{Q^0} is obviously rational. Hence we may pass to the quotient B/Γ and complete the proof by decreasing induction. □

Corollary 5.5. *The Hilbert series of any T-space in a finitely generated relatively free associative PI-algebra is rational.*

The proof of Theorem 5.3 implies

Corollary 5.6. *Let $\mathfrak{M}$ be a convenient variety such that all its subvarieties are representable. Then any ascending chain of T-spaces in a finitely generated relatively free algebra from $\mathfrak{M}$ stops.*

So it remains to consider problems related to finite-basedness and local representability, for T-ideals.

Enrich an algebra A by the operation of taking the trace $\delta_e(a)$ as well as the traces of operators from the corresponding operator algebra $D(A)$. Consider the quotient by the ideal generated by elements $\big(\delta(a) - \delta_e(a)\big)(f)$ where $f \in S_\Lambda(g)$ to obtain the algebra $\widehat{A}$. We have

Statement 5.6. *Any algebra from a variety with extended (by δ-type operators) signature, generated by the algebra $\widehat{A}$ is a Noetherian-type algebra. The variety is structurizable. The natural morphism $A \to \widehat{A}$ is an embedding.*

Замечание. The extension of the operator algebra $D(A)$ requires in general an infinite number of operators. However we can ensure that by Shirshov height theorem and lemma on tree 3.8 it suffices to use only a finite number of operators.

Let I be the intersection of ideals generated by R_i and $\big(c(A) - 1\big)$th power of the radical. Recall that the algebra A is irreducible and involves a nonzero polynomial f taking values only in I. We may assume that f is multilinear, and any specialization of variables with nonzero result requires $(c(A) - 1)$ radical specializations and for each semisimple component, a specialization connected with it.

Suppose Γ is a nonzero T-ideal, $\Gamma \subset I$. Then there exists a nonzero T-ideal $\Gamma^0 \subset \Gamma$ generated by polynomials of the form $\big\{g = S_\Lambda(f) \bigm| f \in \Gamma\big\}$ and closed under the operators $\delta(a)$.

Suppose $a_i \in R'_i$. Apply to a_i the refinement procedure, that is, represent a_i as a linear combination of monomials $M_{ij}(\Psi_i, \vec{y})$ containing $c(A)$ occurrences of Ψ_i. Take an arbitrary set of specializations for g and apply to g the product of $\delta(a_i)$. Then using Ψ_i and elements from radical specializations construct $c(A)$ sets of $b(A)+1$ elements each. Alternating in these sets produces additional terms where the radical specialization is inside a_i, as well as terms where the content of Ψ_i is inside Ψ_j $(i \neq j)$. These additional terms vanish.

The reason is that displacement of a variable whose value belongs to $A^{(i)}$ inside a monomial in elements of $A^{(k)}$ for $k \neq i$ results in zero. Moreover, the result of applying δ to a radical element from $f \in \Gamma^0$ is linearly representable by elements of the form $\sum f_i \delta_i$ where $f_i \in \Gamma^0$ involve strictly less than $c(A)-1$ radical specializations, hence each of these polynomials vanishes. See a similar argument in [27] for the case of absence of mixed elements.

Since the T-ideal Γ^0 is closed under multiplication by operators δ then the corresponding ideals in the original and the enriched algebras coincide. Thus we have proved a non-associative analogue of second Kemer lemma.

Statement 5.7 (the second non-associative Kemer lemma)**.** *If there exists a nonzero T-ideal $\Gamma_0 \subset I$ then $d(A) = c(A) - 1$.*

Corollary 5.7. *The ideal $H(D)$ corresponding to the Kemer diagram D of an algebra A is representable.*

Corollary 5.8. *If there exists a nonzero T-ideal $\Gamma_0 \subset I$ then there exists a faithful representation of the algebra such that internal traces coincide with external ones.*

5.2.1. *The spaces H_r. The relative form of second Kemer lemma.* Let Γ be a T-ideal in the algebra A. For it, define *relative* extremal diagrams, Kemer diagrams, parameters b, d and k and all other constructions from the beginning of 3.

If $\Gamma \subseteq \bigcap R'_i$ then by the non-associative analogue of first Kemer lemma 5.2 we have $b(A, \Gamma) = b$. Such T-ideals are of main interest for us.

Let S_r be an alternation operator related to a diagram D_r including r large and not less than $k+2$ columns. Here $r \leq d(A)$.

Definition 5.3. The space H_r is the maximal T-ideal Γ such that $d(\Gamma, A) = r$ and $b(\Gamma, A) = b(A)$. Set $H'_r = H_r \cap S_{D_r}(A)$ where $S_{D_r}(A)$ is a T-ideal generated by polynomials of the form $S_{\Lambda_r}(f)$, here Λ_r is the set of variables corresponding to the diagram D_r.

In other words, alternation in the set of variables corresponding to a diagram which has greater number of large columns than D_r and not less than $k+2$ small columns, being applied to an element of H_r results in zero.

The correctness of the definition is obvious since the joint of any family of ideals with this property retains it.

Proposition 5.5 implies the important

Corollary 5.9. *If* $r_1 < r_2$ *then* $H_{r_1} \cap S_{D_{r_2}}(A) = 0$.

The spaces H'_r have the same properties as the extremal ideal I, and the analogues of the results of the above parts. The formulations are below.

Recall that a space $S \subset B$ *is representable in an algebra* B if there exists a morphism B to a Noetherian-type algebra, such that its restriction to S is an embedding.

Statement 5.8. *Let* $A/S_{D_{r+1}}(A)$ *be an algebra.*

(a) *The space* H'_r *is representable in* $A/S_{D_{r+1}}(A)$.

(b) *The space* H'_r *contains no obstacle for representability of* $A/S_{D_{r+1}}(A)$.

Доказательство. Assertion (b) is a reformulation of assertion (a). Let us prove (a). Let $\widehat{A}'$ be the quotient of $\widehat{A}$ by the ideal generated by $\big(\delta(a)-\delta_e(a)\big)(f)$ where $f \in H'_r$ for some r. Then in view of the above, the natural morphism $A \to \widehat{A}'$ also is an embedding and the analogue of Proposition 5.6 holds.

Note that the space H'_r and its image in $A/S_{D_{r+1}}(A)$ are isomorphic. The same is true for extended algebras. □

The basic lemmas for the relative case are proved similarly.

This results in the following statement.

Statement 5.9.

(a) *The spaces* H'_r *are closed under multiplication by internal traces.*

(b) *If* $b(A,\Gamma) = b$ *then* $\Gamma \subseteq \bigoplus H'_r$.

(c) *Suppose* Γ *is a* T*-ideal,* $\Gamma \subseteq H'_r$ *and* $b(A,\Gamma) = b$. *Then* $d(A,\Gamma) = r$.

(d) *For any nonzero* T*-ideal* $\Gamma \subseteq H'_r$ *there exists a closed* T*-ideal* $0 \neq \Gamma' \subseteq \Gamma$. *Also there exists a polynomial* $h \in \Gamma'$ *such that* $S_{D_r}(h) \neq 0$.

(e) *Suppose* $g \in H'_r$. *Then any value of* g *in* A *is linearly representable by values of letter alternators corresponding to the diagram* D_r *and applied to elements of* $\mathrm{T}(g) \in H'_r$.

Consider the quotient of $\widehat{A}$ by the ideal generated by elements of the form $\big(\delta(a) - \delta_e(a)\big)(f)$ where $f \in H'_r$ $\big(r = 1, \ldots, d(A)\big)$. Denote it by $\widehat{A}'$. Again the algebra A embeds into a Noetherian-type algebra $\widehat{A}'$ which generates (as an enriched algebra) a structurizable variety.

Now we formulate a relative analogue of second Kemer lemma.

Lemma 5.5 (second Kemer lemma for H'_r)**.** *Suppose the projection of a* T*-ideal* Γ *to the algebra* $A(q-1)$ *is zero and to* $A(q)$ *is not, and furthermore* $b(A,\Gamma) = b$. *Then* $\Gamma \subseteq H'_r$ *и* $d(A,\Gamma) = q$.

Since the proof is similar to the absolute case, we do not present it.

5.2.2. *Test algebras.* In this subsubsection we reduce a finite-dimensional algebra determining a variety, to some canonical form. This procedure is related to constructing the algebra $\breve{A}$ when proving local representability of rings as well as algorithmic solvability of the problem of identity inference.

The next proposition specifies the results of Proposition 3.6 for the structurizable case. It follows from the above considerations.

Statement 5.10. (a) *If* $\mathfrak{M} = \mathrm{Var}(A)$ *is a Kurosh variety then the space* $K(A, D)$ *is representable in any quotient algebra* $A' = A/J$. *Furthermore for the natural morphism of* A' *to the «extended» algebra* $\widehat{A}/\widehat{J}$, *its restriction to* $K(A, D)$ *is an embedding.*

(b) $S_\Lambda\big(F(A)\big) \simeq S_\Lambda\big(F\big(K_1(A, D)\big)\big)$ *where* $F(B)$ *is the space of values of polynomials on* B.

The space $K(A, D)$ is naturally isomorphic to the space of letter alternators corresponding to the diagram D.

Statement 5.11. *To any closed* T*-ideal* M *there corresponds a closed* T*-ideal* $K(M)$ *of* $K_1(A, D)$, *which is a subspace of* $K(A, D)$. *This correspondence preserves inclusion and strict inclusion* (*of verbal ideals of the corresponding algebras but not of* T*-ideals*).

The above proposition means that as regards testing ideals for some extremal properties, the algebra A is equivalent to $K(A, D)$. Note that the results of Subsection 3.3 (see Propositions 3.5 and 3.6) mean that the space of polynomials in a relatively free algebra with extremal alternators is representable. On the other hand, Lemma 5.3 on letter thinning enables to return to these spaces when necessary. We will make use of this.

Доказательство. The first part of the proposition is obvious, so it suffices to show that strict inclusion is preserved. The assertion in question means that if the ranges of polynomials from two T-ideals M_1 and M_2 in $K_1(A, D)$ agree then the same is true for their ranges in A.

If we pass from the language of supports to the purely combinatorial language, coincidence of ranges in $K_1(A, D)$ means coincidence of ranges of letter alternators corresponding to the set of variables Λ. But this is already proven (see propositions 5.2 and 5.5). □

Замечание. The above proposition implies isomorphy of lattices of verbal ideals in the algebra A and in its «extension» $K_1(A, D)$. Namely, if $\Gamma_i \subseteq \bigcap_j \mathrm{T}\left(A(j)\right) \cap J_{d(A)}$ then $\big(b(\Gamma_i, A), d(\Gamma_i, A)\big) = \big(b(A), d(A)\big)$, and coincidence of ranges in $K(A, D)$ (that is, of extremal letter alternators of elements from Γ_i) implies coincidence of ranges of corresponding polynomials in A. This does not automatically imply coincidence in rings of polynomials with coefficients from these algebras, in particular, in relatively free algebras, algebras of generic elements.

Proposition 5.11 and lemma 5.3 on letter thinning imply

Corollary 5.10. *Let M be a closed T-ideal in $S_\Lambda(A)$. Then the ideal $S_\Lambda(A)$ contains no obstacle for representability of M.*

Statement 5.12. *Suppose $b(A,\Gamma) = b(a)$ and $d(A,\Gamma) < d(A)$. Then $\Gamma \cap S_\Lambda = 0$ in A (as well as in the extended algebra $\widehat{A}$).*

Доказательство. It is obvious that the intersection in $K(A,D)$ is zero. This implies that the intersection in $\widehat{A}$ of the maximal closed T-ideal $\Gamma^0 \subseteq \Gamma$ and the ideal S_Λ is zero as well. Pass to the quotient by Γ^0 and complete the proof by decreasing induction. □

The results of the present part together with second Kemer lemma for the relative case imply

Lemma 5.6 (on the canonical support)**.** *Suppose $\mathfrak{M} = \mathrm{Var}(A)$ is a convenient variety, $\overline{A} = A/J(A)$, $\{A_i\}$ is a set of quotients of $\overline{A}$ by various sets of prime components. Then $\mathfrak{M}$ is generated by a finite set of algebras of the form $K(A_i, D_i)$ where D_i is a rectangular diagram consisting of $d(D_i)$ columns of length $\dim(A_i) + 1$.*

If $b(A,\Gamma) = b_0$, $d(A,\Gamma) = d_0$ then Γ does not vanish in some of algebras $K(A_i, D_i)$ where $\dim(A_i) = b_0$ and $d(D_i) = d_0$.

We can construct a somewhat different convenient support such that external traces agree with internal ones. Specifically, let A be a relatively free representable algebra from a convenient variety $\mathfrak{M}$. Consider the extended algebra

$$A' = A[\mathrm{Tr}]\Big/ \mathrm{id}\,\Big(S_\Lambda(A)\cdot\big(\delta(a) - \mathrm{Tr}(a)\big)\Big), \quad c(A) = d(\Lambda).$$

Then $\mathrm{Var}(A') = \mathrm{Var}(A)$ and moreover the quotient algebra $A^1 = A'/S_\Lambda(A)$ has smaller complexity parameters: $\big(b(A^1), d(A^1)\big) < \big(b(A), d(A)\big)$, and representability of A^1 implies representability of the relatively free algebra $A^{(1)}$ from $\mathrm{Var}(A^1)$. Starting with $A^{(1)}$, we construct the algebras A^2 and $A^{(2)}$ and so on. The required algebra A_* is $\oplus A^k$.

Statement 5.13. *In the algebra A_* external traces agree with internal ones.*

5.3. *Completion of the proof of Theorem* 1.1. Let A be a free representable algebra from a good structurizable variety $\mathfrak{M}$. Representability of $A/S_\Lambda(A)$ follows easily from second Kemer lemma. The algebra A may be interpreted as the algebra of generic elements in a finite-dimensional algebra with nilpotency degree of the radical equal to $d(A)+1$. Then in the space $S_\Lambda(A)$ external traces agree with internal ones, and the space itself is closed under internal traces. Hence the ideal $S_\Lambda(A)$ is representable.

We will show how this implies Theorem 1.1. First of all, representability of closed ideals is obvious because extension by operators of the form $\delta(a)$ results in a Noetherian-type algebra.

Let Γ be a T-ideal. If $\Gamma \cap S_\Lambda(A) = 0$ then we can pass to the quotient $A/S_\Lambda(A)$ and use decreasing induction on complexity parameters.

If $J = \Gamma \cap S_\Lambda(A) \neq 0$ then there exists a nonzero closed T-ideal $J^0 \subseteq J$. Since it is representable, we may use decreasing induction to complete the proof under the assumption of finite-basedness for T-ideals. Thus we have managed to deduce local representability from local Specht property.

To avoid local Specht property (and moreover to prove this property), we will argue as follows. Take for J^0 the *maximal closed T-ideal included in J*. Consider the quotient $A' = A/J^0$ and define the spaces $S'_\Lambda(A')$ in it as well as J. Decreasing induction shows that it suffices to consider the case $b(A') = b(A)$ and $d(A') = d(A)$.

The set Λ' corresponds to the Kemer diagram D' of A', which may differ from D by greater number of small columns (for getting to the required space when applying the operator S'_Λ).

Замечание. If we take the quotient of A' by the elements of δ-torsion $\big($that is, the elements with nontrivial annihilator in the ring generated by elements of the form $\delta(a)\big)$ then the Kemer diagram for the resulting algebra A'' is included in D. On the other hand, the torsion ideal $\mathrm{Tor}_\delta(A')$ satisfies $b\big(A', \mathrm{Tor}_\delta(A')\big) < b(A)$. Hence instead of enlarging the Kemer diagram we may use the algebra A''.

It suffices to show that in A' the meet of the projection $\pi(\Gamma)$ with $S'_\Lambda(A')$ is zero; otherwise this meet includes a nonzero closed T-ideal the preimage of which is a closed T-ideal sstrictly including the preimage of 0, that is, J^0. But this contradicts maximality of J^0. Thus

$$\big(b(A/J^0, J), d(A/J^0, J)\big) < \big(b(A, J), d(A, J)\big) = \big(b(A), d(A)\big).$$

The proof of the basic theorem $\big($under the assumption of representability of $A/S_\Lambda(A)\big)$ is complete.

On the other hand, representability of $A/S_\Lambda(A)$ follows from closedness of the ideal $S_\Lambda(A)$.

Замечание. Argument related to refinement and closedness of T-ideals as well as to spaces $K_{\mathfrak{M}}(A, D)$ works for the multilinear case in positive characteristic.

Previously, considering closed ideals and passing to quotients by them we reduced the situation to the case when either $b(A, \Gamma) < b(A)$ or $b(A, \Gamma) = b(A)$ and $d(A, \Gamma) < d(A)$. Now we examine this situation in more detail.

So we have $S_\Lambda(\Gamma) = 0$. Furthermore by the induction argument the algebra $A' = A/\big(S_\Lambda(A) + \Gamma\big)$ $\big($here $b(A') \leq b(A)$ and if $b(A') = b(A)$ then $d(A') < d(A)\big)$

may be assumed to be representable. Thus we have a morphism h_1 of the quotient A/Γ to a representable algebra $S_\Lambda(A/\Gamma)$.

Hence to prove representability of A/Γ it suffices to construct another morphism of this algebra h_2 to some representable algebra, such that its kernel is disjoint with $S_\Lambda(A/\Gamma)$. Then the sum of morphisms $h_1 \oplus h_2$ is the required embedding.

Let H be a sum of T-ideals Γ_i such that either $b(A, \Gamma_i) < b(A)$ or $b(A, \Gamma_i) = b(A)$ and $d(A, \Gamma_i) < d(A)$. Since $H \cap S_\Lambda(A) = 0$ and $H \supseteq \Gamma$ $\big($because $S_\Lambda(\Gamma) = 0\big)$, it suffices to establish representability of A/H. Hence we may assume $H = \Gamma$.

Furthermore the ideals H_r are closed and hence representable. Consider the quotient A_1 by the sum of ideals

$$\bigoplus_{r<d(A)} H_r.$$

For this new algebra we have $b(A_1, H) < b(A_1) = b(A)$.

Now by the non-associative analogue of first Kemer lemma, in the algebra A_1 the range of polynomials from H is disjoint with the ideal generated by all prime components R_i of A_1. Consider morphisms $\varphi_i : A_1 \to A_1/R_i$ and $\varphi = \bigoplus_i \varphi_i$. The kernel of the latter morphism is disjoint with the range of polynomials from H, and it determines an embedding of the corresponding spaces of non-commutative polynomials.

Let Q be the space in A_1 generated by homogeneous elements z_i such that each of them generates an ideal disjoint with the R-component of A_1 generated by monomials including elements of each R_i. Then Q and R are ideals in A_1, and A_1 embeds into the sum $A_1/Q \oplus A_1/R$. Hence in view of the non-associative analogue of first Kemer lemma the range of H lies in Q, and the range of $S_\Lambda(A_1)$ lies in R. Now it is clear that if the ring of polynomials with coefficients in A_1 is extended by traces of operators then the extended spaces H and $S_\Lambda(\widehat{A}_1)$ are disjoint.

The proof of local finite-basedness and local representability for convenient structurizable varieties over a field of zero characteristic as well as of rationality of their Hilbert series is complete. □

Замечание. We may extend algebras by systems of absolutely anticommuting variables (any monomial including two identical new variables is zero), consider the canonical algebraical representation of appropriate order for the resulting algebras and deal with them.

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