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ASYMPTOTIC EFFICIENCY OF THE MOMENT METHOD FOR ESTIMATING THE COMPOUND POISSON DISTRIBUTION

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Point estimation of distribution parameters is considered for a three-parameter compound Poisson process. Formulas are derived for moment method estimation, including estimate biases and the covariance matrix. Asymptotic efficiency of the parameter estimates with a series informant is examined. The efficiency of moment method estimation is computed and analyzed for typical parameter values.

Introduction

We investigate numerically the asymptotic efficiency of parameter estimates of the compound Poisson distribution. Let us briefly enumerate some of the previously established properties of compound Poisson [1, 2, 3, p. 15]. The probability generating function is written in the form

$$P(z) = \exp\left\{\sum_{\nu=1}^{k}\lambda_\nu\left((\bar{\varepsilon}+\varepsilon z)^\nu - 1\right)\right\} = \exp\left\{\sum_{\nu=1}^{k}\theta_\nu(z^\nu - 1)\right\},$$

$\lambda_1, \dots, \lambda_k > 0$, $0 < \varepsilon \le 1$ is given, $\bar{\varepsilon} = 1 - \varepsilon$,

$$\theta_\nu = \varepsilon^\nu \sum_{\mu=\nu}^{k} C_\mu^\nu \bar{\varepsilon}^{\mu-\nu}\lambda_\mu, \quad \left(\lambda_\nu = \sum_{\mu=\nu}^{k} C_\mu^\nu \varepsilon^{-\mu}(-\bar{\varepsilon})^{\mu-\nu}\theta_\mu\right), \quad \nu = \overline{1,\ k}, \tag{1}$$

or in matrix form

$$\boldsymbol{\theta} = \mathbf{C}\boldsymbol{\lambda}, \quad \mathbf{C} = \mathbf{C}^{kxk} = \left\|c_{\nu\mu}\right\| = \left\|\varepsilon^\nu C_\mu^\nu \bar{\varepsilon}^{\mu-\nu}\right\|.$$

The distribution function has the form

$$p_n = p_0 \sum \prod_{\nu=1}^{k} \frac{\theta_\nu^{i_\nu}}{i_\nu!}, \quad n = 0, 1, \dots, \tag{2}$$

where

$$p_0 = \exp\left\{-\sum_{\nu=1}^{k}\theta_\nu\right\} = \exp\left\{\sum_{\nu=1}^{k}\lambda_\nu(\bar{\varepsilon}^\nu - 1)\right\}$$

and the sum is over nonnegative integer solutions $i_1, \dots, i_k \ge 0$ of the equation $\sum_{\alpha=1}^{k}\alpha i_\alpha = n$.

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[1] Deceased.

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Below we use the following facts concerning the distribution (2) [1–3]:

– representations of the derivatives with respect to the parameters θ_ν, λ_ν, $\nu = \overline{1,\ k}$:

$$\frac{\partial p_n}{\partial \theta_\nu} = \sum_{j=0}^{\min\{1,[n/\nu]\}} (-1)^{1-j} p_{n-\nu j}, \qquad \frac{\partial p_n}{\partial \lambda_\nu} = \sum_{j=0}^{\min\{\nu,n\}} C_\nu^j \varepsilon^j \bar{\varepsilon}^{\nu-j} p_{n-j} - p_n;$$

– recurrences for p_n, $n = 1, 2, \ldots$:

$$np_n = \sum_{\nu=1}^{\min\{k,n\}} \nu\theta_\nu p_{n-\nu} = \sum_{\mu=1}^{k} \lambda_\mu \sum_{\nu=1}^{\min\{\mu,n\}} \nu C_\mu^\nu \bar{\varepsilon}^{\mu-\nu} p_{n-\nu};$$

– expressions for the cumulants

$$\kappa_r = \sum_{\nu=1}^{k} \nu^r \theta_\nu = \sum_{\mu=1}^{k} \lambda_\mu \sum_{\nu=1}^{\mu} \nu^r C_\mu^\nu \varepsilon^\nu \bar{\varepsilon}^{\mu-\nu}, \qquad r = 1, 2, \ldots, \tag{3}$$

including the initial moments and the five lowest central moments

$$\kappa_1 = \alpha_1 = \theta_1 + 2\theta_2 + 3\theta_3 = \varepsilon\lambda_1 + 2\varepsilon\lambda_2 + 3\varepsilon\lambda_3,$$

$$\kappa_2 = \mu_2 = \theta_1 + 4\theta_2 + 9\theta_3 = \varepsilon\lambda_1 + 2\varepsilon(1+\varepsilon)\lambda_2 + 3\varepsilon(1+2\varepsilon)\lambda_3,$$

$$\kappa_3 = \mu_3 = \theta_1 + 8\theta_2 + 27\theta_3 = \varepsilon\lambda_1 + 2\varepsilon(1+3\varepsilon)\lambda_2 + 3\varepsilon\left(1 + 6\varepsilon + 2\varepsilon^2\right)\lambda_3,$$

$$\kappa_4 = \theta_1 + 16\theta_2 + 81\theta_3 = \mu_4 - 3\mu_2^2,$$

$$\kappa_5 = \theta_1 + 32\theta_2 + 243\theta_3 = \mu_5 - 10\mu_2\mu_3,$$

$$\kappa_6 = \theta_1 + 64\theta_2 + 729\theta_3 = \mu_6 - 15\mu_4\mu_2 - 10\mu_3^2 + 30\mu_2^3;$$

– the elements of the Fisher information matrix [4]

$$\mathbf{B} = \mathbf{B}_\lambda^{kxk} = \left\| E_\lambda \left(\frac{\partial \ln L}{\partial \lambda_\nu}\, x\, \frac{\partial \ln L}{\partial \lambda_\mu} \right) \right\| = \left\| b_{\nu\mu} \right\|, \qquad \nu,\ \mu = \overline{1,\ k},$$

$$b_{\nu\mu} = -E_\lambda \frac{\partial^2 \ln L}{\partial \lambda_\nu \partial \lambda_\mu} = N \sum_{n=0}^{\infty} p_n^{-1} \frac{\partial p_n}{\partial \lambda_\nu} \frac{\partial p_n}{\partial \lambda_\mu} = N i_{\nu\mu},$$

$$i_{\nu\mu} = \bar{\varepsilon}^{\,\nu+\mu} \sum_{i=0}^{\nu} \sum_{j=0}^{\mu} C_\nu^i C_\mu^j \left(\frac{\varepsilon}{\bar{\varepsilon}}\right)^{i+j} \sum_{n=\max\{i,j\}}^{\infty} \frac{p_{n-i} p_{n-j}}{p_n} - 1 = \bar{\varepsilon}^{\,\nu+\mu} \sum_{i=0}^{\nu} \sum_{j=0}^{\mu} C_\nu^i C_\mu^j \left(\frac{\varepsilon}{\bar{\varepsilon}}\right)^{i+j} \bar{i}_{ij},$$

where $L = \prod_{i=1}^{N} p_{n_i}$ is the likelihood function,

$$N\bar{i}_{ij} = N\left(\sum_{n=\max\{i,j\}}^{\infty} \frac{p_{n-i}p_{n-j}}{p_n} - 1\right)$$

are the elements of the information matrix $\mathbf{B}_\theta^{kxk} = N\bar{\mathbf{I}}_\theta^{kxk}$ in the system of parameters θ_ν, $\mathbf{B}_\lambda^{kxk} = N\mathbf{I}_\lambda^{kxk}$;

– the determinant of the matrix $\mathbf{B}_\lambda^{kxk}$ for $k \ge 2$

$$|\mathbf{B}| = \det \mathbf{B}_\lambda^{kxk} = \frac{N^k}{k^2\lambda_k^2}\begin{vmatrix} & & \mathbf{I}_\lambda^{k-1xk-1} & & \begin{matrix}\varepsilon \\ 2\varepsilon \\ \dots \\ (k-1)\varepsilon\end{matrix} \\ \varepsilon & 2\varepsilon & \dots & (k-1)\varepsilon & \mu_2 \end{vmatrix}$$

$$= \varepsilon^{k(k+1)} \det \mathbf{B}_\theta^{kxk} = \frac{N^k\varepsilon^{k(k-1)}}{k^2\lambda_k^2}\begin{vmatrix} & & \bar{\mathbf{I}}_\theta^{k-1xk-1} & & \begin{matrix}1 \\ 2 \\ \dots \\ k-1\end{matrix} \\ 1 & 2 & \dots & k-1 & \mu_2 \end{vmatrix},$$

where for $k = 3$ we have

$$|\mathbf{B}| = \frac{N^3\varepsilon^6}{9\lambda_3^2}\left\{\mu_2(\bar{i}_{11}\bar{i}_{22} - \bar{i}_{12}^2) - 4\bar{i}_{11} - \bar{i}_{22} + 4\bar{i}_{12}\right\},$$

where

$$\bar{i}_{11} = \sum_{n=1}^{\infty}\frac{p_{n-1}^2}{p_n} - 1, \quad \bar{i}_{22} = \sum_{n=2}^{\infty}\frac{p_{n-2}^2}{p_n} - 1, \quad \bar{i}_{12} = \sum_{n=2}^{\infty}\frac{p_{n-1}p_{n-2}}{p_n} - 1.$$

In what follows we investigate the asymptotic efficiency of the consistent estimates $\tilde{\boldsymbol{\lambda}}$ defined as [6, p. 389]

$$e_0(\tilde{\boldsymbol{\lambda}}) = e_0\left(\tilde{\boldsymbol{\lambda}}\middle|\boldsymbol{\lambda}\right) = \left(\lim_{N\to\infty}|\mathbf{B}|\left|\mathbf{V}(\tilde{\boldsymbol{\lambda}})\right|\right)^{-1},$$

where $\mathbf{V}(\tilde{\boldsymbol{\lambda}}) = \left\|\operatorname{cov}(\tilde{\lambda}_\nu, \tilde{\lambda}_\mu)\right\|$ is the covariance matrix of the estimates. As an alternative to the maximum likelihood method [5–7], which produces asymptotically normal and efficient estimates with bias $O(N^{-1})$, we consider the method of moments for estimating the parameters $\boldsymbol{\lambda}$ of distribution (2).

Estimates, Bias, and the Covariance Matrix

For our distribution the estimates $\tilde{\lambda}$ given (3) can be obtained from the linear algebraic system

$$\mathbf{W C \lambda} = \boldsymbol{\kappa},$$

where $\mathbf{W} = \mathbf{W}^{kxk} = \|\mu^{\nu}\|$, $\nu, \mu = \overline{1, k}$ is the Vandermonde matrix, $\boldsymbol{\kappa}$ is the column vector of sample cumulants. Then

$$\mathbf{V}(\tilde{\boldsymbol{\lambda}}) = \mathbf{V}(\mathbf{C}^{-1}\mathbf{W}^{-1}\boldsymbol{\kappa}) = \mathbf{C}^{-1}\mathbf{W}^{-1}\mathbf{V}(\boldsymbol{\kappa})(\mathbf{W}^{-1}\mathbf{C}^{-1})',$$

$$e_0(\tilde{\boldsymbol{\lambda}}) = \frac{|\mathbf{W}|^2|\mathbf{C}|^2}{|\mathbf{B}||\mathbf{V}(\boldsymbol{\kappa})|}.$$

In the important particular case $k = 3$ the moment method system takes the form

$$\begin{cases} \varepsilon\lambda_1 + 2\varepsilon\lambda_2 + 3\varepsilon\lambda_3 = a_1, \\ \varepsilon\lambda_1 + 2\varepsilon(1+\varepsilon)\lambda_2 + 3\varepsilon(1+2\varepsilon)\lambda_3 = m_2, \\ \varepsilon\lambda_1 + 2\varepsilon(1+3\varepsilon)\lambda_2 + 3\varepsilon(1+6\varepsilon+2\varepsilon^2)\lambda_3 = m_3, \end{cases} \tag{4}$$

where a_1 is the sample mean, m_2, m_3 are the sample second and third central moments. The moment method estimates are

$$\begin{cases} \tilde{\lambda}_1 = \dfrac{2(\varepsilon^2+\varepsilon+1)a_1 - (2\varepsilon+3)m_2 + m_3}{2\varepsilon^3}, \\ \tilde{\lambda}_2 = \dfrac{-(2+\varepsilon)a_1 - (\varepsilon+3)m_2 - m_3}{2\varepsilon^3}, \\ \tilde{\lambda}_3 = \dfrac{2a_1 - 3m_2 + m_3}{6\varepsilon^3}, \end{cases}$$

and the restrictions $\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3 > 0$ for $0 < \varepsilon \le 1$ produce the following solvability conditions for the system:

$$1 < \frac{m_2}{a_1} < 1 + 2\varepsilon, \quad 1 < \frac{m_3}{m_2} < \frac{1+6\varepsilon+2\varepsilon^2}{1+2\varepsilon}, \quad 1 < \frac{m_3}{a_1} < 1 + 6\varepsilon + 2\varepsilon^2, \tag{5}$$

where any of the three restrictions is a consequence of the other two.

Noting that [7]

$$E(a_1) = \alpha_1, \quad E(m_2) = \mu_2 - \frac{\mu_2}{N}, \quad E(m_3) = \mu_3 - \frac{3\mu_3}{N} + O(N^{-2}),$$

$$D(a_1) = \frac{\mu_2}{N}, \qquad D(m_2) = \frac{\mu_4 - \mu_2^2}{N} + O(N^{-2}),$$

$$D(m_3) = \frac{\mu_6 - 6\mu_4\mu_2 - \mu_3^2 + 9\mu_2^3}{N} + O(N^{-2}),$$

$$\operatorname{cov}(a_1, m_2) = \frac{\mu_3}{N} + O(N^{-2}),$$

$$\operatorname{cov}(a_1, m_3) = \frac{\mu_4 - 3\mu_2^2}{N} + O(N^{-2}),$$

$$\operatorname{cov}(m_2, m_3) = \frac{\mu_5 - 4\mu_2\mu_3}{N} + O(N^{-2}),$$

and that the variance of the function $y = y(a_1, m_i, m_j)$ of sample moments is

$$D(y) = \left(\frac{\partial y}{\partial a_1}\right)^2 D(a_1) + \left(\frac{\partial y}{\partial m_i}\right)^2 D(m_i) + \left(\frac{\partial y}{\partial m_j}\right)^2 D(m_j) + 2\left(\frac{\partial y}{\partial a_1}\right)\left(\frac{\partial y}{\partial m_i}\right)\operatorname{cov}(a_1, m_i)$$

$$+ 2\left(\frac{\partial y}{\partial a_1}\right)\left(\frac{\partial y}{\partial m_j}\right)\operatorname{cov}(a_1, m_j) + 2\left(\frac{\partial y}{\partial m_i}\right)\left(\frac{\partial y}{\partial m_j}\right)\operatorname{cov}(m_i, m_j)$$

with accuracy $O(N^{-3/2})$, where the derivatives are evaluated at $a_1 = \alpha_1$, $m_i = \mu_i$, $m_j = \mu_j$, we obtain with accuracy $O(N^{-1})$

$$\varepsilon NE(\tilde{\lambda}_1 - \lambda_1) = (\lambda_1 + 2(\varepsilon - 2)\lambda_2 - 3(\varepsilon + 5)\lambda_3),$$

$$2\varepsilon NE(\tilde{\lambda}_2 - \lambda_2) = -(\lambda_1 + 2(5 - \varepsilon)\lambda_2 - 3(4\varepsilon + 11)\lambda_3),$$

$$\varepsilon NE(\tilde{\lambda}_3 - \lambda_3) = -(2\lambda_2 + 3(\varepsilon + 2)\lambda_3)$$

and with accuracy $O(N^{-1/2})$

$$\varepsilon^6 ND(\tilde{\lambda}_1) = \left(\varepsilon^4\theta_1 + 4\varepsilon^2\bar{\varepsilon}^2\theta_2 + 9\bar{\varepsilon}^4\theta_3 + 2\varepsilon(3 + \varepsilon)\mu_2^2 - 6\varepsilon\mu_2\mu_3 + 9A\right),$$

$$\varepsilon^6 ND(\tilde{\lambda}_2) = \left(\varepsilon^2\theta_2 + 9\bar{\varepsilon}^2\theta_3 + \frac{1}{2}\varepsilon(6 + \varepsilon)\mu_2^2 - 3\varepsilon\mu_2\mu_3 + 9A\right),$$

$$\varepsilon^6 ND(\tilde{\lambda}_3) = (\theta_3 + A),$$

where

$$A = (2\theta_2 + 9\theta_3)^2 + (\theta_2 + 9\theta_3)\mu_2 + \frac{1}{6}\mu_2^3.$$

In (4) multiplying the first equality by $(1+2\varepsilon)$ and subtracting the second equality from the first, we obtain for the estimates $\tilde{\lambda}_1$, $\tilde{\lambda}_2$ the equality

$$2\varepsilon^2(\tilde{\lambda}_1 + \tilde{\lambda}_2) = (1+2\varepsilon)a_1 - m_2.$$

This equality combined with the formula for the variance of the function of sample moments and (11) gives with accuracy $O(N^{-1/2})$

$$\varepsilon^6 N\,\mathrm{Cov}(\tilde{\lambda}_1, \tilde{\lambda}_2) = -\left(2\varepsilon^2\bar{\varepsilon}\theta_2 + 9\bar{\varepsilon}^3\theta_3 + \frac{9}{2}\varepsilon\mu_2(\mu_2 - \mu_3) + \varepsilon^2\mu_2^2 + 9A\right).$$

We similarly obtain from (4)

$$\varepsilon^2(\tilde{\lambda}_1 - 3\tilde{\lambda}_3) = (1+\varepsilon)a_1 - m_2, \qquad 2\varepsilon^2(\tilde{\lambda}_2 + 3\tilde{\lambda}_3) = m_2 - a_1$$

and thus end up with the equalities

$$\varepsilon^6 N\,\mathrm{Cov}(\tilde{\lambda}_1, \tilde{\lambda}_3) = \left(3\bar{\varepsilon}^2\theta_3 + \varepsilon\mu_2^2 - \varepsilon\mu_2\mu_3 + 3A\right),$$

$$2\varepsilon^6 N\,\mathrm{Cov}(\tilde{\lambda}_2, \tilde{\lambda}_3) = -\left(6\bar{\varepsilon}\theta_3 + \varepsilon\mu_2^2 - \varepsilon\mu_2\mu_3 + 6A\right).$$

For $\varepsilon = 1$ we obtain the same expressions for the estimates $\tilde{\lambda}_1$, $\tilde{\lambda}_2$, $\tilde{\lambda}_3$, their biases, and the covariance matrix elements as in [8].

Estimate Efficiency

For the asymptotic efficiency of the moment method estimates we obtain the equality

$$e_0(\tilde{\boldsymbol{\lambda}}) = \frac{9\lambda_3^2\varepsilon^{12}}{\left\{\mu_2\left(\bar{i}_{11}\bar{i}_{22} - \bar{i}_{12}^2\right) - 4\bar{i}_{11} - \bar{i}_{22} + 4\bar{i}_{12}\right\}|\boldsymbol{\Delta}|},$$

where

$$|\boldsymbol{\Delta}| = \begin{vmatrix} \varepsilon^6 ND(\tilde{\lambda}_1) & \varepsilon^6 N\,\mathrm{Cov}(\tilde{\lambda}_1, \tilde{\lambda}_2) & \varepsilon^6 N\,\mathrm{Cov}(\tilde{\lambda}_1, \tilde{\lambda}_3) \\ \varepsilon^6 N\,\mathrm{Cov}(\tilde{\lambda}_1, \tilde{\lambda}_2) & \varepsilon^6 ND(\tilde{\lambda}_2) & \varepsilon^6 N\,\mathrm{Cov}(\tilde{\lambda}_2, \tilde{\lambda}_3) \\ \varepsilon^6 N\,\mathrm{Cov}(\tilde{\lambda}_1, \tilde{\lambda}_3) & \varepsilon^6 N\,\mathrm{Cov}(\tilde{\lambda}_2, \tilde{\lambda}_3) & \varepsilon^6 ND(\tilde{\lambda}_3) \end{vmatrix}.$$

Its evaluation thus reduces to computing the sums of three numerical series. The accuracy of approximating the sums of series with partial sums can be estimated either empirically (the result is satisfactory if repeatedly increasing the number of terms in the partial sums leads to small changes in the computed value) or by bounding the residual term through a comparison of the numerical series. In fact, we have the inequalities

$$\sum_{n=1}^{\infty}\frac{p_{n-1}^2}{p_n} \le \frac{1}{\theta_1}\sum_{n=1}^{\infty} np_{n-1} = \frac{\alpha_1+1}{\theta_1}, \qquad \sum_{n=2}^{\infty}\frac{p_{n-2}^2}{p_n} \le \frac{1}{2\theta_2}\sum_{n=2}^{\infty} np_{n-2} = \frac{\alpha_1+2}{2\theta_2},$$

$$\sum_{n=2}^{\infty}\frac{p_{n-1}p_{n-2}}{p_n} \le \frac{1}{\theta_1}\sum_{n=2}^{\infty} np_{n-2} = \frac{\alpha_1+2}{\theta_1},$$

which give bounds for the series residuals.

The values of $e_0(\tilde{\boldsymbol{\lambda}})$ as a function of the parameters λ_1, λ_2, λ_3 for various ε are shown in Figs. 1–6 in the form of level lines and surfaces. The sums of the series $\bar{i}_{11}$, $\bar{i}_{12}$, $\bar{i}_{22}$ are evaluated with accuracy to 10^{-20} in their residuals.

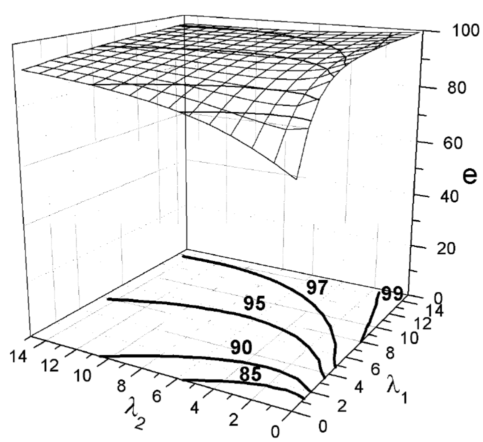

Fig. 1. $\varepsilon = 0.1$, $\lambda_3 = 0.1$.

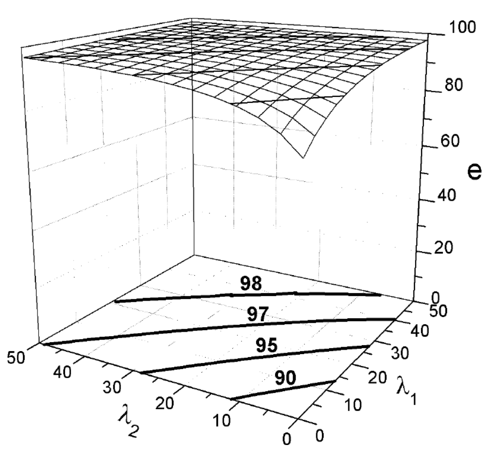

Fig. 2. $\varepsilon = 0.1$, $\lambda_3 = 10$.

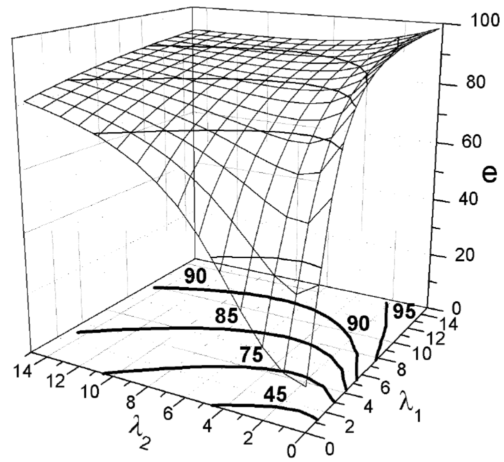

Fig. 3. $\varepsilon = 0.5,\ \lambda_3 = 0.1.$

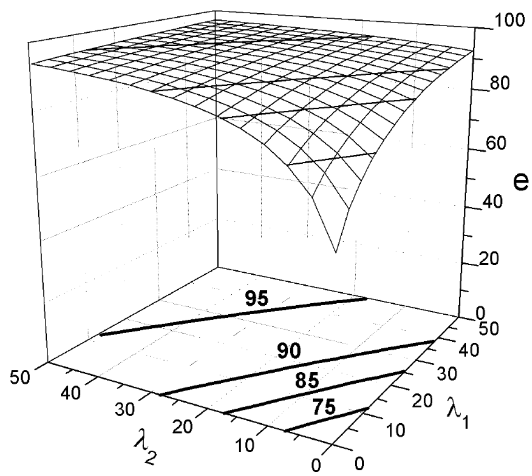

Fig. 4. $\varepsilon = 0.5,\ \lambda_3 = 10.$

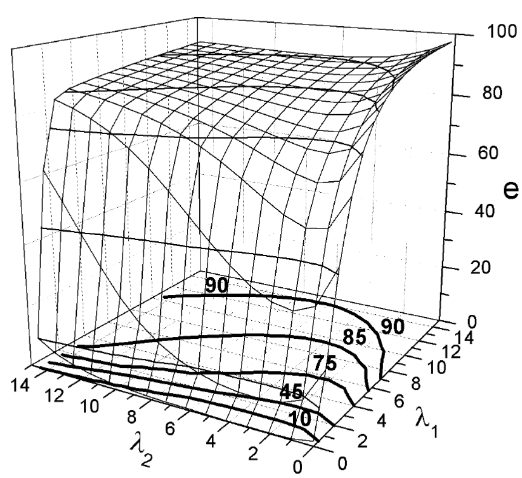

Fig. 5. $\varepsilon = 0.9,\ \lambda_3 = 0.1.$

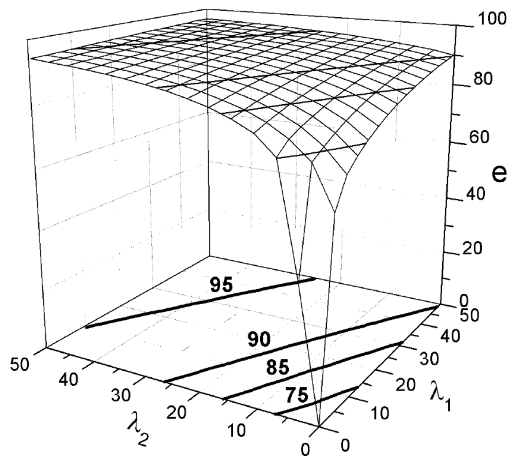

Fig. 6. $\varepsilon = 0.9$, $\lambda_3 = 10$.

We see that for small $\varepsilon = 0.1$ the estimates $\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3$ are inefficient $(e_0 < 0.9)$ only when the values of $\lambda_1, \lambda_2, \lambda_3$ are constrained. In the limiting case $\lambda_3 = 0$ $(k = 2)$ the estimates $\tilde{\lambda}_1$, $\tilde{\lambda}_2$ are asymptotically efficient virtually on the entire set of admissible parameter values [4]. As λ_3 increases $(0 < \lambda_3 < 5-10)$ the low efficiency regions increases to $\{(\lambda_1, \lambda_2): 0 < \lambda_1, \lambda_2 < 13; \lambda_1 = -\lambda_2 + 13\}$ (Figs. 1, 2). It more actively grows in the parameter λ_1 (the surface flattens out). As λ_3 is further increased $(\lambda_3 > 10)$, the efficiency zone $e_0 < 0.9$ shrinks (the surface becomes more horizontal), and for $\lambda_3 > 25$ there is virtually no region of efficiency (the surface is almost totally horizontal). This demonstrates the asymptotic efficiency of the moment method estimates $\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3$.

As ε increases, the low-efficiency region clearly expands while preserving the previously noted features. Thus, in the limiting case $\lambda_3 = 0$ $(k = 2)$ [4] we have the following: for $\varepsilon = 0.5$ this region almost reaches $\{(\lambda_1, \lambda_2): 0 < \lambda_1 < 3, 0 < \lambda_2 < 8\}$; for $\varepsilon = 0.9$ it approximately equals $\{(\lambda_1, \lambda_2): 0 < \lambda_1 < 5, 0 < \lambda_2 < 15\}$; and for $\varepsilon = 1$ [4, 9] it virtually reduces to the strip $\{(\lambda_1, \lambda_2): 0 < \lambda_1 < 4, 0 < \lambda_2\}$. As λ_3 increases $(0 < \lambda_3 < 5-10)$, the zone $e_0 < 0.9$ expands and reaches its maximum for $\lambda_3 = 5$–10: for $\varepsilon = 0.5$ (Figs. 3, 4) it expands approximately to $\{(\lambda_1, \lambda_2): 0 < \lambda_1, \lambda_2 < 30; \lambda_1 = -\lambda_2 + 30\}$; for $\varepsilon = 0.9$, 1.0 (Figs. 5, 6) it expands to $\{(\lambda_1, \lambda_2): 0 < \lambda_1 < 50,\ 0 < \lambda_2 < 25;\ \lambda_1 = -2\lambda_2 + 50\}$. As λ_3 is further increased $(\lambda_3 > 10)$, the efficiency zone $e_0 < 0.9$ shrinks and for large λ_3 it is virtually not observed for $\varepsilon = 0.5$, $\lambda_3 > 43$ and $\varepsilon = 0.9$, $\lambda_3 > 40$. However, for $\varepsilon = 1$, $\lambda_3 > 40$ the low-efficiency region $\{(\lambda_1, \lambda_2): 0 < \lambda_1 < 10,\ 0 < \lambda_2 < 10;\ \lambda_1 = -\lambda_2 + 10\}$ remains unchanged.

Our calculations and comparative analysis provide new insights and revise earlier notions [8] of the efficiency of using the moment method for point estimation of the parameters λ_1, λ_2, λ_3 of distribution (2) for $k = 3$ and various ε.

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