PAPER

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Few-cycle solitons in a quasiequilibrium nanodispersed medium of asymmetric molecules

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Abstract

The nonlinear dynamics of extremely short electromagnetic pulses in a nanodispersed birefringent medium is investigated. The matrix of the medium and chaotically mixed nanogranules form an amorphous photon crystal containing asymmetric molecules of identical chemical composition. The concentrations of molecules in the nanogranules and in the matrix are different. This nanodispersed (discrete) structure of the medium leads to spatial dispersion. The wave equation for the ordinary component of pulses propagating in such a medium is derived under the conditions of the sudden perturbation approximation. This equation generalizes the sine-Gordon and Rabelo–Fokas equations and appears to be integrable in the frameworks of the inverse scattering transformation method if an additional restriction on the parameters characterizing the spatial dispersion and anisotropy of the medium is imposed. This restriction implies that the medium is prepared in the quasiequilibrium state before the pulse effect, when the molecules in the matrix or the ones in the nanogranules are in the excited state. The soliton and breather solutions of the integrable wave equation are investigated.

Keywords: soliton, few-cycle pulses, permanent dipole moment, integrability

(Some figures may appear in colour only in the online journal)

1. Introduction

Theoretical nonlinear optics is one of the sources of equations of mathematical physics, which are of interest from the point of view of their integrability by the inverse scattering transformation method (ISTM) [1–4]. Integrable nonlinear wave equations possess solutions of the solitonic type. Such types of solutions are the most important in nonlinear optics [5] and in other physical applications of the equations [4]. The investigation of different generalizations of soliton solutions of integrable and nonintegrable nonlinear equations has attracted a lot of attention in recent years [6–15].

One of the tendencies of the development of laser physics and nonlinear optics is the generation of electromagnetic pulses with shorter duration. The term 'few-cycle pulses' (FCPs) has been widely approved of. Pulses of this type contain a small number (up to one) of periods of electromagnetic oscillations. It is impossible to apply to them the concept of an envelope for the electromagnetic field, which is exploited commonly for quasi-monochromatic pulses (QMPs). Therefore, mathematical approximations and approaches which are valid for QMPs are not correct in the FCP case. First of all, it concerns the approximation of slowly varying amplitudes and phases. Since the concept of the envelope is not applicable to FCPs, it is necessary to use here the equations for a full electric field.

Different theoretical approaches have been developed for the nonlinear optics of FCPs. Among the basic approximations used to investigate the dynamics of FCPs in nonlinear media, we distinguish the approximation of unidirectional

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propagation [16] and those of sudden perturbations and optical transparency [17–23]. Various nonlinear regimes of propagation of FCPs both in isotropic and in anisotropic media have been revealed with the help of these approximations. The progress of the nonlinear optics of FCPs is summarized in [24–28].

A separate problem is the derivation of the material equations of the response of the medium on an influence of the FCP. It is necessary for this to choose a theoretical model of the medium that is valid in other cases [21]. One of the elementary models is that of a medium consisting of quantum two-level atoms or molecules. Although this model is rough, it describes well certain physically significant cases [17]. This model will be used here.

Anisotropy in the two-level model can be introduced by means of permanent dipole moments (PDMs) of the molecules. Below, we will refer to molecules having PDMs as asymmetric ones. Quantum dots, wells and rods and natural asymmetric quantum objects can be treated as such molecules [29–31].

A PDM appears if the diagonal matrix elements of the dipole moment operator in the basis of the eigenfunctions of the Hamiltonian of a molecule differ from zero. In that case, the electric field of the pulse fulfills two functions simultaneously: it causes transitions between stationary quantum states and dynamically shifts the frequencies of these transitions due to the Stark effect. It is necessary to note that the investigation of the nonlinear dynamics of FCPs in the case of anisotropic models of the medium has led to various systems of wave and material equations integrable by the ISTM [32–44]. These equations generalize the well-known integrable models of nonlinear optics such as the self-induced transparency equations [45], the sine-Gordon (SG) equation [46] and the reduced Maxwell–Bloch equations [16].

The optical axis can be formed in a medium containing asymmetric molecules by the direction of their PDMs. In this case, 'scalar' and 'vector' models of FCP propagation exist. The 'scalar' one is characterized by a single component of the polarization of the electric field E of the FCP. Accordingly, this component causes quantum transitions and shifts of their frequencies. 'Vector' models are characterized by ordinary E_o and extraordinary E_e components of the electric field. In the case of FCPs propagating perpendicularly to the optical axis, the ordinary component excites quantum transitions only, while the extraordinary one causes a dynamic shift of their frequencies [39–43]. A general case of the vector model was considered in [44].

A change in the populations of the quantum levels of asymmetric molecules under an effect of powerful FCPs on the medium modifies its state. This makes the process significantly nonlinear. A short duration of the pulse leads inevitably to the temporal nonlocality (dispersion) of the polarization response of the medium. Also, if the characteristic spatial scale of the FCP is of the same order of magnitude as the scale of structural heterogeneity of the medium, then the spatial dispersion becomes essential. This dispersion takes place, for example, in a medium consisting of evenly mixed microgranules containing active molecules [43, 47]. Such a two-component medium is called microdispersed. It is important to note that a microdispersed structure of the medium is able to influence qualitatively the character of optical dispersion. For example, the optical dispersion of separate microgranules can be normal. At the same time, the dispersion of a system of microgranules can have an abnormal character [47]. It essentially affects the transverse self-focusing or defocusing of the pulse [47].

Let L_g be the characteristic size of a microgranule and l_g be the average distance between microgranules. Assume $L_g \ll l_g$. Taking the duration of the pulse to be $\tau_p \sim 10^{-15}$ s, we find its characteristic spatial scale to be $\lambda \sim c\tau_p \sim 10^{-5}$ cm ~ 100 nm, where c is the speed of light in vacuum. It is seen from this estimation that spatial dispersion becomes essential at $l_g \sim 100$ nm. Thus, the size of the granules is $L_g \sim 10$ nm. For this reason, below we use 'nanogranules' (and, accordingly, 'nanodispersed medium') instead of 'microgranules'.

In our case, it is possible to speak about the amorphous photon crystal. We will assume that such a medium is placed into a constant electric field, due to which the PDMs of asymmetric molecules are built parallel to each other. Thus identical asymmetric molecules are found in the nanogranules and in the main matrix. However, their concentrations are different. For this reason, the values of the constant electric field in the nanogranules and in the matrix are also different. In turn, this causes different Stark shifts of the frequencies. Therefore, the frequencies of the quantum transitions of molecules in the nanogranules and in the matrix differ [48]. It is also possible to prepare a nanodispersed medium in such a way that its main matrix and nanogranules contain different isotopes of the same chemical element. In this case, the isotopic shift of quantum levels can be essential. This shift arises, for example, due to the difference of the masses of the nuclei of isotopes or because of the distinctiveness of the nucleus shell structure [49].

The difference noted above of the frequencies of the same quantum transitions can be used to prepare a medium in different states before the impact of the FCP. For example, one can shift molecules in the main matrix or in nanogranules to the excited state, by adjusting a quasi-monochromatic laser signal resonant with the corresponding quantum transitions. Thus, the medium can be prepared in a quasiequilibrium initial state before the FCP effect.

In the present report, we investigate the solitonic modes of propagation of vector FCPs in a quasiequilibrium nanodispersed medium consisting of asymmetric molecules. In section 2, a self-consistent system of the material and wave equations is derived for the offered physical model of a two-component medium consisting of the two sorts of two-level asymmetric molecules. In section 3, the procedure of exclusion of material variables is carried out using the accepted physical approximations. This allows us to obtain the nonlinear wave equation for the ordinary component of the pulse. The ordinary and extraordinary components of the pulse are connected here by a simple algebraic relation. In section 4, the case of the nonlinear wave equation obtained, which is integrable by the ISTM, is considered. Some solutions of the

solitonic type are also constructed and studied in this section. In section 5, the main conclusions are given and some research prospects are discussed.

2. Self-consistent system of material and wave equations

Let us consider a medium with nanogranules which contain separately two sorts of two-level molecules. Following [43], we will refer to the two-level molecules of the matrix as 1-molecules and to the two-level ones of the granules as 2-molecules. The frequencies of their quantum transitions are equal to ω_1 and ω_2, respectively. Hereafter, quantities related to 1-molecules and 2-molecules are supplied with subscripts 1 and 2 or with superscripts in parentheses (1) and (2), respectively.

We apply the semiclassical approach to describe the interaction of powerful electromagnetic pulses with the matter.

Let FCPs propagate along the z-axis of the Cartesian system of coordinates, which is perpendicular to the optical axis (y-axis) formed by the PDMs of molecules of both types. The ordinary E_o and extraordinary E_e components of the electric field of the pulse are polarized in the case considered of a birefringent medium along the x-axis and y-axis, respectively. Then, the wave equation is written in vector form as

$$\frac{\partial^2 \mathbf{E}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}(\mathbf{P_1} + \mathbf{P_g}), \tag{1}$$

where $\mathbf{P_1}$ and $\mathbf{P_g}$ are the polarization responses of the matrix and the system of granules, respectively.

It is assumed that the sizes of molecules and granules are much smaller than the distance between them. In discrete representation, we then write

$$\mathbf{P}_{1,g} = \sum_j \mathbf{d}_{1,g}(\mathbf{r}_j, t)\delta(\mathbf{r} - \mathbf{r}_j). \tag{2}$$

Here $\mathbf{d}_1(\mathbf{r}_j, t)$ and $\mathbf{d}_g(\mathbf{r}_j, t)$ are the nonstationary dipole moments of the molecules and granules, respectively; $\delta(\mathbf{r} - \mathbf{r}_j)$ is Dirac's delta function; and the summation is carried out over each molecule and each nanogranule.

We emphasize that the values of the nonstationary dipole moments depend on the electric field of the pulse. In particular, these values vary for free atoms and those located in the field of the light pulse. Therefore, the dipole moments in (2) depend on both the time t and position $\mathbf{r}_j$ of the atom (nanogranule).

Obviously, the distances between 1-molecules are much smaller than those between granules containing 2-molecules. We assume also that the average distance l_1 between 1-molecules is much smaller than the characteristic spatial scale λ of the pulse, i.e.

$$\varepsilon_1 = \frac{l_1}{\lambda} \ll 1. \tag{3}$$

This condition is fulfilled well in the case of $l_1 \sim 10^{-7}$ cm. At the same time, it is seen from the estimates given above that the characteristic distance l_g between granules containing 2-molecules satisfies the condition $l_g \gg l_1$. Therefore, we put

$$\varepsilon_g = \left(\frac{l_g}{\lambda}\right)^2 \ll 1. \tag{4}$$

Note that condition (3) imposes a stricter constraint on parameter l_1 than condition (4) on parameter l_g. Indeed, inequality (4) is satisfied if we assume $l_g < \lambda$ rather than $l_g \ll \lambda$ (see the numerical estimates given in the introduction).

Condition (3) allows us to apply to equation (2) the approximation of a continuous medium in the case of 1-molecules. Then

$$\mathbf{P}_1 = \sum_j \mathbf{d}_1(\mathbf{r}_j, t)\delta(\mathbf{r} - \mathbf{r}_j) \approx \sum_l \mathbf{d}_1(\mathbf{r}_l, t)\delta(\mathbf{r} - \mathbf{r}_l)\Delta N_l^{(1)}.$$

Here, we pass from the summation over the separate molecules to that over the clusters. It is assumed also that the nonstationary dipole moments of all 1-molecules in the separate clusters are equal; the lth cluster contains $\Delta N_l^{(1)}$ molecules.

Let $\Delta V_l^{(1)}$ be the volume of the lth cluster. Then, we have

$$\begin{aligned}\mathbf{P}_1 &= \sum_l \mathbf{d}_1(\mathbf{r}_l, t)\delta(\mathbf{r} - \mathbf{r}_l)\frac{\Delta N_l^{(1)}}{\Delta V_l^{(1)}}\Delta V_l^{(1)} \\ &= n_1 \sum_l \mathbf{d}_1(\mathbf{r}_l, t)\delta(\mathbf{r} - \mathbf{r}_l)\Delta V_l^{(1)} \\ &\to n_1 \int \mathbf{d}_1(\mathbf{r}', t)\delta(\mathbf{r} - \mathbf{r}')\mathrm{d}^3\mathbf{r}' = n_1 \mathbf{d}_1(\mathbf{r}, t),\end{aligned}$$

where n_1 is the average concentration of 1-molecules, which is supposed to be constant in the continuous limit. Thus,

$$\mathbf{P}_1 = n_1 \mathbf{d}_1. \tag{5}$$

Taking into account (4), we consider approximately an influence of the discrete character of the medium in the case of 2-molecules in granules by applying the approximation of a quasicontinuous medium. To this aim, we write relation (2) approximately as

$$\mathbf{P}_g \approx \frac{1}{N_n}\sum_{m=1}^{N_n}\sum_j \mathbf{d}_g(-\mathbf{l}_{mj} + \mathbf{r}_m, t)\delta(\mathbf{r} - \mathbf{r}_j), \tag{6}$$

where $\mathbf{r}_m$ is the radius vector of the mth granule, one of the next neighbors of the jth granule; $\mathbf{l}_{mj} = \mathbf{r}_m - \mathbf{r}_j$ is the vector connecting the jth granule with the mth next neighbor; N_n is the average number of such next neighbors; and index j runs over all granules.

Introducing vector $\mathbf{q}_j = \mathbf{r} - \mathbf{r}_j$, we rewrite equation (6) in the following form:

$$\mathbf{P}_g = \frac{1}{N_n}\sum_{m=1}^{N_n}\sum_j \mathbf{d}_g(\mathbf{r} - \mathbf{q}_m + \mathbf{l}_{mj}, t)\delta(\mathbf{q}_j). \tag{7}$$

Considering inequality (4), we expand $\mathbf{d}_g(\mathbf{r} - \mathbf{q}_m + \mathbf{l}_{mj}, t)$ in the Taylor series with respect to $\mathbf{l}_{mj}$ and keep the first three

terms only. Then

$$\mathbf{d}_{\rm g}(\mathbf{r}-\mathbf{q}_m+\mathbf{l}_{mj},\,t)=\mathbf{d}_{\rm g}(\mathbf{r}-\mathbf{q}_m,\,t)+(\mathbf{l}_{mj}\nabla)\mathbf{d}_{\rm g}(\mathbf{r}-\mathbf{q}_m,\,t)+\frac{1}{2}(\mathbf{l}_{mj}\nabla)^2\mathbf{d}_{\rm g}(\mathbf{r}-\mathbf{q}_m,\,t). \tag{8}$$

Substitution of expression (8) into equation (7) gives

$$\mathbf{P}_{\rm g}=\mathbf{P}_{\rm g}^{(1)}+\mathbf{P}_{\rm g}^{(2)}+\mathbf{P}_{\rm g}^{(3)}. \tag{9}$$

Here

$$\mathbf{P}_{\rm g}^{(1)}=\frac{1}{N_n}\sum_j\sum_{m=1}^{N_n}\mathbf{d}_{\rm g}(\mathbf{r}-\mathbf{q}_m,\,t)\delta(\mathbf{q}_j),$$

$$\mathbf{P}_{\rm g}^{(2)}=\frac{1}{N_n}\sum_j\sum_{m=1}^{N_n}(\mathbf{l}_{mj}\nabla)\mathbf{d}_{\rm g}(\mathbf{r}-\mathbf{q}_m,\,t)\delta(\mathbf{q}_j),$$

$$\mathbf{P}_{\rm g}^{(3)}=\frac{1}{2N_n}\sum_j\sum_{m=1}^{N_n}(\mathbf{l}_{mj}\nabla)^2\mathbf{d}_{\rm g}(\mathbf{r}-\mathbf{q}_m,\,t)\delta(\mathbf{q}_j).$$

Since the next neighbors are distributed isotropically around the allocated jth nanogranule with good accuracy, we have in quasicontinuous long-wave approximation (4)

$$\frac{1}{N_n}\sum_{m=1}^{N_n}\mathbf{d}_{\rm g}(\mathbf{r}-\mathbf{q}_m,\,t)\approx\mathbf{d}_{\rm g}(\mathbf{r}-\mathbf{q}_j,\,t)=\mathbf{d}_{\rm g}(\mathbf{r}_j,\,t).$$

Then, as in deriving expression (5), we find

$$\mathbf{P}_{\rm g}^{(1)}=\sum_j\mathbf{d}_{\rm g}(\mathbf{r}_j,\,t)\delta(\mathbf{r}-\mathbf{r}_j)=n_{\rm g}\mathbf{d}_{\rm g}(\mathbf{r},\,t), \tag{10}$$

where $n_{\rm g}$ is the concentration of nanogranules.

In a similar manner, we then write

$$\mathbf{P}_{\rm g}^{(2)}=(\langle\mathbf{l}_{mj}\rangle\nabla)\sum_j\mathbf{d}_{\rm g}(\mathbf{r}-\mathbf{q}_j,\,t)\delta(\mathbf{q}_j),$$

where $\langle\mathbf{l}_{mj}\rangle$ is the average value of vector $\mathbf{l}_{mj}$. Since the medium of the nanogranules is isotropic, we have $\langle\mathbf{l}_{mj}\rangle=0$. Thus

$$\mathbf{P}_{\rm g}^{(2)}=0. \tag{11}$$

In the approximations accepted, we also have

$$\mathbf{P}_{\rm g}^{(3)}=\frac{1}{2}\langle(\mathbf{l}_{mj}\nabla)^2\rangle\sum_j\mathbf{d}_{\rm g}(\mathbf{r}-\mathbf{q}_j,\,t)\delta(\mathbf{q}_j).$$

In the case of an isotropic distribution of the next neighbors, we find

$$\langle(\mathbf{l}_{mj}\nabla)^2\rangle=\frac{1}{3}\langle\mathbf{l}_{mj}^2\rangle\Delta,$$

where Δ is the Laplace operator. It is obvious that $\langle\mathbf{l}_{mj}^2\rangle=l_{\rm g}^2$. Then

$$\mathbf{P}_{\rm g}^{(3)}=\frac{1}{6}n_{\rm g}l_{\rm g}^2\Delta\mathbf{d}_{\rm g}(\mathbf{r},\,t). \tag{12}$$

In the frameworks of the long-wave approximation used here, we have $\mathbf{d}_{\rm g}(\mathbf{r},\,t)=N\mathbf{d}_2(\mathbf{r},\,t)$, where N is the average quantity of molecules in a granule, and $\mathbf{d}_2(\mathbf{r},\,t)$ is the non-stationary dipole moment of 2-molecules. Besides, according to equation (1), electromagnetic pulses are one-dimensional, i.e. all dynamic variables depend only on the coordinate z and time t. Summarizing this and taking into account equations (9)–(12), we obtain

$$\mathbf{P}_{\rm g}=n_{\rm g}N\left(1+\frac{l_{\rm g}^2}{6}\frac{\partial^2}{\partial z^2}\right)\mathbf{d}_2. \tag{13}$$

The density operator $\hat{\rho}^{1,2}$ and the operator of the dipole moment $\hat{\mathbf{m}}$ of the molecules have the following form in the basis of the eigenfunctions of the Hamiltonian $\hat{H}_0^{1,2}$ of free molecules:

$$\hat{\rho}^{1,2}=\begin{pmatrix}\rho_{\rm bb}^{1,2} & \rho_{\rm ba}{}^{1,2}\\ \rho_{\rm ab}^{1,2} & \rho_{\rm aa}^{1,2}\end{pmatrix},\quad \hat{\mathbf{m}}=\begin{pmatrix}\mathbf{m}_{\rm bb} & \mathbf{m}_{\rm ba}\\ \mathbf{m}_{\rm ab} & \mathbf{m}_{\rm aa}\end{pmatrix}. \tag{14}$$

Here, diagonal elements $\mathbf{m}_{\rm aa}$ and $\mathbf{m}_{\rm bb}$ form the PDMs of molecules, and nondiagonal elements $\mathbf{m}_{\rm ab}=\mathbf{m}_{\rm ba}^*$ describe the transitions between the quantum levels; indices a and b correspond to the lower and upper quantum levels, respectively.

Below, it is supposed for simplicity that the nondiagonal elements of the dipole moment operator are real: $\mathbf{m}_{\rm ab}=\mathbf{m}_{\rm ba}=\mathbf{m}$. Also, distinctions of the operator of the dipole moment for both sorts of molecules are neglected.

Using the relations in (14), we obtain

$$\mathbf{d}_{1,2}={\rm Sp}(\hat{\rho}^{1,2}\hat{\mathbf{m}})=\frac{\mathbf{m}_{\rm aa}+\mathbf{m}_{\rm bb}}{2}+\mathbf{D}W_{1,2}+2\mathbf{m}U_{1,2},$$

where $\mathbf{D}=\mathbf{m}_{\rm bb}-\mathbf{m}_{\rm aa}$ is the PDM of the transition, $W_{1,2}=\frac{\rho_{\rm bb}^{1,2}-\rho_{\rm aa}^{1,2}}{2}$ is the inversion of the populations of the quantum levels, and $U_{1,2}=\frac{\rho_{\rm ab}^{1,2}+\rho_{\rm ba}^{1,2}}{2}$.

Taking into account that $\mathbf{m}_{\rm bb}$ and $\mathbf{m}_{\rm aa}$ are parallel to the y-axis, while $\mathbf{m}$ is parallel to the x-axis [39–43], we write the operator of interaction of the FCP field with molecules in the electro-dipole approximation as

$$\hat{V}=-\hat{\mathbf{m}}\mathbf{E}=-\begin{pmatrix}m_{\rm bb}E_{\rm e} & mE_{\rm o}\\ mE_{\rm o} & m_{\rm aa}E_{\rm e}\end{pmatrix},$$

where $m_{\rm bb}$, $m_{\rm aa}$ and m are the projections of $\mathbf{m}_{\rm bb}$, $\mathbf{m}_{\rm aa}$ and $\mathbf{m}$ on the corresponding axes.

Using the von Neumann equation for the density operator gives us the following system of material equations [43]:

$$\begin{aligned}\frac{\partial U_{1,2}}{\partial t}&=-(\omega_{1,2}-\Omega_{\rm e})V_{1,2},\\ \frac{\partial V_{1,2}}{\partial t}&=(\omega_{1,2}-\Omega_{\rm e})U_{1,2}+\Omega_{\rm o}W_{1,2},\\ \frac{\partial W_{1,2}}{\partial t}&=-\Omega_{\rm o}V_{1,2}.\end{aligned} \tag{15}$$

Here

$$V_{1,2}=\frac{\rho_{\rm ab}^{1,2}-\rho_{\rm ba}^{1,2}}{2i},\quad \Omega_{\rm o}=\frac{2mE_{\rm o}}{\hbar},\quad \Omega_{\rm e}=\frac{DE_{\rm e}}{\hbar},$$

where D is the projection of $\mathbf{D}$ on the y-axis, and $\hbar$ is Planck's constant.

As seen from equation (15), the ordinary component of the FCP causes quantum transitions in the accepted geometry of propagation, whereas the extraordinary one shifts dynamically the frequencies of these transitions.

Assume that the asymmetry of molecules is weak:

$$\varepsilon_2 = \frac{\Omega_e}{\omega_{1,2}} \ll 1. \tag{16}$$

The substitution of expressions (5) and (13) into equation (1) and scalar multiplications of the resulting equation on $2\mathbf{m}/\hbar$ at first and then on $\mathbf{D}/\hbar$ give us two wave equations on components Ω_o and Ω_e, respectively. Taking into account inequality (16), we retain on the right-hand side of the wave equation on component Ω_e the local polarization response of both sorts of molecules only. Thus, we obtain the following system:

$$\frac{\partial^2\Omega_o}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\Omega_o}{\partial t^2} = \frac{16\pi m^2}{\hbar c^2}\frac{\partial^2}{\partial t^2} \times \left(n_1 U_1 + N n_g U_2 + \frac{N n_g l_g^2}{6}\frac{\partial^2 U_2}{\partial z^2}\right), \tag{17}$$

$$\frac{\partial^2\Omega_e}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\Omega_e}{\partial t^2} = \frac{4\pi D^2}{\hbar c^2}\frac{\partial^2}{\partial t^2}(n_1 W_1 + N n_g W_2). \tag{18}$$

The subsequent consideration is based on the approximate analysis of the self-consistent system of equations (15), (17) and (18).

3. Nonlinear wave equation

To except the material variables, we apply the approximation of sudden perturbations [17–19] to equation (15). Let the condition

$$\varepsilon_3 = (\omega_{1,2}\tau_p)^2 \ll 1 \tag{19}$$

be valid. The tunnel quantum transitions of nuclei in molecules, and electron transitions in quantum dots, wells, etc, can satisfy this condition, for example. Taking $\omega_{1,2} \sim 10^{14}\ \mathrm{s}^{-1}$ and $\tau_p \sim 10^{-15}$ s, we find $\varepsilon_2 \sim 0.1$.

In the zeroth-order approximation with respect to the small parameter (19), we obtain from equation (15)

$$W_{1,2} = W_{1,2}^{(0)}\cos\theta, \quad V_{1,2} = W_{1,2}^{(0)}\sin\theta, \tag{20}$$

where

$$\theta = \int_{-\infty}^{t} \Omega_o dt', \tag{21}$$

and $W_{1,2}^{(0)}$ is the inversion of the population of the quantum levels before an influence of the FCP. In the case of an equilibrium medium, we have $-1/2 \leqslant W_{1,2}^{(0)} < 0$. If the medium is nonequilibrium, then $0 < W_{1,2}^{(0)} \leqslant 1/2$.

Substituting the second expression in (20) into the first equation of the system in (15), we find in the following order with respect to the small parameter (19)

$$\frac{\partial U_{1,2}}{\partial t} = W_{1,2}^{(0)}(\omega_{1,2} - \Omega_e)\sin\theta. \tag{22}$$

Substituting equations (20) and (22) into equations (17) and (18), we obtain the following system of wave equations:

$$\frac{\partial^2\Omega_o}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\Omega_o}{\partial t^2} = -\frac{16\pi m^2}{\hbar c^2}\frac{\partial}{\partial t}\left\{[n_1 W_1^{(0)}(\omega_1 - \Omega_e) + N n_g W_2^{(0)}(\omega_2 - \Omega_e)]\sin\theta + l_g^2\frac{\omega_2 N n_g W_2^{(0)}}{6}\frac{\partial^2}{\partial z^2}\sin\theta\right\}, \tag{23}$$

$$\frac{\partial^2\Omega_e}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\Omega_e}{\partial t^2} = \frac{4\pi D^2}{\hbar c^2}\frac{\partial^2}{\partial t^2}[(n_1 W_1^{(0)} + N n_g W_2^{(0)})\cos\theta]. \tag{24}$$

If the right-hand side of the nonuniform one-dimensional wave equation is less than both terms in its left-hand side, then the velocity of wave perturbation is close to that of light. In this case, it is possible to apply the unidirectional propagation approximation [16, 18]. It is easy to see from equation (23) that this condition is valid if $\Omega_o/\tau_p \gg \eta_o\omega_1^2$, where $\eta_o = 16\pi m^2 n_1/(\hbar\omega_1)$. It follows from (21) that $\Omega_o \sim \partial\theta/\partial t \sim 1/\tau_p$. Then, $\eta_o\varepsilon_2^2 \ll 1$. Similarly, we find from equation (24) that $\eta_o(D/2m)^2 \ll 1$. It is important to note that the last two inequalities can be satisfied without the fulfillment of the condition of a low-density medium $\eta_o \ll 1$ [16]. Indeed, these two inequalities are valid in the case of $\eta_o \sim 1$ if condition (19) of sudden perturbation and one of weak asymmetry of the molecules,

$$\varepsilon_4 = \left(\frac{D}{2m}\right)^2 \ll 1, \tag{25}$$

are fulfilled.

We set $\Omega_o = \Omega_o(\tau, \zeta)$, $\Omega_e = \Omega_e(\tau, \zeta)$, where $\tau = t - z/c$, $\zeta = \varepsilon z$, $\varepsilon = \max\{\eta_o\varepsilon_2, \eta_o\varepsilon_4\}$, in accordance with the unidirectional propagation approximation. Then, neglecting the term of the order ε^2, we have

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial\tau}, \quad \frac{\partial^2}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2}{\partial\tau^2} - \frac{2\varepsilon}{c}\frac{\partial^2}{\partial\tau\partial\zeta}.$$

Since the right-hand side of equation (23) is proportional to ε, it is necessary to put

$$\frac{\partial^2}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2}{\partial\tau^2}$$

there. As a result, integrating with respect to τ the wave equations obtained in this case from equations (23) and (24), we find

$$\frac{\partial\Omega_o}{\partial z} = a[\omega_1 n_1 W_1^{(0)} + \omega_2 N n_g W_2^{(0)} - (n_1 W_1^{(0)} + N n_g W_2^{(0)})\Omega_e]\sin\theta + W_2^{(0)} b\tau_s^2\frac{\partial^2}{\partial\tau^2}\sin\theta, \tag{26}$$

$$\frac{\partial\Omega_e}{\partial z} = g(n_1 W_1^{(0)} + N n_g W_2^{(0)})\Omega_o\sin\theta, \tag{27}$$

where

$$a = \frac{8\pi m^2}{\hbar c}, \quad b = \frac{8\pi m^2}{\hbar c} N n_g \omega_2, \quad g = \frac{2\pi D^2}{\hbar c},$$
$$\tau_s = \frac{l_g}{\sqrt{6}\, c}. \tag{28}$$

It is taken into account under the derivation of equation (27) that $\partial\theta/\partial\tau = \Omega_o$ (see relation (21)). Obviously, the characteristic time τ_s of spatial dispersion is of the same order of magnitude as the time of propagation of light between the next nanogranules.

We have from equations (26) and (27)

$$\frac{\partial}{\partial z}\left[\Omega_0^2\left(1 + \eta_o \frac{nN_g}{n_1}\omega_1\omega_2\tau_s^2 W_2^{(0)}\cos\theta\right) + \frac{a}{g}\Omega_e^2 - 2\frac{a}{g}\omega_{ef}\Omega_e\right] = 0, \tag{29}$$

where

$$\omega_{ef} = \frac{\omega_1 n_1 W_1^{(0)} + \omega_2 N n_g W_2^{(0)}}{n_1 W_1^{(0)} + N n_g W_2^{(0)}}.$$

Under the derivation of the conservation law (29), we put

$$\frac{\partial^2}{\partial\tau^2} \approx \frac{1}{c}\frac{\partial^2}{\partial\tau\partial z}$$

in the last term of the right-hand side of equation (26).

Supposing $N \sim n_1 L_g^3$ and $n_g \sim l_g^{-3}$, we obtain

$$\eta_o \frac{nN_g}{n_1}\omega_1\omega_2\tau_s^2 \sim \eta_o(\omega_1\tau_s)^2\left(\frac{L_g}{l_g}\right)^3.$$

Taking the values of η_o, l_g, L_g and ω_1 given above, we have $\tau_s \sim 10^{-16}$ s and $\eta_o(nN_g/n_1)\omega_1\omega_2\tau_s^2 \sim 10^{-8}$. Therefore, the second term in the parentheses of equation (29) can be neglected. Also, due to inequality (16), we neglect the second term in the square brackets of equation (29). Then, we find after integration

$$\Omega_e = \frac{g}{2a\omega_{ef}}\Omega_o^2 = \left(\frac{D}{2m}\right)^2 \frac{\Omega_o^2}{2\omega_{ef}}. \tag{30}$$

It is seen from this relation that

$$\varepsilon_4 \sim \sqrt{\varepsilon_2\varepsilon_3}$$

in the order of magnitude.

Substituting (30) into equation (26) and taking into account relation (21), we obtain the nonlinear wave equation

$$\frac{\partial^2\theta}{\partial z\partial\tau} = -\alpha\left[1 - \frac{\tau_c^2}{2}\left(\frac{\partial\theta}{\partial\tau}\right)^2\right]\sin\theta + \beta\tau_s^2\frac{\partial^2}{\partial\tau^2}\sin\theta, \tag{31}$$

where

$$\alpha = -a(\omega_1 n_1 W_1^{(0)} + \omega_2 N n_g W_2^{(0)}), \quad \beta = W_2^{(0)} b,$$
$$\tau_c = \frac{1}{2}\left|\frac{D}{m\omega_{ef}}\right|. \tag{32}$$

Parameter τ_c has the dimension of time and characterizes the asymmetry of molecules. Note that the definitions of the characteristic times τ_s and τ_c are similar to those introduced in [43]. In the case of $\tau_c = \tau_s = 0$ (absence of asymmetry of molecules and spatial dispersion), equation (31) is reduced to the famous SG equation [46].

If $\tau_c = 0$, then equation (31) coincides with the Rabelo–Fokas equation [50–53] integrable by the ISTM. Another generalization of the Rabelo–Fokas equation was investigated in [43]. In the designations used here, it is written in the following form:

$$\frac{\partial^2\theta}{\partial z\partial\tau} = a\left[\omega_1 n_1 W_1^{(0)}\sqrt{1 - \tau_c^2\left(\frac{\partial\theta}{\partial\tau}\right)^2} + \omega_2 N n_g W_2^{(0)}\right]\sin\theta + \beta\tau_s^2\frac{\partial^2}{\partial\tau^2}\sin\theta. \tag{33}$$

Supposing in this equation that the asymmetry is weak:

$$\tau_c^2\left(\frac{\partial\theta}{\partial\tau}\right)^2 \ll 1, \quad \sqrt{1 - \tau_c^2\left(\frac{\partial\theta}{\partial\tau}\right)^2} \approx 1 - \frac{\tau_c^2}{2}\left(\frac{\partial\theta}{\partial\tau}\right)^2,$$

we do not reduce it to equation (31). The reason is that 1-molecules were assumed in [43] to be asymmetric only. Here, both sorts of molecules possess the property of asymmetry. Conditions (3), (16) and (25) signify that the asymmetry and spatial dispersion are rather low. This allows us to neglect the asymmetry of molecules in the nanogranules.

4. Integrable case and soliton solutions

As was shown in [43], if an additional condition on the parameters of equation (33) is imposed, then this equation is integrable by the ISTM and is connected by means of the change of dependent and independent variables with the modified SG (MSG) equation [54–57]:

$$\frac{\partial^2\xi}{\partial\tilde{x}\partial\tilde{t}} = -\sqrt{1 - \kappa^2\left(\frac{\partial\xi}{\partial\tilde{x}}\right)^2}\,\sin\xi. \tag{34}$$

Here $\xi = \xi(\tilde{x}, \tilde{t})$, κ is a constant. A quite similar situation occurs in the case of equation (31).

Let the condition

$$\alpha\tau_c^2 = \beta\tau_s^2 \tag{35}$$

be valid. Then, equation (31) is written as

$$\frac{\partial^2\theta}{\partial z\partial\tau} = -\alpha\left[1 - \frac{\tau_c^2}{2}\left(\frac{\partial\theta}{\partial\tau}\right)^2\right]\sin\theta + \alpha\tau_c^2\frac{\partial^2}{\partial\tau^2}\sin\theta. \tag{36}$$

Equation (36) is integrable by the ISTM and is represented as a zero-curvature condition

$$\frac{\partial\hat{L}}{\partial z} - \frac{\partial\hat{A}}{\partial\tau} + [\hat{L}, \hat{A}] = 0,$$

where

$$\hat{L} = \frac{1}{2}\begin{pmatrix} \mathrm{i}\frac{\partial\theta}{\partial\tau} & \alpha\lambda\left(1 + \frac{\mathrm{i}\tau_c}{\sqrt{2}}\frac{\partial\theta}{\partial\tau}\right)^2 \\ \alpha\lambda\left(1 - \frac{\mathrm{i}\tau_c}{\sqrt{2}}\frac{\partial\theta}{\partial\tau}\right)^2 & -\mathrm{i}\frac{\partial\theta}{\partial\tau} \end{pmatrix},$$

$$\hat{A} = -\frac{1}{2}\begin{pmatrix} \mathrm{i}\sqrt{2}\,\alpha\tau_c \sin\theta & \frac{\mathrm{e}^{\mathrm{i}\theta}}{\lambda} \\ \frac{\mathrm{e}^{-\mathrm{i}\theta}}{\lambda} & -\mathrm{i}\sqrt{2}\,\alpha\tau_c \sin\theta \end{pmatrix} + \alpha\tau_c^2 \cos\theta\, \hat{L}$$

(λ is the spectral parameter).

Using relations (28) and (32) and the expressions for parameters α and β, we write the integrability condition (35) as follows:

$$W_1^{(0)} = -W_2^{(0)} \frac{\omega_2 N n_g}{\omega_1 n_1}\left(1 + \frac{\tau_s^2}{\tau_c^2}\right). \tag{37}$$

It is seen that the initial inversions $W_1^{(0)}$ and $W_2^{(0)}$ of the populations of the quantum levels should have opposite signs for equation (31) to be integrable. This means that the medium is in a quasiequilibrium state before an influence of the electromagnetic pulse.

Note that condition (37) differs from the integrability condition of equation (33) in essence. Namely, equation (33) is integrable in cases of an equilibrium or nonequilibrium medium, i.e., when parameters $W_1^{(0)}$ and $W_2^{(0)}$ have identical signs. Therefore the soliton solutions of equation (31) (or equation (36)) constructed below are not a limiting case of the soliton solutions of equation (33) at $\tau_c^2\left(\frac{\partial\theta}{\partial\tau}\right)^2 \ll 1$. Thus, the soliton solutions considered here are not incorporated into to the ones found in [43].

Let the parameters of equations (34) and (36) be connected by the relation

$$\kappa = -\sqrt{2}\,\alpha\tau_c.$$

Then, there exists a change of variables transforming independent and dependent variables of the MSG equation (34) into those of equation (36). This change of variables has the following form:

$$\begin{aligned} &\tau = \frac{1}{2\alpha}\int^{\tilde{x}} (1 + \sqrt{1 - \kappa^2 u^2})\,\mathrm{d}\tilde{x}', \\ &z = \tilde{t}, \\ &\theta(\tau, z) = \xi(\tilde{x}, \tilde{t}), \end{aligned} \tag{38}$$

where we use the notation

$$u = \frac{\partial\xi}{\partial\tilde{x}}, \tag{39}$$

and variable τ satisfies the equation

$$\frac{\partial\tau}{\partial\tilde{t}} = -\alpha\tau_c^2 \cos\xi. \tag{40}$$

The last equation removes the functional arbitrariness in the definition of τ.

From equations (38) and (21), we have

$$\Omega_o = \frac{2\alpha u}{1 + \sqrt{1 - \kappa^2 u^2}}. \tag{41}$$

Given the multi-soliton solutions of the MSG equation (34), we obtain the multi-soliton solutions of equation (36) with the help of the change of variables in (38). The multi-soliton solutions of the MSG equation were studied in detail in [40] (see [43] also).

Without loss of generality, the one-soliton solution of the MSG equation (34) is written as follows:

$$\xi = (-1)^k 2\arccos\frac{\mu\kappa - \tanh\varphi}{\sqrt{1 - 2\mu\kappa\tanh\varphi + \mu^2\kappa^2}}, \tag{42}$$

where

$$\varphi = \mu\tilde{x} - \frac{\tilde{t}}{\mu} + \varphi^{(0)},$$

μ and $\varphi^{(0)}$ are real constants, and $k \in \{0, 1\}$.

It is seen that the topological charge $S = (\xi|_{\tilde{x}\to\infty} - \xi|_{\tilde{x}\to-\infty})/\pi$ of the one-soliton solution (42) of the MSG equation is

$$S = \begin{cases} (-1)^k \operatorname{sgn}(\mu) & \text{if } |\mu\kappa| < 1, \\ 0 & \text{if } |\mu\kappa| > 1. \end{cases}$$

Consequently, the one-soliton solutions are divided into three families having topological charges of 1, −1 and 0, respectively. The first two families correspond to the kinks and antikinks (2π-pulses) of the SG equation. Solitons of the third family are called neutral kinks [40].

It follows from equations (42) and (39) that

$$u = 2\mu\,\operatorname{sech}\varphi\,\frac{1 - \mu\kappa\tanh\varphi}{1 - 2\mu\kappa\tanh\varphi + \mu^2\kappa^2}. \tag{43}$$

Then, we have

$$\max|u| = \begin{cases} 2|\mu|\sqrt{1 - \mu^2\kappa^2} & \text{if } |\mu\kappa| < \frac{1}{\sqrt{2}}, \\ \frac{1}{|\kappa|} & \text{if } |\mu\kappa| \geqslant \frac{1}{\sqrt{2}}. \end{cases} \tag{44}$$

It is seen that the amplitude of u is independent of parameter μ if $|\mu| > 1/\sqrt{2}\,|\kappa|$. In this case, the profile of u consists of two peaks with an amplitude of $1/|\kappa|$, which are separated by an interval depending on parameter μ [40]. These peaks have the same polarity if $1/\sqrt{2} < |\mu\kappa| < 1$, and they have opposite polarities in the case of neutral kinks ($|\mu| > 1/|\kappa|$).

Substituting expressions (42) and (43) into equation (38), we find the one-soliton solution of equation (36). Note that the square root in formulas (38) and (41) changes the branch in the points where $|u|$ achieves its maximum value $1/|\kappa|$ (see equation (44)). Since variable u changes the sign in the case of neutral kinks, corresponding solutions of equation (36) are singular.

From the first relation in (38) and equations (40) and (43), we obtain the following expression for variable τ:

$$\tau = \frac{\tilde{x}}{\alpha} + \frac{\kappa}{2\alpha}\ln\left[1 - 2\mu\kappa\tanh\varphi + \mu^2\kappa^2\right] - \alpha\tau_c^2\tilde{t}. \tag{45}$$

It is seen from this expression that the one-soliton solution of equation (36) is steady.

Let variable Ω_o of the one-soliton solution of equation (36) decrease by $\Omega_o \sim \exp(-|\,t - z/v|/\tau_p)$ on the tails. Here, parameters τ_p and v are the characteristic duration

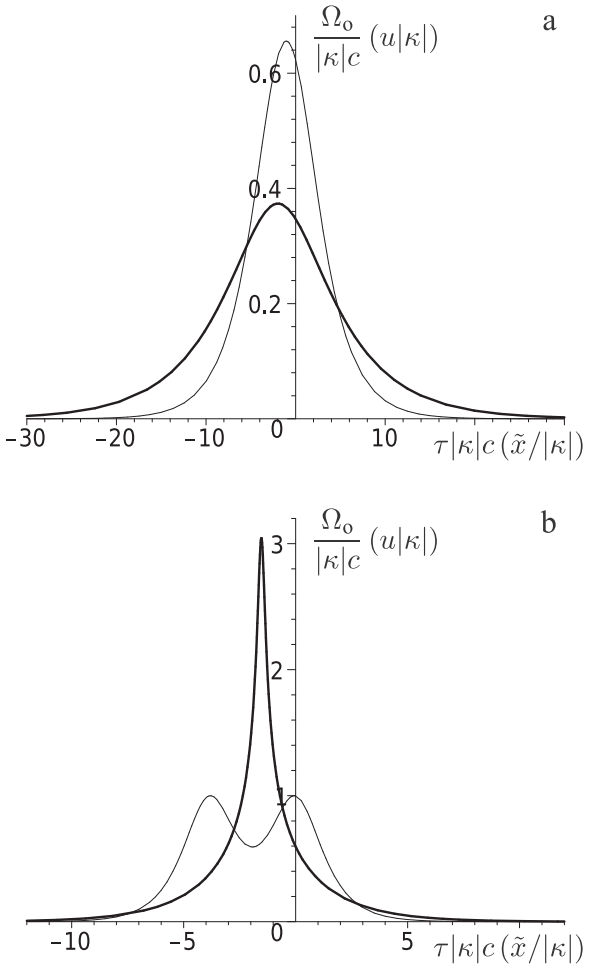

Figure 1. Profiles of variables Ω_o and u (thin line; the corresponding designations of the axes are given in parentheses) of one-soliton solutions with parameters $\alpha = 0.5\kappa^2 c$, $\tau_c = \sqrt{2}/|\kappa|c$, $\varphi^{(0)} = 0$, $k = 0$ and $\mu = 0.35/|\kappa|$ (a) and $\mu = 0.95/|\kappa|$ (b).

of the one-soliton pulse and its velocity in the laboratory system of coordinates (z, t), respectively. It follows from equations (41), (43), and (45) and the expressions for κ and φ that

$$\tau_p = \frac{1}{|\mu\alpha|}, \quad v = \frac{c}{1 + c\alpha(\tau_p^2 - \tau_c^2)}. \tag{46}$$

It follows from these relations that the one-soliton solution of equation (36) is nonsingular and has a velocity v lower than c if $W_1^{(0)} < 0$ ($\alpha > 0$). In the case of $W_1^{(0)} > 0$ ($\alpha < 0$), the condition $v < c$ is fulfilled for singular solitons only. Hence, the velocity of a nonsingular steady-state soliton satisfies the condition $v < c$ for the case $W_1^{(0)} < 0$ (i.e. $W_2^{(0)} > 0$) only. Note that the duration of a nonsingular one-soliton pulse satisfies the condition $\tau_p > \sqrt{2}\,\tau_c$.

Figure 1 shows the profiles of variable Ω_o of the one-soliton solution of equation (36). The profiles of variable u connected with Ω_o by relation (41) are depicted here by thin lines. In this case, the designations of the axes are presented in the parentheses. It is seen that the amplitude of $|\Omega_o|$ tends to infinity as $|\mu|$ tends to $1/|\kappa|$ (figure 1(b)).

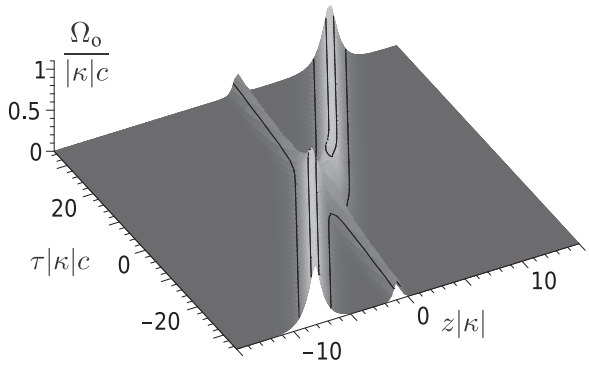

Figure 2. Profile of variable Ω_o of two-soliton solution with parameters $\alpha = 0.5\kappa^2 c$, $\tau_c = \sqrt{2}/|\kappa|c$, $\mu_1 = 0.25/|\kappa|$, $\mu_2 = -0.6/|\kappa|$, $\varphi_1^{(0)} = \varphi_1^{(0)} = 0$, and $k_1 = k_2 = 0$.

The two-soliton solution of the MSG equation (34) is written as

$$\xi = -2\arctan\frac{\mu_+\sinh\varphi_-}{\mu_-\cosh\varphi_+} - 2\arctan\frac{\mu_+[\varepsilon_-\sinh\varphi_- - 2\mu_-\kappa\cosh\varphi_-]}{\mu_-[\varepsilon_+\cosh\varphi_+ - 2\mu_+\kappa\sinh\varphi_+]}, \tag{47}$$

where

$$\mu_\pm = \frac{\mu_1 \pm \mu_2}{2}, \quad \varphi_\pm = \frac{\varphi_1 \pm \varphi_2}{2}, \quad \varepsilon_\pm = 1 \pm \mu_1\mu_2\kappa^2,$$

$$\varphi_{1,2} = \mu_{1,2}\tilde{x} - \frac{\tilde{t}}{\mu_{1,2}} + \varphi_{1,2}^{(0)} + \mathrm{i}k_{1,2}\pi,$$

$\mu_{1,2}$ and $\varphi_{1,2}^{(0)}$ are real constants, and $k_{1,2} \in \{0, 1\}$.

The two-soliton solution of equation (36) is obtained by the substitution of expression (47) into equations (38) and (39). This solution describes the elastic collision of two one-soliton pulses with durations and velocities defined by the relations in (46), where $\mu = \mu_1$ and $\mu = \mu_2$.

A plot of variable Ω_o of the two-soliton solution of equation (36) in the case of solitons having the same polarity is presented in figure 2. The character of the interaction of such solitons is similar to that in the SG equation case. However, if the solitons have opposite polarities, then their interaction can lead to the appearance of a short-lived pulse with an extraordinarily large amplitude (see figure 3). The dynamics of this pulse resembles that of rogue waves [58, 59].

The breather solution of the MSG equation (34) is defined as

$$\xi = 2\arctan\frac{\mu_R\sin\varphi_I}{\mu_I\cosh\varphi_R} + 2\arctan\frac{\mu_R[(1 - |\mu|^2\kappa^2)\sin\varphi_I - 2\mu_I\kappa\cos\varphi_I]}{\mu_I[(1 + |\mu|^2\kappa^2)\cosh\varphi_R - 2\mu_R\kappa\sinh\varphi_R]}, \tag{48}$$

where

$$\varphi_R = \mu_R\left(\tilde{x} - \frac{\tilde{t}}{|\mu|^2}\right) + \varphi_R^{(0)}, \quad \varphi_I = \mu_I\left(\tilde{x} + \frac{\tilde{t}}{|\mu|^2}\right) + \varphi_I^{(0)},$$

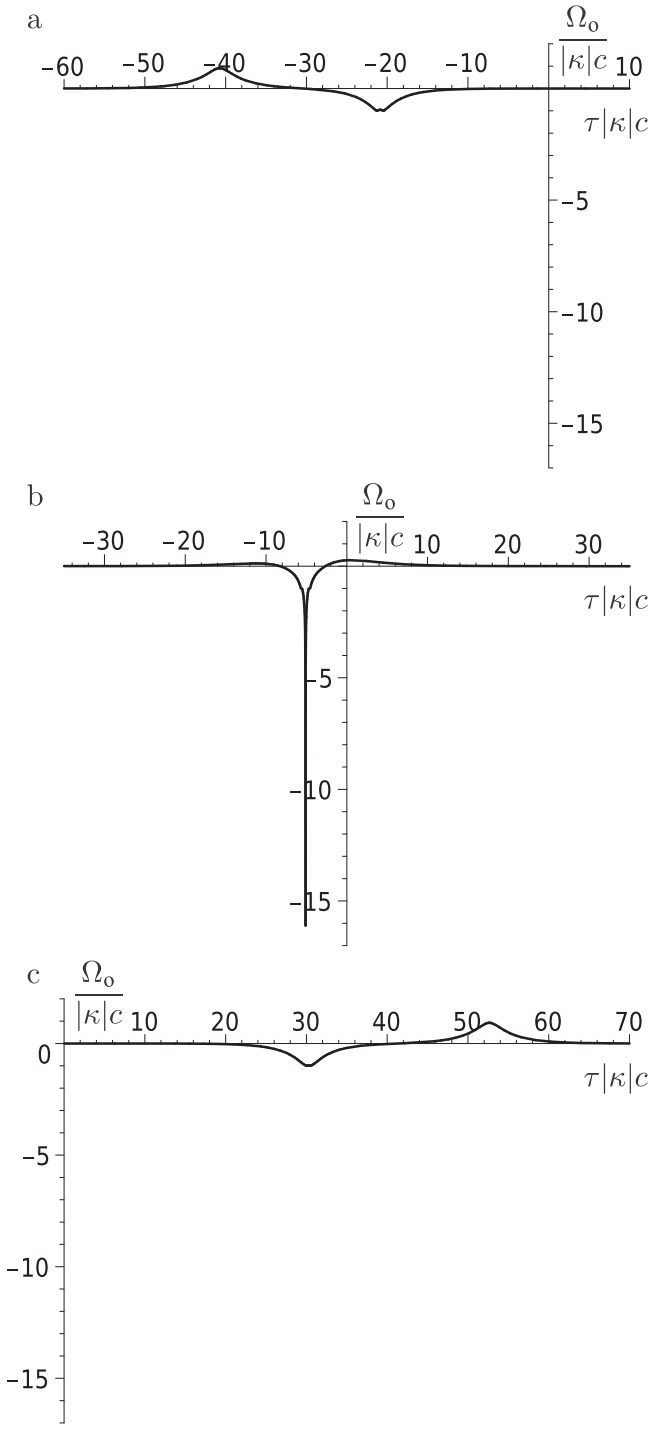

Figure 3. Profiles of variable Ω_o of two-soliton solution with parameters $\alpha = 0.5\kappa^2 c$, $\tau_c = \sqrt{2}/|\kappa|c$, $\mu_1 = 0.69/|\kappa|$, $\mu_2 = 0.7/|\kappa|$, $\varphi_1^{(0)} = \varphi_1^{(0)} = 0$, $k_1 = k_2 = 0$ and $z = -10/|\kappa|$ (a), $z = 0$ (b) and $z = 15/|\kappa|$ (c).

μ_R, μ_I, $\varphi_R^{(0)}$ and $\varphi_I^{(0)}$ are real constants, and $\mu = \mu_R + i\mu_I$.

Substitution of expression (48) into equations (38) and (39) gives the breather solution of equation (36). Plots of variable Ω_o of this solution are presented in figure 4.

Let us determine the characteristic parameters of the breather solution of equation (36) by supposing that variable

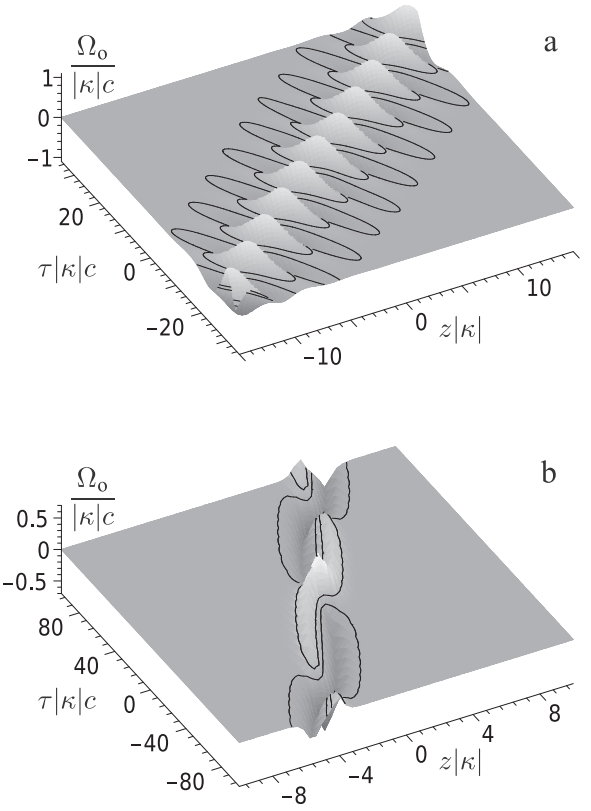

Figure 4. Profiles of variable Ω_o of breather solution with parameters $\alpha = 0.5\kappa^2 c$, $\tau_c = \sqrt{2}/|\kappa|c$, $\varphi_R^{(0)} = \varphi_I^{(0)} = 0$ and $\mu_R = 0.4/|\kappa|$, $\mu_I = 0.7/|\kappa|$ (a) and $\mu_R = 0.4/|\kappa|$, $\mu_I = 0.04/|\kappa|$ (b).

Ω_o decreases by $\exp(-|\,t - z/v_g|/\tau_b)\cos(\omega_b(t - z/v_{ph}))$ at the tails. Parameters τ_b, ω_b, v_g and v_{ph} are the duration of the breather, its frequency, the group velocity and the phase velocity in the laboratory system of coordinates (z, t), respectively. From the first relation in (38), equations (39), (41), and (48) and the expressions for φ_R and φ_I, we have

$$\tau_b = \frac{1}{|\mu_R \alpha|}, \quad \omega_b = |\mu_I \alpha|,$$

$$v_g = c\left[1 - c\alpha\left(\tau_c^2 - \frac{\tau_b^2}{1 + \omega_b^2\tau_b^2}\right)\right]^{-1}, \tag{49}$$

$$v_{ph} = c\left[1 - c\alpha\left(\tau_c^2 + \frac{\tau_b^2}{1 + \omega_b^2\tau_b^2}\right)\right]^{-1}. \tag{50}$$

In the case of $\tau_c = 0$, these formulas coincide with the ones of the SG equation.

As $\mu_I \to 0$, expression (48) reduces to

$$\xi = 2\arctan\frac{\mu\tilde{x} + \tilde{t}/\mu}{\cosh\varphi} + 2\arctan\frac{(1 - \kappa^2\mu^2)(\mu\tilde{x} + \tilde{t}/\mu) - 2\kappa\mu}{(1 + \kappa^2\mu^2)\cosh\varphi - 2\kappa\mu\sinh\varphi},$$

where we put $\mu = \mu_R$. Substituting this expression into equations (38) and (39) gives us the solution of equation (36), which describes the interaction of solitons with equal velocities. A plot of variable Ω_0 of this solution is presented in

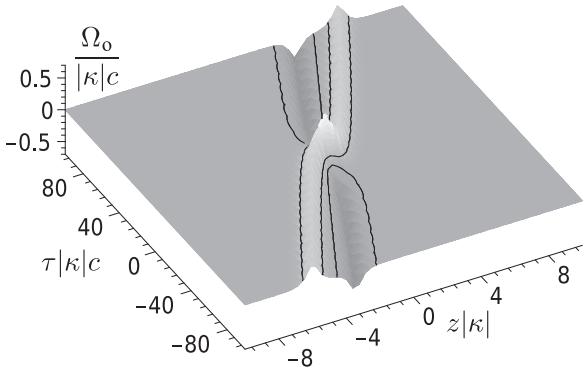

Figure 5. Profile of variable Ω_o in the case of interaction of solitons having equal velocities for parameter values of $\alpha = 0.5\kappa^2 c$, $\tau_c = \sqrt{2}/|\kappa|c$, $\varphi^{(0)} = 0$ and $\mu = 0.3/|\kappa|$.

figure 5. It is remarkable that the polarities of variable Ω_o of the colliding solitons are opposite. The breather solution with $\omega_b\tau_b < 1$ can be interpreted as a bound state of two one-soliton pulses having equal durations and opposite polarities (see figures 4(b) and 5).

In the case of $\omega_b\tau_b \gg 1$, variable Ω_o of the breather turns into an envelope soliton. In that case, τ_b and ω_b are the duration of the envelope soliton and its carrier frequency, respectively. Using expression (28) and the equality $\lambda = 2\pi c/\omega_b$, we rewrite condition (3) into the form $(\omega_b\tau_s)^2 \ll 1$. From this inequality and equation (35), we find $(\omega_b\tau_c)^2 \ll b/a = n_g\omega_2/(n_1\omega_1)$. Assuming that $\omega_1 \sim \omega_2$, we have $(\omega_b\tau_c)^2 \ll n_g/n_1 \sim (l_1/l_g)^3 \ll 1$. From equations (49) and (50), we obtain the following expressions for the phase and group velocities of the envelope soliton:

$$v_g = \frac{c}{1 + c\alpha/\omega_b^2}, \quad v_{ph} = \frac{c}{1 - c\alpha/\omega_b^2}.$$

According to the remark after equation (24), the values of the group and phase velocities differ a little from c. Then $c|\alpha|/\omega_b^2 \ll 1$ and

$$v_g \approx c(1 - c\alpha/\omega_b^2), \quad v_{ph} \approx c(1 + c\alpha/\omega_b^2).$$

In an equilibrium medium of 1-molecules ($W_1^{(0)} < 0$, $\alpha > 0$), we have $v_g < c$. If the medium of 1-molecules is nonequilibrium ($W_1^{(0)} > 0$, $\alpha < 0$), then $v_g > c$. This superluminal regime does not contradict the special theory of relativity and is due to the mechanism of pulse reshaping during propagation in the nonequilibrium medium [60–62].

The multi-soliton solutions of equation (36) describe the elastic collisions of the steady-state solitons and breathers considered above. They are constructed by applying the change of variables in (38) to the multi-soliton solutions of the MSG equation (34).

5. Conclusion

In this work, the wave equation (31) is obtained under an investigation of the propagation of extremely short electromagnetic pulses through a nonlinear nanodispersed medium of asymmetric molecules. Under the imposition of condition (35) (see also (37)) on the coefficients of this equation, it becomes integrable by the ISTM. This allows us to find and investigate the soliton and breather solutions. Curiously, the integrability condition (35) of equation (31) corresponds from the physical point of view to an initial quasiequilibrium condition of the medium. Perhaps, such a medium can be a working element of the laser. This circumstance distinguishes radically the soliton solutions obtained here from the ones found in [43] for another integrable generalization of the SG equation. Owing to this, the soliton solutions of equation (31) are not incorporated into the soliton solutions studied in [43].

In the case of an isotropic homogeneous medium ($\tau_c = \tau_s = 0$), the integrability condition (35) is satisfied obviously, while equation (31) is reduced to the SG equation describing, in particular, the self-induced transparency phenomenon. It is well known that the soliton solution of this equation is unstable if the molecules of the matrix are in the excited initial state ($W_1^{(0)} > 0$) [1]. As a result, a natural question arises about the stability of the soliton and breather solutions of equation (31) under such a condition. On the other hand, the integrability condition (35) can be fulfilled if the molecules of the granules are excited first ($W_2^{(0)} > 0$), while the molecules of the matrix are in the ground state ($W_1^{(0)} < 0$). What is possible to say about the stability of solitons then? Also, the integrability of the systems obtained from the self-consistent system (15), (17) and (18) by applying different approximations (e.g., the unidirectional propagation approximation) has to be investigated. Answers to these questions are beyond the scope of this study and need separate consideration. Nevertheless, they can shed light on the mechanisms of formation and propagation of solitons in quasiequilibrium media.

Acknowledgments

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