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SHORT COMMUNICATIONS

The Doss Method for the Stochastic Schrödinger–Belavkin Equation

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As is well known (see [1]), for the Cauchy problem

$$\lambda\frac{\partial\Psi}{\partial t}(t,x)=\frac{\lambda^2}{2m}\Delta\Psi(t,x)+V(x)\Psi(t,x),\qquad \Psi(0,x)=f(x),\quad (t,x)\in[0,T]\times\mathbb{R}^n,$$

where $\lambda>0$, the solution is defined by the *Feynman–Kac formula*

$$\Psi(t,x)=\mathsf{E}f\left(x+\sqrt{\frac{\lambda}{m}}B_t\right)\exp\left(\int_0^t\frac{V}{\lambda}\left(x+\sqrt{\frac{\lambda}{m}}B_s\right)ds\right).$$

In [2], it was proved that, in the case of the Schrödinger equation

$$ih\frac{\partial\Psi}{\partial t}(t,x)=\frac{-h^2}{2m}\Delta\Psi(t,x)+V(x)\Psi(t,x),\qquad \Psi(0,x)=f(x),\quad (t,x)\in[0,T]\times\mathbb{R}^n,$$

the solution of a similar Cauchy problem is defined by the equality

$$\Psi(t,x)=\mathsf{E}\widetilde{f}(x+\sqrt{ih}B_t)\exp\left(\frac{1}{ih}\int_0^t\widetilde{V}(x+\sqrt{ih}B_s)\,ds\right),$$

where E is the expectation; the functions $\widetilde{f}(\,\cdot\,)$ and $\widetilde{V}(\,\cdot\,)$ are analytic continuations in the argument of the functions $f(\,\cdot\,)$ and $V(\,\cdot\,)$ to a suitable domain (see [2]).

In this paper, it is shown that the solution of the stochastic Schrödinger–Belavkin equation can be written in a similar way if the randomized Feynman–Kac formula for the Euclidean analog of this equation is known.

The randomized Feynman–Kac formula will be used in the following theorem.

Theorem 1. *Let $C_0(t_1,t_2)$ be the space of continuous functions vanishing at a point t_1 with standard Wiener measure w_{t_1,t_2}. Then the function $\Psi_\omega(\,\cdot\,)(\,\cdot\,)$ defined by the equality*

$$\Psi_\omega(t)(q)=\int_{C_0(0,t)}\exp\left(b\int_0^t V(\tau,q+\xi(\tau))\,d\tau+c\int_0^t R(q+\xi(\tau))\,dB_\omega(\tau)\right)\varphi_0(q+\xi(t))w_{0,t}\,(d\xi),$$

is a solution of the Cauchy problem

$$d\Psi_\omega(t)(\,\cdot\,)=a(\Psi_\omega(t))''(\,\cdot\,)+bV(t,\,\cdot\,)\Psi_\omega(t)(\,\cdot\,)+cR(\,\cdot\,)\Psi_\omega(t)(\,\cdot\,)\,dB_\omega(t),\qquad \Psi_\omega(0)(\,\cdot\,)=\varphi_0(q),$$

where a, b, and c are positive number parameters, $R(q)>0$ for all $q\in Q$, the functions $R(\,\cdot\,)$, φ_0 are real and continuous, and $V(t,\,\cdot\,)$ is integrable and bounded on Q (Q is the configuration space on which $\Psi_\omega(\,\cdot\,)(\,\cdot\,)$ is defined).

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Scheme of proof. Just as in the deterministic case, using the Itô formula, we obtain the following equality:

$$
\begin{aligned}
&d\biggl(\exp\biggl(b\int_t^{t+s} V(\tau, q+\xi(t)+\xi_1(\tau))\,d\tau + c\int_t^{t+s} R(q+\xi(t)+\xi_1(\tau))\biggr)\,dB_\omega(\tau)\\
&\qquad\times\int_{C_0(0,t)}\exp\biggl(b\int_0^t V(\tau, q+\xi(\tau))\,d\tau + c\int_0^t R(q+\xi(\tau))\,dB_\omega(\tau)\biggr)\varphi_0(q+\xi(t+s))w_{0,t}\,(d\xi)\biggr)\\
&\quad=\exp\biggl(b\int_t^{t+s} V(\tau, q+\xi(t)+\xi_1(\tau))\,d\tau + c\int_t^{t+s} R(q+\xi(t)+\xi_1(\tau))\,dB_\omega(\tau)\biggr)\\
&\qquad\times\int_{C_0(0,t)}\exp\biggl(b\int_0^t V(\tau, q+\xi(\tau))\,d\tau + c\int_0^t R(q+\xi(\tau))\,dB_\omega(\tau)\biggr)\varphi_0(q+\xi(t+s))w_{0,t}\,(d\xi)\\
&\qquad\times\bigl(bV(\tau, q+\xi(t)+\xi_1(t+s)) + cR(q+\xi(t)+\xi_1(t+s))\bigr)\,dB_\omega\\
&\qquad+\frac{d}{ds}\biggl(\int_{C_0(0,t)}\exp\biggl(b\int_0^t V(\tau, q+\xi(\tau))\,d\tau + c\int_0^t R(q+\xi(\tau))\,dB_\omega(\tau)\biggr)\biggr)\\
&\qquad+\frac{1}{2}\frac{d^2}{ds^2}\biggl(\int_{C_0(0,t)}\dots\biggr).
\end{aligned}
$$

Further, we integrate both sides over $C_0(t, t+s)$ and pass to the limit as $s \to 0$ (for details, see [3]). □

In what follows, of importance will be the case where the Cauchy problem in Theorem 1 is of the form

$$
\begin{gathered}
d\Psi_\omega(t)(q) = \alpha\,\frac{d^2\Psi_\omega(t)(q)}{dq^2}\,dt + \biggl(\alpha V(q) - \frac{\lambda}{4}\,q^2\biggr)\Psi_\omega(t)(q)\,dt + \sqrt{\frac{\lambda}{2}}\,q\Psi_\omega(t)(q)\,dB_\omega(t),\\
\Psi_\omega(0,\,\cdot\,) = \varphi_0(\,\cdot\,),
\end{gathered}
\tag{1}
$$

where $\varphi_0(\,\cdot\,) \in C_b(\mathbb{R}^1)$ and $\alpha \in \mathbb{C} \setminus \{0\}$, $\operatorname{Re}\alpha \ge 0$. In this case, its solution is defined by the following Feynman–Kac formula:

$$
\begin{aligned}
\Psi_\omega(t)(q) = \int \exp\biggl\{\int_0^t \alpha V(q+\xi(\tau))d\tau - \int_0^t \frac{\lambda}{2}(q+\xi(\tau))^2 d\tau\biggr\}&\\
\times\exp\biggl\{\sqrt{\frac{\lambda}{2}}\int_0^t (q+\xi(\tau))\,dB_\omega(t)\biggr\}\varphi_0(q+\xi(t))w_{0t}^{\alpha}\,(d\xi).&
\end{aligned}
\tag{2}
$$

In the Feynman–Kac formula itself, the parameter α is real and positive. For such values of the parameter, we obtain precisely the Euclidean analog of the stochastic Schrödinger equation.

In [2], the Feynman–Kac formula for the (nonstochastic) Schrödinger equation was obtained from the Feynman–Kac formula for the heat equation.

In [2] and [4], representations of the solution of the Schrödinger equation as a path integral were obtained by analytic continuation in a parameter; see also [5] and [6]. For such a variant of analytic continuation, a countably additive measure does not appear, which was noted in [4]. Instead, there arises a generalized measure, called the *Feynman measure*, defined on a suitable function space. It should be noted that, in important (for applications) cases, no analytic continuation of a countably additive measure defined on an infinite-dimensional space can be a countably additive measure. In addition, note that the Feynman measure can be obtained by an essentially different method by using Chernoff's theorem or other closely related propositions (see [7] and [8]).

In what follows, we also another approach, going back to Doss'es paper [2]; it is based on the analytic continuation of an integrable function. Here the measure with respect to which integration is performed is replaced by its image, which again turns out to be a countably additive measure.

Namely, let

$$\varphi(t,q)=\int_{C_0([0,t],Q)}\exp\biggl(\int_0^t V(q+\xi(\tau))\,d\tau\biggr)\varphi_0(q+\xi(t)\,w(d\xi));$$

then, after a change of variable, we obtain the following chain of equalities:

$$\begin{aligned}\varphi\biggl(t,\frac{q}{\sqrt{-i}}\biggr)&=\int_{C_0([0,t],Q)}\exp\biggl(\int_0^t V\biggl(\frac{q}{\sqrt{-i}}+\xi(\tau)\biggr)\,d\tau\biggr)\varphi_0\biggl(\frac{q}{\sqrt{-i}}+\xi(t)\biggr)\,w(d\xi)\\&=\int_{C_0([0,t],Q)}\exp\biggl(\int_0^t V\biggl(\frac{q}{\sqrt{-i}}+\frac{\sqrt{-i}\xi(\tau)}{\sqrt{-i}}\biggr)\,d\tau\biggr)\varphi_0\biggl(\frac{q}{\sqrt{-i}}+\frac{\sqrt{-i}\xi(t)}{\sqrt{-i}}\biggr)\,w(d\xi)\\&=\int_{C_0([0,t],Q)}\exp\biggl(\int_0^t V\biggl(\frac{1}{\sqrt{-i}}(q+\xi_1(\tau))\,d\tau\biggr)\varphi_0\biggl(\frac{1}{\sqrt{-i}}(q+\xi_1(\tau)\biggr)wf^{-1}\,(d\xi_1),\end{aligned}$$

where $f(\xi)=\xi_1=\sqrt{-i}\xi$. Thus,

$$\varphi(t,q_1)=\int_{C_0([0,t],Q)}\exp\biggl(\int_0^t V\biggl(q_1+\frac{\xi_1(\tau)}{\sqrt{-i}}\biggr))\,d\tau\biggr)\varphi_0\biggl(q_1+\frac{\xi_1(\tau)}{\sqrt{-i}}\biggr))wf^{-1}\,(d\xi_1),\tag{3}$$

where $q_1=q/\sqrt{-i}$.

In what follows, we shall describe a generalization of the Doss method to the stochastic case. Using this generalization, we shall derive a Feynman–Kac formula for the Schrödinger–Belavkin equation from the Feynman–Kac formula for the stochastic heat equation. This method for deriving the Feynman–Kac formula for the Schrödinger–Belavkin equation differs from that used in [9].

It consists in that all functions appearing in the Feynman–Kac formula corresponding to the heat equation are analytically continued to a suitable domain and then extended by continuity to its closure, after which the change-of-variable formula is applied to the resulting integral. The fact that the resulting integral with respect to a countably additive measure yields a representation of the solution of the Schrödinger–Belavkin equation follows from Theorem 1 and the uniqueness of analytic continuation.

In the proof of the main theorem given below, we use the following lemma.

Lemma. *Let $\psi\colon[0,\infty)\to L_2(\mathbb{R}^1)$ be a solution of the equation*

$$\psi(t)-\psi(0)=\int_0^t((\psi(\tau))''-iV\psi(\tau))\,d\tau$$

and, for each $t\ge 0$, let the function $x\mapsto\psi(t)(x)$ admit the analytic continuation to the domain

$$\biggl\{z\in\mathbb{C}:z=\rho e^{-i\alpha},\ \alpha\in\biggl[0,\frac{\pi}{4}\biggr),\ \rho>0\biggr\}$$

and the extension by continuity to its closure. Let $\varphi\colon[0,\infty)\to L_2^{\mathbb{C}}(\mathbb{R}^1)$ (here $L_2^{\mathbb{C}}(\mathbb{R}^1)$ is the complexification of the space $L_2(\mathbb{R}^1)$) be the function defined as follows:

$$\varphi(t)(x)=\psi(t)(\sqrt{-i}x),\qquad\sqrt{-i}=e^{-i\pi/4},$$

where the differentiation is performed with respect to the space variable. Then the function φ is a solution of the equation

$$i\varphi(t)-i\varphi(0)=\int_0^t(-(\varphi(\tau))''+V\varphi(\tau))\,d\tau.$$

Proof. This fact is verified as follows:

$$\begin{aligned}i\varphi(t)(x)-i\varphi(0)(x)=i\psi(t)(\sqrt{-i}x)-i\psi(0)(\sqrt{-i}x)&=\int_0^t(i(\psi(\tau))''(\sqrt{-i}x)+V\psi(\tau)(\sqrt{-i}x))\,d\tau\\&=\int_0^t(-(\varphi(\tau))''(x)+V\varphi(\tau)(x))\,d\tau;\end{aligned}$$

here we have used the equalities

$$\varphi'(t) = \psi'(t), \qquad (\varphi(t))'(x) = \exp\left(-\frac{\pi}{4}\right)(\phi(t))'(x), \qquad (\varphi(t))''(x) = -i(\psi(t))''(x),$$
$$i(\psi(t))''(x) - i \cdot iV\psi(t)(x) = -(\varphi(t))''(x) + V\varphi(t)(x). \quad \square$$

$\square$

By definition, the solution of the stochastic heat equation (1) is the solution of the following stochastic integral equation:

$$\Psi_\omega(t)(q) - \Psi_\omega(0)(q) = \int_0^t \alpha\Delta(\Psi_\omega(\tau))(q)\,d\tau + \int_0^t \alpha V(q)\Psi_\omega(\tau))(q)\,d\tau + \int_0^t -\frac{\lambda}{4}q^2(\Psi_\omega(\tau))(q)\,d\tau + \int_0^t \sqrt{\frac{\lambda}{2}}q\Psi_\omega(\tau))(q)\,dB_\omega(\tau). \tag{4}$$

If the function $(t, q, \omega) \mapsto \Psi_\omega(\tau))(q)$ is a solution of this equation, then the function η_μ defined by the equality

$$\eta_\mu(t)(q,\omega) = \Psi_\omega(t)(\sqrt{-\mu}q)$$

is a solution of the following equation:

$$\begin{aligned}\mu(\eta_\mu(t)(q,\omega) - \eta_\mu(0)(q,\omega))&\\ = \int_0^t -\frac{1}{2}\alpha\Delta(\eta_\mu(\tau))(q,\omega)\,d\tau &+ \int_0^t \alpha\mu V(\sqrt{-\mu}q)\eta_\mu(\tau)(q,\omega)\,d\tau\\ - \int_0^t -\frac{\lambda}{4}(\sqrt{-\mu}q)^2(\eta_\mu(\tau))(q,\omega)\,d\tau &- \int_0^t \sqrt{\frac{\lambda}{2}}\sqrt{-\mu}\sqrt{-\mu}q\eta_\mu(\tau)(q,\omega)\,dB_\omega(\tau).\end{aligned} \tag{5}$$

This is verified by differentiation. Thus, the following theorem holds.

Theorem 2. *Let* Ψ *be a solution of the following Cauchy problem*:

$$\begin{gathered}d\Psi_\omega(t)(q) = i\frac{d^2\Psi_\omega(t)(q)}{dq^2}\,dt + \left(iV(q) - \frac{\lambda}{4}q^2\right)\Psi_\omega(t)(q)\,dt + \sqrt{\frac{\lambda}{2}}q\Psi_\omega(t)(q)\,dB_\omega(t),\\ \Psi_\omega(0,\,\cdot\,) = \varphi_0(\,\cdot\,).\end{gathered} \tag{6}$$

Then

$$\begin{aligned}\Psi(t, q_1) = \int \exp\biggl\{\int_0^t iV\left(q_1 + \frac{\xi_1(\tau)}{\sqrt{-i}}\right)d\tau + \int_0^t -i\frac{\lambda}{4}\left(q_1 + \frac{\xi_1(\tau)}{\sqrt{-i}}\right)^2 d\tau\biggr\}&\\ \times \exp\biggl\{\int_0^t i\sqrt{\frac{\lambda}{2}}\left(q_1 + \frac{\xi_1(\tau)}{\sqrt{-i}}\right)dB_\omega(\tau)\biggr\}\varphi_0\left(q_1 + \frac{1}{\sqrt{-i}}\xi_1(t)\right)wf^{-1}\,(d\xi_1)&\end{aligned} \tag{7}$$

(*in the notation of* (3)).

Proof. Let us extend the function $\mu \mapsto \eta_\mu$ to the domain

$$\left\{z \in \mathbb{C} : z = \rho e^{-i\alpha},\ \alpha \in \left(-\frac{\pi}{2}, 0\right],\ \frac{3}{2} > \rho > \frac{1}{2}\right\}$$

and to its closure by continuity. Equation (6) is obtained from (5) by setting $\mu = i$. For an arbitrary real μ, the function η_μ satisfies the Feynman–Kac formula. Further, equality (2) always holds in view of the uniqueness of analytic continuation. By Theorem 1, the solution of Eq. (1) for a real α is the function (2). It is easy to see that the function (7) is obtained from the function (2) by the change specified in the lemma. Then it follows from the lemma that the function (7) is a solution of the Schrödinger–Belavkin equation. $\square$

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