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ORDINARY DIFFERENTIAL EQUATIONS

Asymptotic Behavior of Singular Solutions of Emden–Fowler Type Equations

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Abstract—We study the behavior of singular solutions of the Emden–Fowler type equation $y^{(n)} = p(x, y, y', \dots, y^{(n-1)})|y|^k \operatorname{sgn} y$, $n > 2$, with a regular ($k > 1$) or singular ($0 < k < 1$) nonlinearity. A singular solution is a solution that has a vertical (possibly, resonance) asymptote (for $k > 1$) or a solution that vanishes together with derivatives of order $\le n$ at some point or has a point of accumulation of zeros (for $0 < k < 1$).

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1. INTRODUCTION. STATEMENT OF THE PROBLEM

Consider the Emden–Fowler type equation

$$y^{(n)} = p(x, y, y', \dots, y^{(n-1)})|y|^k \operatorname{sgn} y, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}, \quad n \ge 2, \quad k \in (0, +\infty)\backslash\{1\}, \tag{1}$$

where the function $p : \mathbb{R}^{n+1} \to \mathbb{R}$ is jointly continuous in all the variables, satisfies the inequalities $m \le p(x, \xi_1, \dots, \xi_n) \le M$ with some positive constants m and M, and is Lipschitz continuous in the variables $\xi_1, \dots, \xi_n$. The class of such functions is denoted by $\mathcal{P}_n$. We also consider the equation

$$y^{(n)} = p_0|y|^k \operatorname{sgn} y, \qquad n \ge 2, \qquad k \in (0, +\infty)\backslash\{1\}, \qquad p_0 \in \mathbb{R}\backslash\{0\}, \tag{2}$$

which is a special case of Eq. (1) if $p_0 > 0$.

The monograph [1, Sec. 11] defines singular solutions of the first and second kind. The terminology is as follows. For the regular nonlinearity ($k > 1$), a *singular solution* is either a nonoscillating solution that has a vertical asymptote at the right endpoint x^* of its existence interval or an oscillating solution that has a *resonance* asymptote at that point (i.e., satisfies the conditions $\overline{\lim}_{x\to x^*-0}\, y(x) = +\infty$ and $\underline{\lim}_{\,x\to x^*-0}\, y(x) = -\infty$). For the singular nonlinearity ($0 < k < 1$), a *singular solution* is either a solution that, together with the derivatives of order $\le n$, vanishes at some point or an oscillating solution that has a point of accumulation of zeros.

The present paper considers the asymptotic behavior of singular solutions of Eqs. (1) and (2) near the boundary of their existence intervals or near a point where the uniqueness of solutions is violated. Note that we obtain a complete asymptotic classification of solutions, including a description of singular solutions, of Eq. (1) with $p = p(x)$ for $n = 3$ and of Eq. (2) for $n = 4$.

2. REGULAR NONLINEARITY. CONSTANT-SIGN SOLUTIONS

We say that the function $p(x, \xi_1, \dots, \xi_n)$ satisfies the condition $\mathfrak{A}(x^*, p_0)$ if it tends to the number p_0 as $x \to x^* - 0$, $\xi_1 \to +\infty$, ..., $\xi_n \to +\infty$. It is known for $n \in \{2, 3, 4\}$ that if p satisfies the condition $\mathfrak{A}(x^*, p_0)$, then each positive solution of Eq. (1) with a vertical asymptote at the point x^* has the power-law asymptotics

$$y(x) = C(x^* - x)^{-\alpha}(1 + o(1)) \quad \text{as} \quad x \to x^* - 0, \tag{3}$$

$$\alpha = \frac{n}{k-1}, \tag{4}$$

$$C = \left(\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{p_0}\right)^{1/(k-1)} \tag{5}$$

(see [1, Ch. V] for $n = 2$ and [2; 3, Ch. 5.3] for $n \in \{3, 4\}$).

It was also proved in [4] that for each n there exist positive constants C_1 and C_2 such that any solution with a vertical asymptote at $x = x^*$ satisfies the inequalities $C_1(x^* - x)^{-\alpha} \le y(x) \le C_2(x^* - x)^{-\alpha}$ in a left neighborhood of x^*. Under some additional assumptions about p, the paper [3, Ch. 5.1] establishes that Eq. (1) has a solution with the power-law asymptotics (3) for any n, while for $5 \le n \le 11$ there exists an $(n-1)$-parameter family of such solutions (see also [5]). The natural conjecture that all positive solutions of Eq. (1) with a vertical asymptote at $x = x^*$ exhibit the power-law asymptotic behavior (3) [1, Problem 16.4] is false even for Eq. (2). It was proved in [6] that for any N and $K > 1$ there exist an integer $n > N$ and a real number $k \in (1, K)$ such that Eq. (2) has a solution of the form

$$y = p_0^{-1/(k-1)}(x^* - x)^{-\alpha} h(\ln(x^* - x)), \tag{6}$$

where h is a nonconstant continuous positive periodic function on $\mathbb{R}$. The existence of a $k > 0$ for which Eq. (2) has a solution of the form (6) was proved in [5] for $12 \le n \le 14$ and in [7] for arbitrary $n \ge 12$.

However, Kiguradze's above-mentioned conjecture that solutions with a vertical asymptote have power-law asymptotic behavior is true for weakly nonlinear equations; namely, the following theorem holds.

Theorem 1. *Let $n > 4$, and let $p \in \mathcal{P}_n$ satisfy the condition $\mathfrak{A}(x^*, p_0)$. There exists a $K > 1$ such that if $k \in (1, K)$, then each positive solution of Eq.* (1) *with a vertical asymptote at the point x^* has the power-law asymptotic behavior* (3)–(5).

The proof of Theorem 1 for Eq. (2) can be found in [8] and for Eq. (1) in [9].

To state further results on the properties of solutions of Eq. (2), we need the following definition.

Definition 1. A solution $y(x)$ of an nth-order differential equation is said to be *n-positive at a point x_0* if $y(x_0) > 0$, $y'(x_0) > 0$, ..., $y^{(n-1)}(x_0) > 0$.

Note that if $k > 1$, then any right maximal solution of Eq. (1) that is n-positive at some point has a vertical asymptote at the right end of its existence interval [1, Sec. 11; 2, Ch. 5, Lemma 5.2].

It turns out that the power-law behavior of n-positive solutions may be atypical even for Eq. (2) (see [10]). Namely, the following assertions hold.

Theorem 2. *If the equation*

$$\prod_{j=0}^{n-1}(\lambda + a + j) = \prod_{j=0}^{n-1}(a + j + 1)$$

has no pure imaginary roots and there exists at least one nonunit root with positive real part, then the set of Cauchy data of asymptotically power-law solutions of Eq. (2) *has Lebesgue measure zero for each initial point $x_0 \in \mathbb{R}$.*

Corollary 1. *If the equation*

$$\prod_{j=0}^{n-1}(\lambda + j) = \prod_{j=0}^{n-1}(j + 1)$$

has no pure imaginary roots and there exists at least one nonunit root with positive real part, then there exists a $k_n > 1$ such that for any $k > k_n$ and any initial point $x_0 \in \mathbb{R}$ the set of Cauchy data of asymptotically power-law solutions of Eq. (2) *has Lebesgue measure zero.*

Theorem 3. *For any integer $n \in [12, 203]$, there exists a $k_n > 1$ such that for any real $k > k_n$ the set of Cauchy data of asymptotically power-law solutions of Eq.* (2) *at any point $x_0 \in \mathbb{R}$ has Lebesgue measure zero.*

Note that some singular solutions of higher-order equations were studied in [11]. Equations with a nonpower-law nonlinearity were considered, e.g., in [12].

3. REGULAR AND SINGULAR NONLINEARITIES. SIGN-ALTERNATING SOLUTIONS

The existence of oscillating solutions of Eq. (1) was proved in [1, Sec. 15] (see also [3, Ch. 6]). The behavior of such solutions was described in [3, Ch. 6] for third- and fourth-order equations and in [13–21] for higher-order equations. Let us present a result showing that formula (6), which describes nonpower-law behavior of singular solutions of Eq. (2), can also describe oscillating solutions of this equation if h is a sign-alternating periodic function (see [22]).

Theorem 4. *For any integer $n > 2$ and real $k > 1$, there exists a periodic oscillating function h on $\mathbb{R}$ such that for all $p_0 < 0$ and $x^* \in \mathbb{R}$ the function*

$$y(x) = |p_0|^{-1/(k-1)}(x^* - x)^{-\alpha} h(\ln(x^* - x)) \tag{7}$$

is a solution of Eq. (2) *on $(-\infty, x^*)$.*

Corollary 2. *For any even $n > 2$ and real $k > 1$, there exists a sign-alternating periodic function h such that for all $p_0 < 0$ and $x^* \in \mathbb{R}$ the function*

$$y(x) = |p_0|^{-1/(k-1)}(x - x^*)^{-\alpha} h(\ln(x - x^*)) \tag{8}$$

is a solution of Eq. (2) *on $(x^*, +\infty)$.*

Corollary 3. *For any odd $n > 2$ and real $k > 1$, there exists a sign-alternating periodic function h such that for all $p_0 > 0$ and $x^* \in \mathbb{R}$ the function* (8) *is a solution of Eq.* (2) *on $(x^*, +\infty)$.*

Theorem 5 [23]. *For any integer $n > 2$ and positive $k < 1$, there exists a sign-alternating periodic function h such that for any p_0 satisfying the inequality $(-1)^n p_0 < 0$ and all $x^* \in \mathbb{R}$ the function* (7) *is a solution of Eq.* (2) *on $(-\infty, x^*)$.*

Note that the behavior of oscillatory solutions was also described in detail when producing the asymptotic classification of solutions of third- and fourth-order equations (2) (see [24–27]). The oscillatory solutions turn out to have the form (7) (see [28]).

4. SINGULAR NONLINEARITY. CONSTANT-SIGN SOLUTIONS

Now let $0 < k < 1$. Let us prove an analog of Theorem 1. Set

$$m = n - 1, \qquad \beta = \frac{1-k}{n} = -\frac{1}{\alpha} > 0.$$

Let us extend the definition of the condition $\mathfrak{A}(x^*, p_0)$ by allowing x^* to be infinite. We say that the condition $\mathfrak{A}(+\infty, p_0)$ is satisfied for a function $p(x, \xi_1, \dots, \xi_n)$ if $p(x, \xi_1, \dots, \xi_n)$ tends to p_0 as $x \to +\infty$, $\xi_1 \to +\infty$, ..., $\xi_n \to +\infty$.

Theorem 6. *Let $n \geq 2$, and let $p \in \mathcal{P}_n$. If p satisfies the condition $\mathfrak{A}(+\infty, p_0)$, then there exists a $k_* \in (0, 1)$ such that for any real $k \in (k_*, 1)$ any right maximal solution of Eq.* (1) *that is n-positive at some point has the power-law asymptotic behavior*

$$y(x) = \left(p_0 \beta^n \prod_{l=1}^{m} (1 - \beta l)^{-1}\right)^{1/(1-k)} x^{1/\beta}(1 + o(1)) \quad \text{as} \quad x \to +\infty. \tag{9}$$

Proof. Let us construct a dynamical system with parameter β on the m-dimensional sphere $\mathbb{S}^m$. This sphere will be viewed as the quotient space of the space $\mathbb{R}^n \setminus \{\mathbf{0}\}$ by the equivalence relation

$$(z_0, \dots, z_m) \sim (\lambda z_0, \dots, \lambda z_m), \qquad \lambda > 0.$$

The equivalence class of a point $(z_0, \dots, z_m) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ will be denoted by $(z_0 : \dots : z_m)$. Any nontrivial solution $y(x)$ of Eq. (2) with $p_0 = 1$ generates the curve in $\mathbb{S}^m$ formed by the points

$$\left(y(x) : \left| \frac{y'(x)}{a_1} \right|^{1/(1-\beta)} \operatorname{sgn} y'(x) : \dots : \left| \frac{y^{(j)}(x)}{a_j} \right|^{1/(1-\beta j)} \operatorname{sgn} y^{(j)}(x) : \dots : \left| \frac{y^{(m)}(x)}{a_m} \right|^{1/(1-\beta m)} \operatorname{sgn} y^{(m)}(x) \right),$$

where $x \in \operatorname{dom} y$ and

$$a_1 = \prod_{l=1}^{m} (1 - \beta l)^{-1/n}, \qquad a_{j+1} = (1 - \beta j) a_1 a_j = a_1^{j+1} \prod_{l=1}^{j} (1 - \beta l), \qquad j = 1, \dots, m-1. \tag{10}$$

In the chart that covers the part of $\mathbb{S}^m_+ \subset \mathbb{S}^m$ where all z_j are positive and which has the coordinate functions $v_j : (z_0 : \dots : z_m) \mapsto (z_j/z_0)^{1-\beta j}$, $j = 1, \dots, m$, this curve, when locally parametrized by the variable

$$\tau = a_1 \int_{x_0}^{x} y(\xi)^{-\beta} d\xi,$$

can be described by the system of equations

$$\begin{aligned} \frac{dv_1}{d\tau} &= (1 - \beta)(v_2 - v_1^2), \\ \frac{dv_j}{d\tau} &= (1 - \beta j)(v_{j+1} - v_1 v_j), \qquad j = 2, \dots, m-1, \\ \frac{dv_m}{d\tau} &= (1 - \beta m)(1 - v_1 v_m). \end{aligned} \tag{11}$$

Once such a trajectory enters the domain $\mathbb{S}^m_+$, it remains there forever. We denote the unique fixed point of system (11) in $\mathbb{S}^m_+$ by v^*. (All the coordinates of this point are equal to unity.) Similar formulas define the curve in other charts, which, taken together, cover the entire sphere. Different variables parametrizing the curve in different charts can be merged into one variable with the use of a partition of unity. In this way, we arrive at a dynamical system $\mathfrak{S}$ depending on a parameter β and globally defined on the entire sphere $\mathbb{S}^m$.

We need the following three lemmas.

Lemma 1. *There exist numbers $\beta_1 > 0$ and $r > 0$ such that for all $\beta \in [0, \beta_1]$ the Jacobian matrix of system* (11) *at the point $v^* = (1, \dots, 1)$ has m distinct eigenvalues with negative real parts and with absolute value $\geq r$.*

Proof. This Jacobian matrix is an $m \times m$ matrix, and for $\beta = 0$ it has the form

$$\begin{pmatrix} -2 & 1 & 0 & \dots & 0 & 0 \\ -1 & -1 & 1 & \dots & 0 & 0 \\ -1 & 0 & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -1 & 0 & 0 & \dots & -1 & 1 \\ -1 & 0 & 0 & \dots & 0 & -1 \end{pmatrix}.$$

Let us prove by induction on $m \in \mathbb{N}$ that its characteristic polynomial $P_m(\lambda)$ can be represented in the form

$$P_m(\lambda) = \frac{(1+\lambda)^{m+1} - 1}{(-1)^m \lambda}. \tag{12}$$

For $m = 1$, we have

$$P_1(\lambda) = -2 - \lambda = -\frac{(1+\lambda)^2 - 1}{\lambda} = \frac{(1+\lambda)^{1+1} - 1}{(-1)^1 \lambda}.$$

Assume that (12) has been proved for some positive integer m. To compute $P_{m+1}(\lambda)$, we expand the corresponding Jacobian by the last row,

$$P_{m+1}(\lambda) = (-1)(-1)^m + (-1-\lambda)P_m(\lambda) = (-1)^{m+1} - (1+\lambda)\frac{(1+\lambda)^{m+1} - 1}{(-1)^m \lambda} = \frac{(1+\lambda)^{m+2} - 1}{(-1)^{m+1} \lambda},$$

which implies relation (12) for m replaced with $m + 1$.

The roots of the polynomial $P_m(\lambda)$ are

$$\lambda_j = -1 + \cos\frac{2\pi j}{n} + i \sin\frac{2\pi j}{n}, \qquad j = 1, \ldots, m,$$

where the value $j = 0$ is omitted in view of the denominator in the representation (12).

The real parts of these roots do not exceed

$$-1 + \cos\frac{2\pi}{n} = -1 + \cos\frac{2\pi m}{n} = -2\sin^2\frac{\pi}{n}.$$

Since all the roots of the polynomial are distinct and hence simple, it follows from the implicit function theorem that they continuously depend on the coefficients of the polynomial. Hence the real parts of all eigenvalues of the Jacobian matrix of system (11) at the point $(1, \ldots, 1)$ are smaller than $-\sin^2(\pi/n)$ for sufficiently small $\beta > 0$. This proves the lemma.

Lemma 2. *There exists a $\beta_2 > 0$ and an open neighborhood U of the point v^* such that for all positive $\beta < \beta_2$ any trajectory of the dynamical system $\mathfrak{S}$ passing through the closure $\overline{U}$ tends to v^*. If such a trajectory does not coincide with v^*, then it crosses the boundary ∂U transversally at some time.*

Proof. Let us change the local coordinates from $(v_j)_{1\le j\le m}$ to $(w_j)_{1\le j\le m}$ to describe the system $\mathfrak{S}$ on $\mathbb{S}_+^m$ more conveniently. To this end, first we apply a shift that continuously depends on β and moves the fixed point to the point $\mathbf{0}$. Then we make a $\mathbb{C}$-linear transformation continuously depending on β such that the linearization of the right-hand side of the system is given by a diagonal matrix. In the new (complex) coordinates, our system is written in the form

$$\frac{dw_j}{d\tau} = \lambda_j(\beta) w_j + q_j(w, \beta), \qquad j = 1, \ldots, m,$$

with continuous functions $q_j(w, \beta)$ quadratic in β.

Let Q be a positive constant such that $|q_j(w,\beta)|^2 \le Q|w|^2$ for all $j \in \{1, \ldots, m\}$, any $w \in \mathbb{C}^m$, and any positive $\beta \le \beta_1$, where β_1 is the constant in Lemma 1 and $|w|^2 = \sum_{j=1}^m |w_j|^2$.

The τ-derivative of the quadratic function $|w|^2$ can be estimated as

$$\frac{d|w|^2}{d\tau} = 2\sum_{j=1}^m \operatorname{Re}\left(\lambda_j(\beta)|w_j|^2 + q_j(w,\beta)\overline{w}_j\right) < 2|w|^2(-r + Q|w|),$$

where $r > 0$ is the constant in Lemma 1. Thus,

$$\frac{d\ln|w|^2}{d\tau} < -r \quad \text{for} \quad |w| < \frac{r}{2Q}.$$

The last inequality specifies the desired neighborhood U of the fixed point v^*. One has $\ln|w|^2 \to -\infty$ as $\tau \to +\infty$ on any trajectory passing through the closure $\overline{U}$; i.e., all such trajectories tend to v^*.

Since the function $\ln|w|^2$ is defined at all points of the set $\overline{U}\setminus\{v^*\}$, we see that the above estimate of the derivative $d\ln|w|^2/d\tau$ proves the last assertion of the lemma. The proof of the lemma is complete.

Now consider a solution $y(x)$ of Eq. (1) assuming that $p \to 1$ as $x \to +\infty$, $\xi_1 \to +\infty$, $\dots$, $\xi_n \to +\infty$. This solution defines a curve in $\mathbb{S}^m$, which is described in the same chart by the system

$$\begin{aligned} \frac{dv_1}{d\tau} &= (1-\beta)(v_2 - v_1^2), \\ \frac{dv_j}{d\tau} &= (1-\beta j)(v_{j+1} - v_1 v_j), \qquad j \in \{2,\dots,m-1\}, \\ \frac{dv_m}{d\tau} &= (1-\beta m)(q(\tau) - v_1 v_m), \end{aligned} \tag{13}$$

where the function $q(\tau)$ is obtained by an appropriate substitution into the function p and tends to unity as $\tau \to +\infty$.

Lemma 3. *The set of all ω-limit points of a trajectory described by system* (13) *with a function $q(\tau)$ tending to unity as $\tau \to +\infty$ is the union of some entire trajectories of the system $\mathfrak{S}$.*

Proof. Let $\hat v$ be a limit point of a trajectory $v(\tau)$ of system (13). Then there exists a sequence $\{\tau_j\}$ such that $v(\tau_j) \to \hat v$ and $\tau_j \to +\infty$ as $j \to +\infty$. Let us draw a trajectory of system (11) through $\hat v$ and show that any of its points is a limit point of the trajectory $v(\tau)$ as well. Let $\tilde v(\tau)$ be the solution of system (11) with $\tilde v(0) = \hat v$, let $\tilde v(T) = \check v$, and assume that the point $\check v$ is not a limit point of $v(\tau)$. Then there exists a number $\varepsilon > 0$ such that the inequality $|v(\tau) - \check v| > \varepsilon$ is satisfied for all sufficiently large τ.

On the other hand, by the theorem on the continuous dependence of the solution on the initial data and the right-hand side, there exists a number $\delta > 0$ such that if $|q(\tau) - 1| < \delta$, then $|v(\tau_0 + T) - \tilde v(T)| < \varepsilon$ for any trajectory $v(\tau)$ of system (13) satisfying the condition $|v(\tau_0) - \hat v| < \delta$ at some point τ_0.

Therefore, we arrive at a contradiction by taking τ_* from the sequence $\{\tau_j\}$ to be large enough that $|v(\tau_*) - \hat v| < \delta$ and $|q(\tau) - 1| < \delta$ for all $\tau > \min(\tau_*, \tau_* + T)$. The proof of the lemma is complete.

Since the sphere $\mathbb{S}^m$ is compact, it follows that each trajectory $s(\tau)$ on it has at least one ω-limit point. If this ω-limit point is unique, then it is the limit of points of the trajectory as $\tau \to +\infty$. Thus, if a trajectory does not tend to the point v^*, then it necessarily contains at least one ω-limit point $v^{**} \neq v^*$. If the trajectory $s(\tau)$ is defined by a solution of Eq. (1) tending to $+\infty$ as $x \to +\infty$, then we can assume that $v^{**} \in \mathbb{S}^m_+$. By Lemma 2, the trajectory $s_1(\tau)$ of the system $\mathfrak{S}$ through the point v^{**} crosses ∂U transversally for some $\beta \in (0, \beta_2)$. If the function $q(\tau)$ is sufficiently close to unity, then the trajectory $s(\tau)$ crosses ∂U transversally as well. In this case, it can enter U but can never leave it. Hence the points of $s_1(\tau)$ outside U cannot be ω-limit points of $s(\tau)$. This contradiction with Lemma 3 shows that $s(\tau) \to v^*$ as $\tau \to +\infty$. In particular,

$$v_1 = (z_1/z_0)^{1-\beta} \to 1 \quad \text{as} \quad \tau \to +\infty,$$

which implies that the corresponding solution $y(x)$ of Eq. (1) satisfies the condition

$$\frac{y'}{a_1 y^{1-\beta}} \to 1 \quad \text{as} \quad x \to +\infty.$$

Hence $(y^\beta)' \sim a_1\beta$ as $x \to +\infty$, and we obtain $y \sim (a_1\beta x)^{1/\beta} = C_\beta x^{1/\beta}$, where

$$C_\beta = (a_1\beta)^{1/\beta} = \beta^{1/\beta}\prod_{l=1}^{m}(1-\beta l)^{-1/(1-k)} = \left(\beta^n \prod_{l=1}^{m}(1-\beta l)^{-1}\right)^{1/(1-k)}$$

in view of (10).

This proves Theorem 6 for the case of $p_0 = 1$.

If y is a solution of Eq. (1) with a function p tending to some number $p_0 > 0$, $p_0 \neq 1$, as $x \to +\infty$, $\xi_1 \to +\infty$, ..., $\xi_n \to +\infty$, then the function ${p_0}^{-1/(1-k)} y$ is a solution of the same equation (1) with a different but similar function p tending to unity. Thus, ${p_0}^{-1/(1-k)} y \sim C_\beta x^{1/\beta}$, and hence relation (9) holds. The proof of Theorem 6 is complete.

5. ASYMPTOTIC CLASSIFICATION OF SOLUTIONS OF THIRD- AND FOURTH-ORDER EQUATIONS WITH A SINGULAR NONLINEARITY

A complete asymptotic classification of solutions has been obtained for the second-, third-, and fourth-order equations. In particular, a more accurate asymptotic representation of singular solutions has been indicated in all of these cases. This classification can be found in [1, Ch. V; 29] for second-order equations with various restrictions on the function p and in [3, Ch. 7; 25] for third- and fourth-order equations with a regular nonlinearity. The papers [14–16] deal with various aspects of qualitative behavior of solutions of third- and fourth-order equations. In the case of the regular nonlinearity ($k > 1$), the asymptotic classification only deals with maximally extended solutions, because the solutions can only have singular behavior near the endpoints of their existence intervals. For $k < 1$, singular behavior can also be observed at an interior point of the existence interval. This necessitates introducing the following definition to classify these solutions.

Definition 2. A *maximally unique solution*, or a *μ-solution*, is a solution $y\colon (a, b) \to \mathbb{R}$, where $-\infty \le a < b \le +\infty$, such that the following two conditions are satisfied:

(i) The equation has no solution that coincides with y on some subinterval of (a, b) and is not equal to y at some point in (a, b).

(ii) The equation either has no solutions defined on another interval containing (a, b) and equal to y on (a, b) or has at least two such solutions that are not equal to each other at points arbitrarily close to the boundary of the interval (a, b).

Note that the assumptions of the classical solution uniqueness theorem for the Cauchy problem are not satisfied in this case. However, the following assertion holds [2, Ch. 7.3].

Theorem 7. *Let $p \in \mathcal{P}_n$. Then the Cauchy problem*

$$y^{(i)}(x_0) = y_i^0, \qquad i = 0, \dots, n-1,$$

for Eq. (1) *has a unique solution for any numbers $x_0, y_0^0, \dots, y_{n-1}^0$ such that at least one y_i^0 is nonzero.*

Set $\gamma = n/(1-k) > 0$.

Theorem 8. *Let $n = 3$, let $0 < k < 1$, and let $p(x, \xi_1, \dots, \xi_n) = p(x)$, where $p(x)$ is a nonnegative continuous function defined on the entire real line $\mathbb{R}$ and having finite positive limits p^* and p_* as $x \to \pm\infty$, respectively.*

Then any μ-solution of Eq. (1) *is one of the following solutions.*

1. *A constant-sign solution with asymptotically power-law behavior on $(x^*, +\infty)$; namely,*

$$\begin{aligned} &y(x) = \pm C(p(x^*))(x - x^*)^\gamma (1 + o(1)) \quad \text{as} \quad x \to x^* + 0, \\ &y(x) = \pm C(p^*) x^\gamma (1 + o(1)) \quad \text{as} \quad x \to +\infty, \end{aligned} \tag{14}$$

where

$$C(p) = \left(\frac{(1-k)^3 p}{3(k+2)(2k+1)} \right)^{1/(1-k)}.$$

2. *A solution oscillating on $(-\infty, x_*)$ whose points x_j, $j \in \mathbb{Z}$, of local extremum satisfy the conditions*

$$\begin{aligned} &x_j \to -\infty, \quad |y(x_j)| = |x_j|^{\gamma + o(1)} \quad \text{as} \quad j \to -\infty, \\ &x_j \to x_* - 0, \quad |y(x_j)| = |x_* - x_j|^{\gamma + o(1)} \quad \text{as} \quad j \to +\infty. \end{aligned} \tag{15}$$

3. *A solution that satisfies relations* (14) *as* $x \to +\infty$ *and* (15) *as* $x \to -\infty$ *and does not vanish together with its derivatives* y' *and* y'' *at any point.*

Remark. If $p_0 > 0$ in Eq. (2), then each solution of the form 2 in Theorem 8 can be written as follows on the interval $(-\infty, x^*)$:

$$y(x) = (x^* - x)^\gamma h(\log(x^* - x))$$

with some nonconstant oscillating periodic function h (see Theorem 5). See also [21, Th. 16].

Theorem 9. *Let* $n = 4$, *let* $0 < k < 1$, *and let* $p_0 < 0$. *Then all* μ*-solutions of Eq.* (2) *can be divided into three types according to their asymptotic behavior*:

1. *Oscillating solutions defined on the half-line* $(-\infty, b)$. *The distance between their neighboring zeros grows infinitely as* $x \to -\infty$ *and tends to zero as* $x \to b$. *These solutions* y *and their derivatives* $y^{(j)}$ *satisfy the conditions*

$$\lim_{x\to b} y^{(i)}(x) = 0 \quad \textit{and} \quad \overline{\lim_{x\to-\infty}}\, |y^{(i)}(x)| = +\infty, \quad i = 0, \dots, 4.$$

The following estimates hold at the points of local extremum:

$$C_1|x - b|^\gamma \le |y(x)| \le C_2|x - b|^\gamma \tag{16}$$

with positive constants C_1 *and* C_2 *depending on* k *and* p_0 *alone.*

2. *Oscillating solutions defined on the half-line* $(b, +\infty)$. *The distance between their neighboring zeros tends to zero as* $x \to b$ *and grows infinitely as* $x \to +\infty$. *These solutions* y *and their derivatives* $y^{(j)}$ *satisfy the conditions*

$$\lim_{x\to b} y^{(i)}(x) = 0 \quad \textit{and} \quad \overline{\lim_{x\to+\infty}}\, |y^{(i)}(x)| = +\infty, \quad i = 0, \dots, 4.$$

The estimates (16) *hold with positive constants* C_1 *and* C_2 *depending on* k *and* p_0 *alone at the points of local extremum.*

3. *Oscillating solutions defined on* $\mathbb{R}$. *These solutions* y *and their derivatives* $y^{(j)}$ *satisfy the relations*

$$\overline{\lim_{x\to-\infty}}\, |y^{(i)}(x)| = \overline{\lim_{x\to+\infty}}\, |y^{(i)}(x)| = \infty, \qquad i = 0, \dots, 4.$$

The following estimates hold at the points of local extremum that are sufficiently large in absolute value:

$$C_1|x|^\gamma \le |y(x)| \le C_2|x|^\gamma$$

with positive constants C_1 *and* C_2 *depending on* k *and* p_0 *alone.*

Theorem 10. *Let* $n = 4$, *let* $0 < k < 1$, *and let* $p_0 > 0$. *Then all* μ*-solutions of Eq.* (2) *are divided into* 13 *types according to their asymptotic behavior*:

1–2. *Solutions defined on the half-line* $(b, +\infty)$ *with power-law asymptotic behavior near the boundaries of the existence interval* (*with the same signs* $\pm$):

$$y(x) \sim \pm C_{4k}(x - b)^\gamma \quad \textit{as} \quad x \to b + 0, \qquad y(x) \sim \pm C_{4k}x^\gamma \quad \textit{as} \quad x \to +\infty,$$

where

$$C_{4k} = \left(\frac{4(k+3)(2k+2)(3k+1)}{p_0(k-1)^4}\right)^{1/(k-1)}.$$

3–4. *Solutions defined on the half-line* $(-\infty, b)$ *with power-law asymptotic behavior near the boundaries of the existence interval* (*with the same signs* $\pm$):

$$y(x) \sim \pm C_{4k}|x|^\gamma \quad \textit{as} \quad x \to -\infty, \qquad y(x) \sim \pm C_{4k}(b - x)^\gamma \quad \textit{as} \quad x \to b - 0.$$

5. *Periodic oscillating solutions defined on the entire line* $\mathbb{R}$. *All of those can be derived from one solution, say,* $z(x)$ *using the relation* $y(x) = \lambda^4 z(\lambda^{k-1}x + x_0)$ *with arbitrary* $\lambda > 0$ *and* x_0.

Thus, there may exist such solutions with an arbitrary maximum $h > 0$ *and an arbitrary period* $T > 0$ *but not with an arbitrary pair* (h, T).

6–7. *Solutions defined on* $\mathbb{R}$ *that are oscillating at* $-\infty$ *and have power-law asymptotic behavior at* $+\infty$:

$$y(x) \sim \pm C_{4k} x^{\gamma} \quad \text{as} \quad x \to +\infty.$$

For every solution of this type, there exists a finite limit of the absolute values of its local extrema as $x \to -\infty$.

8–9. *Solutions defined on* $\mathbb{R}$ *that are oscillating at* $+\infty$ *and have power-law asymptotic behavior at* $-\infty$:

$$y(x) \sim \pm C_{4k} |x|^{\gamma} \quad \text{as} \quad x \to -\infty.$$

For every solution of this type, there exists a finite limit of the absolute values of its local extrema as $x \to +\infty$.

10–13. *Solutions defined on* $\mathbb{R}$ *that exhibit power-law asymptotic behavior at* $-\infty$ *and* $+\infty$ (*with four possible pairs of signs* $\pm$):

$$y(x) \sim \pm C_{4k}(p(b)) |x|^{\gamma} \quad \text{as} \quad x \to \pm\infty.$$

This asymptotic classification supplements the results on the behavior of singular solutions of higher-order equations.

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