Cosmological models with the Gauss–Bonnet term and non-minimally coupled scalar fields

S.Yu. Vernov

Skobeltsyn Institute of Nuclear Physics,
Lomonosov Moscow State University

based on E.O. Pozdeeva, M. Sami, A.V. Toporensky and S.Yu. Vernov, arXiv:1905.05085 [gr-qc].

INTERNATIONAL BOGOLYUBOV CONFERENCE

Moscow, September 9, 2019

THE GAUSS–BONNET TERM

The Gauss–Bonnet term is

$$\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}.$$

The integral

$$S_{\mathcal{G}} = \int d^D x\sqrt{-g}\mathcal{G},$$

is a full derivative at $D = 4$.
So, if we add $S_{\mathcal{G}}$ to a four-dimension action, the equations do not change.
Let us consider, the action

$$S_{\mathcal{R}} = \int d^4x\sqrt{-g}\left(AR^2 + BR_{\mu\nu}R^{\mu\nu} + CR_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right), \qquad (1)$$

where A, B and C are arbitrary constants. You can always put one of these constants is equal to zero without changing the equations of motion. For example, subtracting $CS_{\mathcal{G}}$ we get the action

$$\tilde{S}_{\mathcal{R}} = \int d^4x\sqrt{-g}\left((A - C)R^2 + (B + 4C)R_{\mu\nu}R^{\mu\nu}\right), \qquad (2)$$

that yields the same equations of motion as $S_{\mathcal{R}}$.

There are two basic motivations which lead cosmologists to modify gravity.

The first one is an attempt to connect gravity with quantum physics, at least in a perturbative way, by including quantum correction terms to Einstein's equations.

MODIFIED GRAVITY MODELS WITH THE GAUSS–BONNET TERM

There are two basic motivations which lead cosmologists to modify gravity.

The first one is an attempt to connect gravity with quantum physics, at least in a perturbative way, by including quantum correction terms to Einstein's equations.

The second one is an interest to describe the Universe evolution in a more natural way, without the dark energy and the dark matter components, which turn out to be avoidable in the modified models.

The Gauss–Bonnet models are motivated by α' corrections in string theories. The most general Lagrangian density at the next to leading order in the Regge slope α' reads[1]:

$$L_{string} = -\frac{\lambda}{2}\alpha'\xi(\phi)\left[c_1\mathcal{G} + c_2 G^{\mu\nu}\partial_\mu\phi\partial_\mu\phi + c_3\Box\phi\phi^{;\mu}\phi_{;\mu} + c_4(\phi^{;\mu}\phi_{;\mu})^2\right],$$

where

- $G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$ is the Einstein tensor and $\mathcal{G}$ is the Gauss–Bonnet combination;
- $\alpha' = \lambda_s^2$, where λ_s is the fundamental string length scale;
- c_i are constants (we will consider the case $c_k = 0$, $k = 2, 3, 4$);
- λ is an additional parameter allowing for different species of string theories, $\lambda = -1/4$ for the Bosonic string and $\lambda = -1/8$ for Heterotic string respectively.

[1] D.J. Gross and J.H. Sloan, Nucl. Phys. B **291** (1987) 41;
R.R. Metsaev and A.A. Tseytlin, Nucl. Phys. B **293** (1987) 385.

THE EINSTEIN–GAUSS–BONNET GRAVITY

The model with the Gauss–Bonnet term in a general background described by the following action,

$$S = \int d^4x\sqrt{-g}\left(U(\phi)R - \frac{K}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) - F(\phi)\mathcal{G}\right), \qquad (3)$$

where U, V, and F are differentiable functions and $K = -1, 0, 1$.
Let us consider the action

$$S_{f\mathcal{G}} = \int d^4x\sqrt{-g}f(\mathcal{G}), \qquad (4)$$

where f is a differentiable function. Action $S_{f\mathcal{G}}$ can be linearized with respect to the Gauss–Bonnet term, by adding one more scalar field in the action[2]. Introducing a field ϕ with $K = 0$, we obtain the following action:

$$S_{GB\phi} = \int d^4x\sqrt{-g}\left[\left[\frac{df}{d\phi}(\mathcal{G} - \phi) + f(\phi)\right]\right].$$

Varying over ϕ, one gets $\phi = \mathcal{G}$ and reconstruct $S_{f\mathcal{G}}$.

[2] G. Cognola, E. Elizalde, S. Nojiri, S.D. Odintsov and S. Zerbini, Phys. Rev. D **73** (2006) 084007, [arXiv:hep-th/0601008]

FOUR EPOCHS

Reliable astronomical data support the existence of four distinct epochs of the Universe global evolution:

- an inflation,
- a radiation dominated era,
- a matter dominated era,
- the present dark energy epoch.

FOUR EPOCHS

Reliable astronomical data support the existence of four distinct epochs of the Universe global evolution:

- an inflation,
- a radiation dominated era,
- a matter dominated era,
- the present dark energy epoch.

Initial inflation and dark energy domination are both characterized by an accelerated expansion of the Universe with almost constant Hubble parameter H.
The other epochs of the Universe evolution are described by power-law solutions with $H = J/t$, where J is a positive constant.
In General Relativity, power-law solutions with $H = J/t$ correspond to models with a perfect fluid whose EoS parameter reads
$w_{\rm m} = -1 + 2/(3J)$.
The radiation dominated epoch corresponds to solutions with $J = 1/2$, whereas the matter dominated one corresponds to $J = 2/3$.

INFLATIONARY MODELS

The perturbation theory for such types of models has been developed in C. Cartier, J. c. Hwang and E. J. Copeland, *Evolution of cosmological perturbations in nonsingular string cosmologies*, Phys. Rev. D **64** (2001) 103504 [astro-ph/0106197];
J. c. Hwang and H. Noh, *Classical evolution and quantum generation in generalized gravity theories including string corrections and tachyon: Unified analysis*, Phys. Rev. D **71** (2005) 063536 [gr-qc/0412126]
Inflationary models have been proposed:
C. van de Bruck and C. Longden, Phys. Rev. D **93** (2016) 063519 [arXiv:1512.04768]
K. Nozari and N. Rashidi, Phys. Rev. D **95** (2017) 123518 [arXiv:1705.02617]
S.D. Odintsov and V.K. Oikonomou, Phys. Rev. D **98** (2018) 044039 [arXiv:1808.05045]

DARK ENERGY MODELS

Models the Gauss–Bonnet term successfully generate a dark energy era.
G. Calcagni, S. Tsujikawa and M. Sami, Class. Quant. Grav. **22** (2005) 3977 [arXiv:hep-th/0505193]
S. Tsujikawa and M. Sami, J. Cosmol. Astropart. Phys. **0701** (2007) 006 [arXiv:hep-th/0608178]
S. Nojiri, S.D. Odintsov and M. Sasaki, Phys. Rev. D **71** (2005) 123509 [arXiv:hep-th/0504052]

Models the Gauss–Bonnet term successfully generate a dark energy era.
G. Calcagni, S. Tsujikawa and M. Sami, Class. Quant. Grav. **22** (2005) 3977 [arXiv:hep-th/0505193]
S. Tsujikawa and M. Sami, J. Cosmol. Astropart. Phys. **0701** (2007) 006 [arXiv:hep-th/0608178]
S. Nojiri, S.D. Odintsov and M. Sasaki, Phys. Rev. D **71** (2005) 123509 [arXiv:hep-th/0504052]
Two examples are presented in S.D. Odintsov, V.K. Oikonomou and S. Banerjee, Nuclear Physics B **938** (2019) 935, arXiv:1807.00335

$$\mathrm{I)}\ f_1(\mathcal{G}) \quad = \quad -F_0(1 - e^{\mathcal{G}/\mathcal{G}_0});$$

$$\mathrm{II)}\ f_2(\mathcal{G}) \quad = \quad -F_0(1 - e^{-|\mathcal{G}|/\mathcal{G}_0}),$$

where $\mathcal{G}_0$ corresponds to the present value of the Gauss–Bonnet scalar, F_0 is a constant.

INFLATIONARY MODELS WITH NON-MINIMALLY COUPLED SCALAR FIELDS

Models with non-minimally coupled scalar fields are interesting because of their connection with the particle physics.

Generic quantum corrections to the action of the scalar field minimally coupled to gravity include the term, proportional to $\phi^2 R$.

N.A. Chernikov, E.A. Tagirov, Annales Poincare Phys. Theor. A **9** (1968) 109.

There are models of inflation, where the role of the inflaton is played by the Higgs field non-minimally coupled to gravity.

The first inflationary model with the SM Higgs boson

J.L. Cervantes-Cota and H. Dehnen, *Induced gravity inflation in the standard model of particle physics,* *Nucl. Phys.* **B442** (1995) 391 [astro-ph/9505069] was not realistic.

The inflationary parameters obtained in

F.L. Bezrukov and M. Shaposhnikov, *The Standard Model Higgs boson as the inflaton*, *Phys. Lett.* B **659** (2008) 703–706 [arXiv:0710.3755] are in good agreement with the most recent and accurate observational data (**PLANCK'2018**).

Let us consider firstly model without the Gauss–Bonnet term, describing by the following action:

$$S=\int d^4x\sqrt{-g}\left[U(\sigma)R-\frac{1}{2}g^{\mu\nu}\sigma_{,\mu}\sigma_{,\nu}-V(\sigma)\right],$$

where $U(\sigma)$ and $V(\sigma)$ are differentiable functions.
For the spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) metric with the interval

$$ds^2=\,-\,dt^2+a^2(t)\left(dx_1^2+dx_2^2+dx_3^2\right),$$

the evolution equations are

$$6UH^2+6\dot{U}H=\frac{1}{2}\dot{\sigma}^2+V, \tag{5}$$

$$2U\left(2\dot{H}+3H^2\right)+4\dot{U}H+2\ddot{U}+\frac{1}{2}\dot{\sigma}^2-V=0, \tag{6}$$

$$\ddot{\sigma}+3H\dot{\sigma}+V'=6\left(\dot{H}+2H^2\right)U', \tag{7}$$

where the Hubble parameter $H=\dot{a}/a$.

Let us rewrite equations (5)–(7) in the form similar to the Friedmann equations in the Einstein frame. We introduce a new variables

$$P \equiv \frac{H}{\sqrt{U}} + \frac{U'\dot{\sigma}}{2U\sqrt{U}}, \qquad V_{eff} = \frac{V}{2U^2}.$$

In terms of P we get the following equations

$$3P^2 = \frac{U + 3{U'}^2}{4U^3}\dot{\sigma}^2 + \frac{V}{2U^2} = A\dot{\sigma}^2 + V_{eff}, \tag{8}$$

$$\dot{P} = -A\sqrt{U}\,\dot{\sigma}^2. \tag{9}$$

$$\ddot{\sigma} = -3P\sqrt{U}\dot{\sigma} - \frac{A'}{2A}\dot{\sigma}^2 - \frac{V'_{eff}}{2A}. \tag{10}$$

where $A \equiv (U + 3{U'}^2)/(4U^3)$. If $U(\sigma) > 0$, then $A(\sigma) > 0$.
From Eq. (10) it is clear that de Sitter solutions corresponds to $V'_{eff}(\sigma_{dS}) = 0$.
It has been shown that $V''_{eff}(\sigma_{dS}) > 0$ corresponds to stable de Sitter solutions if $U(\sigma_{dS}) > 0$. M.A. Skugoreva, A.V. Toporensky and S.Yu. Vernov, Phys. Rev. D **90** (2014) 064044 [arXiv:1404.6226].
Similar functions have been obtained in L. Jarv, P. Kuusk, M. Saal and O. Vilson, Class. Quant. Grav. **32** (2015) 235013 [arXiv:1504.02686] by the method of invariant quantities.

MODELS WITH STANDARD SCALAR FIELDS

Let us consider the case $K = 1$:

$$S = \int d^4x\sqrt{-g}\left(U(\phi)R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) - F(\phi)\mathcal{G}\right). \qquad (11)$$

In the spatially flat FLRW universe with the interval

$$ds^2 = -\,dt^2 + a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right),$$

one gets the following equations

$$\ddot{\varphi} + 3H\dot{\varphi} - 6(\dot{H} + 2H^2)U' + V' + 24H^2F'\left(\dot{H} + H^2\right) = 0, \qquad (12)$$

$$6H^2U + 6HU'\dot{\varphi} = \frac{1}{2}\dot{\varphi}^2 + V + 24H^3F'\dot{\varphi}, \qquad (13)$$

$$4\left(U - 4H\dot{F}\right)\dot{H} = -\,\dot{\varphi}^2 - 2\ddot{U} + 2H\dot{U} + 8H^2\left(\ddot{F} - H\dot{F}\right), \qquad (14)$$

where $H = \dot{a}/a$ is the Hubble parameter, primes mean the derivatives with respect to φ and dots mean the derivatives with respect to the cosmic time.

DE SITTER SOLUTUIONS

Let us find de Sitter solutions in the model with the Gauss–Bonnet term and compare them with de Sitter solutions in the corresponding model without the Gauss–Bonnet term.
We restrict ourselves to de Sitter solutions with a constant ϕ.
Substituting $\phi = \phi_{dS}$ and $H = H_{dS}$ into Eqs. (12) and (13), we get:

$$6H_{dS}^2 U_{dS} = V_{dS}, \tag{15}$$

$$F'_{dS} = \frac{3U_{dS}(2U'_{dS}V_{dS} - V'_{dS}U_{dS})}{2V_{dS}^2}. \tag{16}$$

where $V_{dS} = V(\phi_{dS})$, $U_{dS} = U(\phi_{dS})$, and $F_{dS} = F(\phi_{dS})$.
The value of the Hubble parameter at the de Sitter point is the same as in the corresponding model without the Gauss–Bonnet term:

$$H_{dS}^2 = \frac{V_{dS}}{6U_{dS}}. \tag{17}$$

For arbitrary functions U and V with $VU > 0$, we can choose $F(\phi)$ such that the corresponding point becomes a de Sitter solution with the Hubble parameter defined by (17) and the value of $F'(\phi_{dS})$ is fixed by (16).

THE EFFECTIVE POTENTIAL

It would be convenient to obtain position and stability of de Sitter solutions using only one combination of three functions: U, V, and F. To get this combination (the effective potential) we cast Eqs. (12) and (14) as a dynamical system:

$$
\begin{aligned}
\dot{\phi} =& \psi, \\
\dot{\psi} =& \frac{1}{2\left(B - 4F'H\psi\right)} \left\{ 2H\left[3B + 4F'V' - 6U'^2 - 6U\right]\psi - 2\frac{V^2}{U}X \right. \\
& \left. + \left[12\left[(2U''+3)F' + 2U'F''\right]H^2 - 96F'F''H^4 - 3(2U''+1)U'\right]\psi^2 \right\}, \\
\dot{H} =& \frac{8\left(U' - 4F'H^2\right)H\psi - 2\frac{V^2}{U^2}\left(4F'H^2 - U'\right)X + \left(8F''H^2 - 2U'' - 1\right)\psi^2}{4\left(B - 4F'H\psi\right)},
\end{aligned}
$$

where

$$
B = 3\left(4H^2F' - U'\right)^2 + U, \quad X = \frac{U^2}{V^2}\left[24F'H^4 - 12U'H^2 + V'\right].
$$

It would be convenient, if all the necessary information on the existence and stability of de Sitter solutions is obtained from a single combination of functions U, V, and F dubbed effective potential V_{eff}.
The de Sitter solutions would correspond to zeros of the first derivative of V_{eff} and stability of the solutions would correspond to its second derivative being positive.
We achieve this goal if we restrict ourselves to the case of $U > 0$.
We introduce the effective potential $V_{eff}(\phi)$ in the model with the Gauss–Bonnet term, such that

$$V'_{eff}(\phi_{dS}) = X(\phi_{dS}) = 0. \tag{18}$$

Indeed, let

$$V_{eff} = -\frac{U^2}{V} + \frac{2}{3}F. \tag{19}$$

we get

$$X(\phi_{dS}) = \frac{2}{3}F'_{dS} - 2\frac{U'_{dS}U_{dS}}{V_{dS}} + \frac{V'_{dS}U^2_{dS}}{V^2_{dS}} = V'_{eff}(\phi_{dS}) = 0. \tag{20}$$

So, de Sitter solutions correspond to extremum points of the effective potential V_{eff}.

The Lyapunov Stability

To investigate the Lyapunov stability of a de Sitter solution we use the following expansions,

$$H(t) = H_{dS} + \delta H_1(t),\ \phi(t) = \phi_{dS} + \delta\phi_1(t),\ \psi(t) = \delta\psi_1(t),$$

where δ is a small parameter.
In the first order in δ we get the following linear system

$$\dot{\phi}_1 = A_{11}\phi_1 + A_{12}\psi_1 + A_{13}H_1\,, \qquad (21)$$

$$\dot{\psi}_1 = A_{21}\phi_1 + A_{22}\psi_1 + A_{23}H_1\,, \qquad (22)$$

$$\dot{H}_1 = A_{31}\phi_1 + A_{32}\psi_1 + A_{33}H_1\,, \qquad (23)$$

where

$$A = \left\| \begin{array}{ccc} 0 & 1 & 0 \\ -\frac{V^2}{UB}X'_{,\phi} & H_{dS}\left(1 - 4\frac{U}{B}\right) & -\frac{V^2}{UB}X'_{,H} \\ \frac{VX'_{,\phi}}{2BU^2}\left(V'U - U'V\right) & \frac{2H_{dS}}{BV}\left(V'U - U'V\right) & \frac{VX'_{,H}}{2BU^2}\left(V'U - U'V\right) \end{array} \right\|$$

and all functions are taken at $\phi = \phi_{dS}$.

The functions $H_1(t)$, $\phi_1(t)$ and $\psi_1(t)$ are not independent. From Eq. (13), we obtain

$$H_1 = \frac{V'_{dS} U_{dS} - U'_{dS} V_{dS}}{2 U_{dS} V_{dS}} \left(H_{dS}\phi_1 - \psi_1\right). \qquad (24)$$

Substituting (24) into (21) and (22), we get:

$$\dot{\phi}_1 = \tilde{A}_{11}\phi_1 + \tilde{A}_{12}\psi_1 \,, \qquad (25)$$

$$\dot{\psi}_1 = \tilde{A}_{21}\phi_1 + \tilde{A}_{22}\psi_1 \,, \qquad (26)$$

where

$$\tilde{A} = \left\| \begin{array}{cc} 0 & 1 \\ -\frac{V^2 X'_{,\phi}}{UB} - \frac{V X'_{,H}\left(V'\,U - U'\,V\right) H_{dS}}{2U^2 B} & H_{dS}\left(1 - 4\frac{U}{B}\right) + \frac{V X'_{,H}\left(V'\,U - U'\,V\right)}{2U^2 B} \end{array} \right\|$$

The condition on the determinant of the characteristic matrix

$$\det(\tilde{A} - \lambda \cdot I) = 0 \tag{27}$$

gives the following expressions for λ:

$$\lambda_{\pm} = \frac{Z \pm \sqrt{Z^2 + Y}}{4U^2 B}, \tag{28}$$

where

$$Z = -\frac{3U^2}{V^2}\sqrt{\frac{6V}{U}}\left[\frac{7}{9}UV^2 + (V'U - U'V)^2\right],$$

and

$$Y = 8VB\left[X'_{,H} H_{dS} U^2 V U' - X'_{,H} H_{dS} U^3 V' - 2U^3 V X'_{,\phi}\right] = -16U^3 V^2 B V''_{eff}\ .$$

• A de Sitter solution is stable only if both $Z/B < 0$, and $Y < 0$.
• Situation considerably simplifies in the case of a positive U and, therefore, a positive V.
• Indeed, if at the de Sitter point both U and V are positive, then $Z < 0$ and $B > 0$. This means that the condition $Z/B < 0$ at any de Sitter point is satisfied automatically.
• Thus, we finally reach a conclusion that for any $U(\phi_{dS}) > 0$, a de Sitter solution is stable if $V''_{eff}(\phi_{dS}) > 0$ and unstable if $V''_{eff}(\phi_{dS}) < 0$.
• We consider several examples of models and explore the existence and stability of de Sitter solutions.

Models with exponential potential

Let us consider the string theory inspired cosmological model with [3]

$$U = U_0, \quad V = c\mathrm{e}^{-\lambda\phi}, \quad F = \frac{\alpha}{\mu}\mathrm{e}^{\mu\phi}, \tag{29}$$

where U_0, α, c, λ, and μ are positive constants.
In this model, the effective potential is

$$V_{eff} = -\frac{U_0^2}{c}\mathrm{e}^{\lambda\phi} + \frac{2\alpha}{3\mu}\mathrm{e}^{\mu\phi}. \tag{30}$$

The condition $V'_{eff}(\phi_{dS}) = 0$ gives

$$\phi_{dS} = \frac{1}{\lambda - \mu}\ln\left(\frac{2ac}{3U_0^2\lambda}\right). \tag{31}$$

[3] S. Tsujikawa and M. Sami, *String-inspired cosmology: Late time transition from scaling matter era to dark energy universe caused by a Gauss–Bonnet coupling*, J. Cosmol. Astropart. Phys. **0701** (2007) 006 [arXiv:hep-th/0608178]

There exists a de Sitter solution for all $\mu \neq \lambda$. It is easy to see that $V''_{eff} = 0$ at

$$\phi_2 = \frac{1}{\lambda - \mu} \ln\left(\frac{2ac\mu}{3U_0^2\lambda^2}\right) = \phi_{dS} - \frac{\ln(\lambda) - \ln(\mu)}{\lambda - \mu}\,, \qquad (32)$$

and $\phi_{dS} > \phi_2$ for any $\lambda \neq \mu$.

• If $\mu > \lambda$, then V''_{eff} is positive at large ϕ, so the second derivative is positive at the de Sitter point and this point is stable.

• In the opposite case, $\mu < \lambda$, $V''_{eff} < 0$ at large ϕ and the de Sitter solution is unstable. This result coincides with the result obtained in S. Tsujikawa and M. Sami, J. Cosmol. Astropart. Phys. **0701** (2007) 006 [arXiv:hep-th/0608178]
by another method.

Models with generalized exponential potential

• We generalize this result assuming that the constants can be negative:

$$V_{eff} = c_1 \mathrm{e}^{-N_1\phi} + c_2 \mathrm{e}^{-N_2\phi}, \qquad (33)$$

• The same effective potential corresponds to different choice of functions F, V, and U.
• If two of these functions are given, then we can get the third function using the given form of the effective potential.
• It is a way of constructing models with de Sitter solutions.

For example, the model ($V_{eff} = c_1 \mathrm{e}^{-N_1\phi} + c_2 \mathrm{e}^{-N_2\phi}$) with a non-minimally coupled scalar field defined by functions

$$U = U_0\left(\xi\phi^2 + 1\right)\mathrm{e}^{\eta_1\phi}, \quad \text{and} \quad V = V_0\phi^4 e^{\eta_2\phi},$$

has the effective potential given by (33) if

$$F = \frac{3}{2}\left[\frac{4U_0^2\mathrm{e}^{2\,\eta_1\phi-\eta_2\phi}}{V_0}\left(\xi + \frac{1}{\phi^2}\right)^2 + c_1\mathrm{e}^{-N_1\phi} + c_2\mathrm{e}^{-N_2\phi}\right]$$

In this model, c_i and N_i are arbitrary constants. The analysis of the second derivative of V_{eff} gives the following stability conditions:

- if $c_1 > 0$ and $c_2 > 0$, then the de Sitter solution is stable;
- if $c_1 < 0$ and $c_2 < 0$, then the de Sitter solution is unstable;
- if $c_1 > 0$ and $c_2 < 0$, then the de Sitter solution is stable at $|N_1| > |N_2|$ and unstable at $|N_1| < |N_2|$;
- if $c_1 < 0$ and $c_2 > 0$, then the de Sitter solution is stable at $|N_1| < |N_2|$ and unstable at $|N_1| > |N_2|$.

• The effective potential can be used not only to simplify the analysis of the stability of de Sitter solutions in a given model, but also to construct a new model with de Sitter solutions.

MODELS WITH $V = CU^2$

- Let us consider the case $V = CU^2$, where C is a positive constant.
- In this case, a model without the Gauss–Bonnet term transforms to a model with a constant potential in the Einstein frame.
- If the Gauss–Bonnet term is presented, then the function $F(\phi)$ plays a role of the effective potential, fully determining the position and stability of the de Sitter solutions, because

$$V_{eff} = -\frac{1}{C} + \frac{2}{3}F. \tag{34}$$

- So, values of ϕ_{dS} satisfy the condition $F'(\phi_{dS}) = 0$. From Eq. (28), it follows

$$\lambda_{\pm} = -\frac{\sqrt{6CU}}{4} \pm \frac{\sqrt{6CU[9(3U'^2 + U) - 16CU^2F'']}}{12\sqrt{3U'^2 + U}}. \tag{35}$$

- For $U(\phi_{dS}) > 0$, a de Sitter solution is unstable at $F'' < 0$ and
- stable at $F'' > 0$.

• For $F = A_4\,\phi^4 + A_2\phi^2$, de Sitter points defined by the condition $F' = 0$ are

$$\phi_{dS\,\pm} = \pm\sqrt{-\frac{A_2}{2A_4}}, \quad \phi_{dS\,0} = 0. \tag{36}$$

It is evident that $\phi_{dS\,\pm}$ are real only if A_2 and A_4 have different signs. The values of the second derivative of F at the de Sitter points are

$$F''|_{\phi_{dS\,\pm}} = -4A_2, \quad F''|_{\phi_{dS\,0}} = 2A_2.$$

• The de Sitter solution in points $\phi_{dS\,\pm}$ is unstable for any $A_2 > 0$ and $A_4 < 0$ and is stable for any $A_2 < 0$ and $A_4 > 0$.
• At the point $\phi_{dS\,0}$, the de Sitter solution is stable for any $A_2 > 0$ and unstable at $A_2 < 0$.
• At $A_2 = 0$, the only de Sitter point is $\phi_{dS} = 0$ and we get $\lambda_+ = 0$ and $\lambda_- = -\sqrt{6CU}/2$.

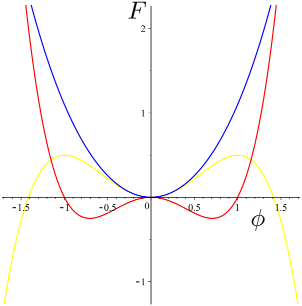

Figure: The function $\tilde{F}(\phi)$ at different values of parameters: $A_2 = 1$ and $A_4 = 0.1$ (blue curve), $A_2 = -1$ and $A_4 = 1$ (red curve), $A_2 = 1$ and $A_4 = -0.5$ (yellow curve).

The working of de Sitter search algorithm through examples in model with $V = CU^2$ and $\tilde{F} = A_4\phi^4 + A_2\phi^2 + \tilde{C}\sin(\omega\phi)$, $\tilde{C}$, ω where $\tilde{C}$, ω are constants

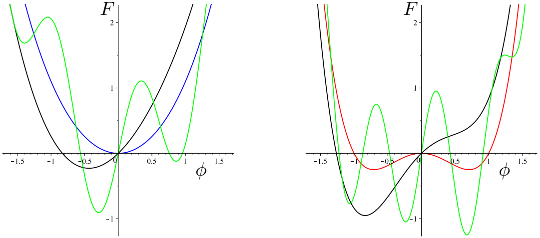

Figure: The function $\tilde{F}(\phi)$ at different values of parameters. In the left picture, $A_2 = 1$, $A_4 = 0.1$, and $\tilde{C} = 0$ (blue curve), $\tilde{C} = 1$ and $\omega = 1$ (black curve), $\tilde{C} = 1$ and $\omega = 5$ (green curve). In the right picture, $A_2 = -1$, $A_4 = 1$, and $\tilde{C} = 0$ (blue curve), $\tilde{C} = 1$ and $\omega = 1$ (black curve), $\tilde{C} = 1$ and $\omega = 7$ (green curve).

Models with a massive scalar field

- Let the potential be of the simplest massive form

$$V = m^2\phi^2, \tag{37}$$

with the coupling function

$$U = \xi\phi^2 \tag{38}$$

- In this situation the effective potential is

$$V_{eff} = -\frac{\xi^2}{m^2}\phi^2 + \frac{2}{3}F. \tag{39}$$

- Without the Gauss–Bonnet contribution the effective potential is a monotonic function, so there are no de Sitter solutions[4].
- However, it is clear that addition of any monomial $F = F_0\phi^n$ with $n > 2$ and $F_0 > 0$ gives us a stable de Sitter solution.

[4] In the point $\phi = 0$ the function $U = 0$, so this extremum of V_{eff} does not corresponds to a de Sitter solution.

• Straightforward calculation shows that

$$\phi_{dS}^{n-2} = \frac{3\xi^2}{nF_0 m^2}$$

and consequently the de Sitter solution exists if $n \neq 2$.

• The second derivative of the effective potential, we easily obtain

$$V''_{eff}(\phi) = \frac{4\xi^2}{m^2}(n-2), \tag{40}$$

which implies that the de Sitter solution is unstable for $n < 2$.

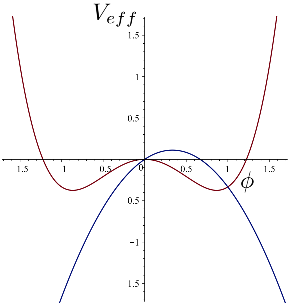

Figure: The effective potential $V_{eff}(\phi)$ for $U = \phi^2$, $V = \phi^2$, $F = \phi^\alpha$ is presented for $\alpha = 4$ (red curve) and $\alpha = 1$ (blue curve).

Models with the Higgs potential

Let us consider model with

$$U = U_0 + \xi\phi^2\,, \qquad V = V_0\phi^4, \quad F = F_0/\phi^4 \tag{41}$$

where U_0, ξ and V_0 are positive constants.The corresponding model without the Gauss–Bonnet term is known as Higgs-driven inflation model[5] Inflationary scenario proposed in Ref.[6] with the same F.

- In this case, the effective potential has the following form,

$$V_{eff} = -\frac{\left(U_0 + \xi\phi^2\right)^2}{V_0\phi^4} + \frac{2F_0}{3\phi^4}. \tag{42}$$

- De Sitter solutions correspond to ϕ real values only if $F_0 > 3U_0^2/(2V_0)$:

$$\phi_{dS} = \pm\frac{\sqrt{(2F_0V_0 - 3U_0^2)}}{\sqrt{3\xi U_0}}. \tag{43}$$

[5] F.L. Bezrukov and M. Shaposhnikov, Phys. Lett. B **659** (2008) 703 [arXiv:0710.3755];
A.O. Barvinsky, A.Y. Kamenshchik, and A.A. Starobinsky,J. Cosmol. Astropart. Phys. **0811** (2008) 021 [arXiv:0809.2104];
F.L. Bezrukov, Class. Quant. Grav. **30** (2013) 214001 [arXiv:1307.0708]

[6] C. van de Bruck and C. Longden, arXiv:1512.04768

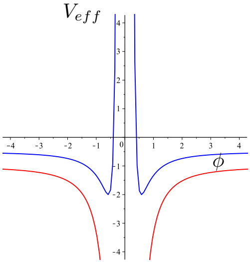

Figure: The red curve corresponds to a model without a de Sitter solution, whereas the blue curve corresponds to a model with de Sitter solutions.The effective potential $V_{eff}(\phi)$ for $U = 1 + \phi^2$, $V = V_0\phi^4$, $F = 1/\phi^4$. $V_{eff}(\phi)$ is presented for $V_0 = 1$ (red curve) and $V_0 = 2$ (blue curve).

Conclusions

• We analyze the Einstein–Gauss–Bonnet gravity model:

$$S = \int d^4x\sqrt{-g}\left(U(\phi)R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) - F(\phi)\mathcal{G}\right),$$

• We have shown that, in the case of $U(\phi) > 0$, it is possible to introduce the effective potential V_{eff} which can be expressed through the coupling function U, the scalar field potential V and the coupling function with the Gauss–Bonnet term F:

$$V_{eff} = -\frac{U^2}{V} + \frac{2}{3}F.$$

• Using this approach, we have studied concrete models with the Gauss–Bonnet term and described a number of situations where de Sitter solutions exist due to the presence of the Gauss–Bonnet term.
• We show that it is convenient to investigate the structure of fixed points using the effective potential, indeed, the stable de Sitter solutions correspond to minima of the effective potential V_{eff}.

Conclusions

• We analyze the Einstein–Gauss–Bonnet gravity model:

$$S = \int d^4x\sqrt{-g}\left(U(\phi)R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) - F(\phi)\mathcal{G}\right),$$

• We have shown that, in the case of $U(\phi) > 0$, it is possible to introduce the effective potential V_{eff} which can be expressed through the coupling function U, the scalar field potential V and the coupling function with the Gauss–Bonnet term F:

$$V_{eff} = -\frac{U^2}{V} + \frac{2}{3}F.$$

• Using this approach, we have studied concrete models with the Gauss–Bonnet term and described a number of situations where de Sitter solutions exist due to the presence of the Gauss–Bonnet term.
• We show that it is convenient to investigate the structure of fixed points using the effective potential, indeed, the stable de Sitter solutions correspond to minima of the effective potential V_{eff}.

Thank for your attention