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A Semi-Analytic Solution for Plane Strain Bending Under Tension of a Strain Hardening Sheet Including Ductile Fracture Prediction

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Abstract. A semi-analytic solution for predicting the evolution of damage in the process of plane strain bending under tension of a sheet of elastic-plastic, isotropic, incompressible, strain hardening material is provided. No restriction is imposed on the strain hardening law. The evolution of damage is described by an arbitrary uncoupled damage mechanics model. The final result is the dependence of the damage variable at the site of fracture initiation on any geometric parameter of the process in parametric form. Having the critical value of the damage variable it is possible to use this dependence for determining the value of the geometric parameter chosen at which the initiation of fracture occurs.

1. Introduction

Plane strain bending under tension at large strains is one of the classical problems in plasticity theory. Its exact analytic solution for rigid perfectly plastic material has been given in [1]. Using additional assumptions concerning the through-thickness distribution of strains analytic plane-strain solutions for more realistic models have been proposed in [2, 3]. Ignoring the transverse stress a new solution has been obtained in [4].

The present paper deals with the prediction of ductile fracture initiation in the process of plane strain bending under tension using uncoupled damage mechanics models. The solution is based on the approach proposed and developed in [5, 6]. The solution is semi-analytic.

2. Kinematics for uncoupled damage mechanics models

Let H and $2L$ be the initial thickness and width of a sheet, respectively. A cross-section of the sheet at the initial instant is illustrated in Fig. 1[a]. It is always possible to choose an Eulerian Cartesian coordinate system (x, y) such that its x- axis coincides with the axis of symmetry of the process and its y-axis with one of the sides of the initial rectangular. Then, the sides A_1B_1, C_1D_1, A_1D_1, and C_1B_1 of the initial rectangular are determined in this coordinate system by the equations

$$x = 0, \quad x = -H, \quad y = -L, \quad y = L, \tag{1}$$

respectively. Introduce a Lagrangian coordinate system (ζ, η) such that

$$\zeta = x/H \quad \text{and} \quad \eta = y/H \tag{2}$$

at the initial instant. It follows from (1) and (2) that the sides A_1B_1, C_1D_1, A_1D_1, and C_1B_1 in the Lagrangian coordinate system are determined by the equations $\zeta=0$, $\zeta=-1$, $\eta=-L/H$ and $\eta=L/H$. Then, the equations of the curves AB, CD, AD, and CB are (Fig. 1[b])

$$\zeta=0,\ \zeta=-1,\ \eta=-L/H \text{ and } \eta=L/H, \tag{3}$$

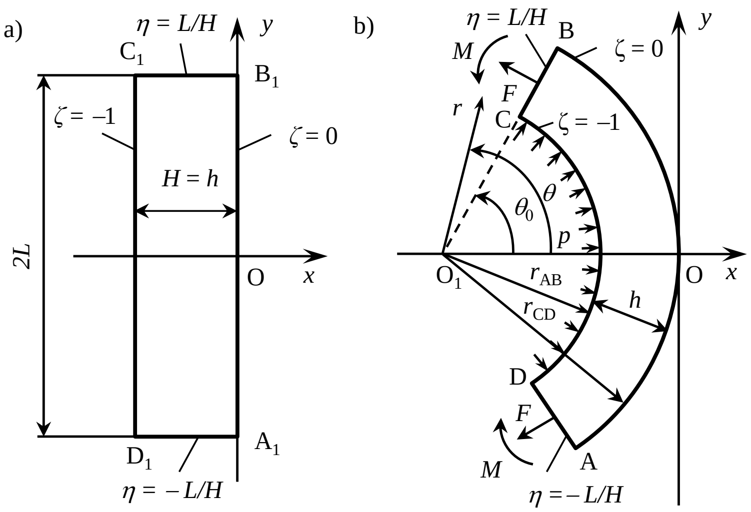

Figure 1. Bending under tension – notation.

respectively. In [5], the following mapping between the Cartesian and Lagrangian coordinate systems has been introduced for describing the process of pure bending:

$$\frac{x}{H}=\sqrt{\frac{\zeta}{a}+\frac{s}{a^2}}\cos(2a\eta)-\frac{\sqrt{s}}{a},\quad \frac{y}{H}=\sqrt{\frac{\zeta}{a}+\frac{s}{a^2}}\sin(2a\eta) \tag{4}$$

Here s is an arbitrary function of a, a is a function of the time, t, and $a=0$ at the initial instant, $t=0$. Moreover, the function s should satisfy the condition

$$s=1/4 \tag{5}$$

at $t=0$. It is evident from (4) that this mapping is symmetric relative to the x- axis. Moreover, it has been demonstrated in [5] that the mapping (4) satisfies the equation of incompressibility, transforms the sides A_1B_1 and C_1D_1 into concentric circular arcs AB and CD and the sides A_1D_1 and C_1B_1 into straight lines AD and CB orthogonal to AB and CD. The coordinate curves of the (ζ,η) coordinate system coincide with principal strain rate trajectories. Then, in the case of coaxial material models, the coordinate curves of this coordinate system coincide with principal stress trajectories. Thus the contour of the sheet is free of the shear stress throughout the process of deformation. The function $s(a)$ should be found from the solution. In the case of pure bending, this function has been found in [5] for several material models. It has been shown in [6] that the mapping (4) describes the process of bending under tension. The total principal strain components are determined from (4) as

$$2\varepsilon_\zeta=-2\varepsilon_\eta=-\ln\left[4(\zeta a+s)\right] \tag{6}$$

It is evident from (5) and (6) that $\varepsilon_\zeta=\varepsilon_\eta=0$ at the initial instant. The further solution is significantly facilitated by introducing a moving cylindrical coordinate system (r,θ) as

$$ar=H\sqrt{\zeta a+s} \quad \text{and} \quad \theta=2a\eta \tag{7}$$

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In this coordinate system, circular arcs AB and CD are determined by the equations $r = r_{AB}$ and $r = r_{CD}$, and straight lines AD and CB by the equations $\theta = \pm\theta_0$. It follows from (7) that

$$r_{AB}/H = \sqrt{s}/a, \quad r_{CD}/H = \sqrt{s-a}/a, \quad \theta_0 = 2aL/H \tag{8}$$

3. Elastic/Plastic Solution for an Arbitrary Hardening Law

Using the mapping (4) in conjunction with the classical Eulerian theory of finite elastoplasticity [7] a semi-analytic solution for the process of plane strain bending under tension has been found in [6]. This solution is summarized in this section to introduce the nomenclature and to provide a basis for integrating damage evolution equations. The yield criterion adopted in [6] reads

$$\sqrt{3}\left|\sigma_\zeta - \sigma_\eta\right| = 2\sigma_0\Phi\left(\varepsilon_{eq}^p\right) \tag{9}$$

Here σ_ζ and σ_η are the normal stresses in the (ζ,η) system of coordinates, σ_0 is the initial yield stress in uniaxial tension, $\Phi\left(\varepsilon_{eq}^p\right)$ is an arbitrary function of the equivalent plastic strain, ε_{eq}^p, satisfying the conditions $\Phi(0)=1$ and $d\Phi\left(\varepsilon_{eq}^p\right)\big/d\varepsilon_{eq}^p \ge 0$ for all ε_{eq}^p. In general, the process consists of three stages. The first stage corresponds to a purely elastic solution. This stage ends when $s = s_e = \exp\left(\sqrt{3}k\right)\big/4$ where $k = \sigma_0/(3G)$ and G is the shear modulus of elasticity. The corresponding value of $a = a_e$ is determined from the equation

$$6kfa_e\big/\sqrt{s_e - a_e} = \ln^2(4s_e) - \ln^2\left[4(s_e - a_e)\right] \tag{10}$$

where $f = F/(\sigma_0 H)$ and F is the tensile force per unit length (Fig. 1b). In what follows, it is assumed that f is constant and that $a > a_e$.

A plastic region starts to develop from the surface $\zeta = 0$ at $a = a_e$. Let ζ_1 be the elastic/plastic boundary and ε_1 be the value of ε_{eq}^p at $\zeta = 0$. It has been found in [6] that

$$\begin{gathered} 4s = \exp\left\{\sqrt{3}\left[\varepsilon_1 + k\Phi(\varepsilon_1)\right]\right\}, \quad \zeta_1 = \left[\exp\left(\sqrt{3}k\right) - 4s\right]\big/(4a), \\ fa\big/\sqrt{s-a} + \ln^2\left[4(s-a)\right]\big/(6k) = \psi(\varepsilon_1) - \psi(0) + (k/2)\Phi^2(\varepsilon_1) \end{gathered} \tag{11}$$

Here $\psi\left(\varepsilon_{eq}^p\right)$ is an anti-derivative of $\Phi\left(\varepsilon_{eq}^p\right)$. Eliminating s in the third equation by means of the first equation leads to the equation for determining ε_1 as a function of a. Then, s and ζ_1 as functions of a can be immediately found from the first and second equation, respectively. This stage ends when another plastic region starts to develop from the surface $\zeta = -1$. The corresponding equation is

$$\exp\left(-\sqrt{3}k\right) - \exp\left\{\sqrt{3}\left[\varepsilon_1 + k\Phi(\varepsilon_1)\right]\right\} + 4a = 0 \tag{12}$$

Since ε_1 has been found as a function of a, equation (12) can be solved to determine the values of a and ε_1 at the end of the second stage. These values will be denoted by a_1 and $\varepsilon_1^{(1)}$, respectively.

There are two plastic regions, $0 \ge \zeta \ge \zeta_1$ and $\zeta_2 \ge \zeta \ge -1$, and an elastic region, $\zeta_2 \le \zeta \le \zeta_1$, at $a > a_1$. Let ε_2 be the value of ε_{eq}^p at $\zeta = -1$. It has been found in [6] that

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$$\begin{gathered}2\left[\psi(\varepsilon_2)-\psi(\varepsilon_1)\right]+k\left[\Phi^2(\varepsilon_2)-\Phi^2(\varepsilon_1)\right]+2fa/\sqrt{s-a}=0,\\ 4a=\exp\left\{\sqrt{3}\left[\varepsilon_1+k\Phi(\varepsilon_1)\right]\right\}-\exp\left\{-\sqrt{3}\left[\varepsilon_2+k\Phi(\varepsilon_2)\right]\right\},\\ 4(s-a)=\exp\left\{-\sqrt{3}\left[\varepsilon_2+k\Phi(\varepsilon_2)\right]\right\},\\ 4a\zeta_2=\exp\left(-\sqrt{3}k\right)-\exp\left\{-\sqrt{3}\left[\varepsilon_2+k\Phi(\varepsilon_2)\right]\right\}-4a\end{gathered} \tag{13}$$

Using the second and third equations it is possible to eliminate a and s in the first equation. The resulting equation should be solved numerically to find ε_2 as a function of ε_1. Then, a, s and ζ_2 as functions of ε_1 can be determined from the second, third and forth equations, respectively. The equation for ζ_2 in (11) is valid and can be used to find ζ_2 as a function of ε_1.

4. Fracture prediction at the surface $\zeta = 0$

The uncoupled damage mechanics models usually involve the maximum tensile stress, σ_m, and the hydrostatic stress, σ. In particular, damage functions usually take the form

$$D=\int g(\sigma,\sigma_m)\,d\varepsilon_{eq}^{p} \tag{14}$$

where g is some function of its arguments. The value of D is supposed to attain its critical value, D_{cr}, at fracture. The through thickness distribution of stress has been found in [6]. However, it is not necessary to use that solution to predict the evolution of damage at $\zeta = 0$ (It is however important to know that the solution exists). Since $\zeta = 0$ is a traction free surface, it follows from (9) that

$$\sigma_m=\sigma_\eta=2\sigma_0\Phi(\varepsilon_1)/\sqrt{3} \tag{15}$$

Moreover, since the material is incompressible, $2\sigma=\sigma_\zeta+\sigma_\eta$ or using (15)

$$\sigma=\sigma_0\Phi(\varepsilon_1)/\sqrt{3} \tag{16}$$

Substituting (15) and (16) into (14) at the surface $\zeta = 0$ yields

$$D=\int_0^{\varepsilon_1} g\left[\sigma_0\Phi(z)/\sqrt{3},\,2\sigma_0\Phi(z)/\sqrt{3}\right]dz \tag{17}$$

Consider several widely used uncoupled damage mechanics models. In the case of the model proposed by Cockcroft and Latham [8] equation (17) becomes

$$D=\frac{2\sigma_0}{\sqrt{3}}\int_0^{\varepsilon_1}\Phi(z)\,dz=\frac{2\sigma_0}{\sqrt{3}}\left[\psi(\varepsilon_1)-\psi(0)\right] \tag{18}$$

In the case of the model proposed in [9], equation (17) becomes

$$D=\int_0^{\varepsilon_1}\left[1+1/\left(\sqrt{3}B\right)\right]dz=\left[1+1/\left(\sqrt{3}B\right)\right]\varepsilon_1 \tag{19}$$

Here B is a constitutive parameter. In the case of the model proposed in [10], equation (17) becomes

$$D=\int_0^{\varepsilon_1}4/3\,dz=4\varepsilon_1/3 \tag{20}$$

In the case of the model proposed in [11], equation (17) becomes

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$$D = \sqrt{3}\int_0^{\varepsilon_1}\left[\sqrt{3} - c\sigma_0\Phi(z)\right]^{-1} dz \tag{21}$$

Here c is a constitutive parameter.

Any of the equations for D derived and (8) in conjunction with the solution given in Section 3 provide the dependence of D on r_{CD} (or any other geometric parameter). Then, the equation $D = D_{cr}$ allows the value of r_{CD} corresponding to the initiation of fracture to be calculated.

It is seen from (19) and (20) that the prediction of fracture initiation based on the models [9] and [10] is independent of the strain hardening law. The function ψ involved in (18) can be found analytically for all widely used hardening laws. The integral involved in (21) is represented in terms of hypergeometric functions for Swift's and Ludwik's hardening laws.

5. Conclusions

The general semi-analytic solution for plane strain bending under tension for elastic plastic, strain hardening material found in [6] has been adopted to predict the initiation of ductile fracture by means of uncoupled damage mechanics models. No restriction on the isotropic hardening and damage evolution laws is imposed. The final result is the dependence of the damage variable on any geometric parameter of the process in parametric form. Having a critical value of the damage variable this dependence can be used to immediately determine the value of the geometric parameter chosen at which the initiation of ductile fracture occurs.

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