Telescopic shearing of non-Newtonian fluids with a saturation stress

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Abstract: A closed form solution for quasi-static telescopic shearing of non-Newtonian fluids between two concentric cylinders is derived and then analysed. The paper focuses on qualitative features of the solution. In particular, it is shown that the qualitative behaviour of the solution in the vicinity of the inner friction surface is controlled by the dependence of the quadratic invariant of the stress tensor on the quadratic invariant of the strain rate tensor as the latter approaches infinity. In particular, no solution at sticking may exist if the quadratic invariant of the stress tensor approaches a finite value as the quadratic invariant of the strain rate tensor approaches infinity. In this case, it is necessary to construct the solution at sliding. This solution is singular, and the exact asymptotic representation of the quadratic invariant of the strain rate tensor in the vicinity of the friction surface depends on the exact dependence of the quadratic invariant of the stress tensor on the quadratic invariant of the strain rate tensor as the latter approaches infinity.

Keywords: sticking; sliding; singularity; saturation stress.

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1 Introduction

Simple shearing for several viscoplastic models including the effect of thermal softening and heat conduction has been studied in Wright (1987). The primary objective of the present paper is to reveal some qualitative features of the solution for telescopic shearing of non-Newtonian fluids between two concentric circular cylinders, which are frictional surfaces for the flow. The flow is supposed to be fully developed and quasi-static. Non-Newtonian fluids are classified by the dependence of the quadratic invariant of the stress tensor on the quadratic invariant of the strain rate tensor or the strain rate dependent viscosity. The present solution demonstrates that the exact asymptotic behaviour of this dependence as the quadratic invariant of the strain rate tensor approaches infinity affects the qualitative behaviour of solutions in the vicinity of rough walls. Of particular interest are dependencies involving a saturation stress of finite magnitude. The saturation stress is the limit of the quadratic invariant of the stress tensor as the quadratic invariant of the strain rate tensor approaches infinity. Under certain conditions, the material model with the saturation stress requires the regime of sliding at the inner friction surface (i.e., the solution at sticking does not exist). It is worthy of note that there is a vast amount of literature on the slip boundary condition in viscous flow (see, for example Yeow et al., 2006; Karapetsas and Mitsoulis, 2013, where further references can be found). In contrast to these works, the present solution does not require any slip boundary condition to describe the transition between the regime of sticking and the regime of sliding. This transition is controlled by the constitutive equation. This behaviour of the solution is similar to that for rigid plastic solids (Alexandrov and Richmond, 2001a). The quadratic invariant of the strain rate tensor approaches infinity in the vicinity of the friction surface where the regime of sliding occurs. This feature of the solution is also similar to that of rigid perfectly plastic solutions (Alexandrov and Richmond, 2001b). Moreover, a general analysis of solution behaviour near frictional interfaces for viscoplastic models involving the saturation stress has also shown that the quadratic invariant of the strain rate tensor approaches infinity in the vicinity of the friction surface where the regime of sliding occurs (Alexandrov and Mustafa, 2015). However, the exact asymptotic representation of this invariant near the friction surface found in Alexandrov and Mustafa (2015) is deferent from that found in the present paper. This difference is associated with the assumption accepted in Alexandrov and Mustafa (2015) that the strain rate component normal to the friction surface does not vanish.

Since the quadratic invariant of the stress tensor is bounded, it is evident that the strain rate dependent viscosity approaches zero as the quadratic invariant of the strain rate tensor approaches infinity. In general, this is a property of shear thinning fluids (for example, Chapman et al., 1997; Haines et al., 2013). However, commonly accepted functions for the strain rate dependent viscosity approach zero too slow for the transition between the regimes of sticking and sliding without a slip boundary condition. On the other hand, it is unlikely that any direct experiment can be used for determining the exact behaviour of the rate dependent viscosity as the quadratic invariant of the strain rate tensor approaches infinity. The present solution can be used in conjunction with an indirect experiment, for example Tsai and Soong (1985), to reveal the qualitative behaviour of the rate dependent viscosity as the quadratic invariant of the strain rate tensor approaches infinity.

An applied aspect of the solution found is that it is be used for approximate analysis of the pull-out test. The single-fibre pull-out test is widely employed to model the failure of fibre-reinforced composites. A comprehensive review of works devoted to this test and published before 1996 has been presented in DiFrancia et al. (1996). An inverse method, which includes a numerical routine, has been proposed in Banholzer et al. (2006) for the interpretation of experimental data. The rate effects on the interpretation of results of pull-out tests have been evaluated in Banholzer et al. (2006). It is worthy of note that the constitutive equations adopted in the present paper are also rate dependent. Analytical solutions for various kinds of the single pull-out test have been proposed in Gurung (2001), Brameshuber and Jung (2005) and Martin et al. (2011). None of these solutions is based on the material model employed in the present paper.

2 Statement of the boundary value problem

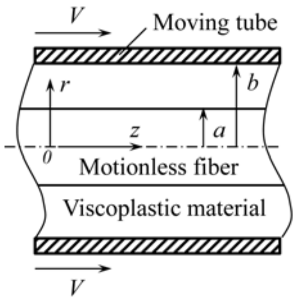
A non-Newtonian fluid is confined between an infinite round rigid fibre of radius a and an infinite round rigid tube of inner radius b (Figure 1). The axes of symmetry of the fibre and tube coincide. A cylindrical coordinate system (r, θ, z) is taken, with its z-axis coinciding with the axis of symmetry of the fibre and the tube. The position of origin O of the coordinate system is immaterial.

The fibre is motionless, and the tube translates along its axis with velocity V. It is assumed that the regime of sticking occurs at both friction surfaces, $r = a$ and $r = b$, if such a solution exists. In the case of non-Newtonian fluids, the quadratic invariant of the stress tensor, τ_e, is a prescribed function of the quadratic invariant of the strain rate tensor, ξ_e:

$$\frac{\tau_e}{\tau_0} = \eta\left(\frac{\xi_e}{\xi_0}\right), \tau_e = \sqrt{\frac{1}{2}\tau_{ij}\tau_{ij}}, = \xi_e = \sqrt{\frac{1}{2}\xi_{ij}\xi_{ij}} \text{ s} \tag{1}$$

Here $\eta(\xi_e / \xi_0)$ is an arbitrary monotonically increasing function of its argument, τ_{ij} are the deviatoric components of the stress tensor, ξ_{ij} are the components of the strain rate tensor, τ_0 and ξ_0 are constants introduced for convenience.

Figure 1 Geometry of the boundary value problem



For fully developed flow the deformation induced by the motion of the tube is telescopic shearing. In particular, the only non-zero stress component in the cylindrical coordinate system is τ_{rz} and the only non-zero strain rate component is ξ_{rz}. The direction of the motion of the tube (Figure 1) demands

$$\tau_{rz} > 0 \tag{2}$$

Therefore, the constitutive equation (1) can be rewritten as

$$\frac{\tau_{rz}}{\tau_0} = \eta\left(\frac{\xi_{rz}}{\xi_0}\right) \tag{3}$$

or

$$\frac{\xi_{rz}}{\xi_0} = \mu\left(\frac{\tau_{rz}}{\tau_0}\right) \tag{4}$$

where $\mu(\tau_{rz} / \tau_0)$ is the function inverse to $\eta(\xi_{rz} / \xi_0)$.

In the case of telescopic shearing, the axial velocity, u_z is the only non-zero velocity component in the cylindrical coordinate system chosen. Therefore,

$$\xi_{rz} = \frac{1}{2}\frac{\partial u_z}{\partial r} \tag{5}$$

The flow considered is quasi-static. The only non-trivial equilibrium equation is

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\tau_{rz}}{r} = 0 \tag{6}$$

If the regime of sticking occurs at both friction surfaces then the axial velocity should satisfy the following conditions:

$$u_z = 0 \tag{7}$$

For $r = a$ and

$$u_z = V \tag{8}$$

For $r = b$.

It will be seen later that the qualitative behaviour of the solution essentially depends on the function $\eta(\xi_e / \xi_0)$ involved in (1).

3 General solution at sticking

Equation (6) can be immediately integrated to give

$$\tau = \frac{\tau_{rz}}{\tau_0} = \frac{aC_1}{r} \tag{9}$$

where C_1 is constant of integration and the first equation defines τ. Let τ_a be the value of τ at $r = a$ and τ_b be the value of τ at $r = b$. Then, it follows from (9) that

$$C_1 = \tau_a \quad \text{and} \quad \tau_b = \tau_a \frac{a}{b} \tag{10}$$

Equation (9) can be rewritten as

$$\tau = \frac{\tau_a}{\rho} \tag{11}$$

where $\rho = r / a$ and $1 \le \rho \le b / a$. Equations (4) and (5) combine to give

$$\frac{\partial u_z}{\partial \rho} = 2a\xi_0 \mu\left(\frac{\tau_{rz}}{\tau_0}\right) \tag{12}$$

Using (9) and (11) equation (12) can be transformed to

$$\frac{\partial u_z}{\partial \rho} = 2a\xi_0 \mu\left(\frac{\tau_a}{\rho}\right) \tag{13}$$

The solution of this equation satisfying the boundary condition (7) is

$$u_z = 2a\xi_0 \int_1^{\rho} \mu\left(\frac{\tau_a}{\beta}\right) d\beta \tag{14}$$

where β is a dummy variable of integration. The solution (14) and the boundary condition (8) combine to give

$$\int_1^{b/a} \mu\left(\frac{\tau_a}{\rho}\right) d\rho = \frac{V}{2a\xi_0} \tag{15}$$

This equation should be solved for τ_a. Using (11) equation (15) can be rewritten as

$$\tau_a \int_{\tau_b}^{\tau_a} \frac{\mu(\tau)}{\tau^2} d\tau = \frac{V}{2a\xi_0} \tag{16}$$

Here τ_b can be eliminated by means of (10). Therefore, (16) is also the equation for determining τ_a. The solution of equations (15) and (16) may or may not exist. In the latter case, the regime of sticking is impossible.

4 Analysis of the solution

The existence of the solution of equation (15) (or (16)) is controlled by the function $\mu(\tau)$. Typical models of fluid mechanics, for example Newtonian fluids, satisfy the condition:

$$\mu(\tau) \to \infty \tag{17}$$

As $\tau \to \infty$. Since $\mu(\tau)$ is a monotonically increasing function of its argument, $\mu(\tau_a) \le \mu\left(\frac{\tau_a}{\rho}\right) \le \mu\left(\frac{a\tau_a}{b}\right)$.

Then,

$$\left(\frac{b}{a}-1\right)\mu(\tau_a) \le \int_1^{b/a} \mu\left(\frac{\tau_a}{\rho}\right) d\rho \le \left(\frac{b}{a}-1\right)\mu\left(\frac{a\tau_a}{b}\right) \tag{18}$$

Using (15) this inequality can be rewritten as

$$\left(\frac{b}{a}-1\right)\mu(\tau_a) \le \frac{V}{2a\xi_0} \le \left(\frac{b}{a}-1\right)\mu\left(\frac{a\tau_a}{b}\right) \tag{19}$$

Therefore, it follows from (17) and (19) that $V \to \infty$ if $\tau_a \to \infty$ and the solution at sticking always exist.

Assume that $\mu(\tau) \to \infty$ as $\tau \to \tau_s$. Here τ_s is a saturation stress. It is possible to put $\tau_0 = \tau_s$ in (1), (3) and (4). Then, $\tau_a \le 1$. Differentiating (15) with respect to V yields

$$\frac{dV}{d\tau_a} - 2a\xi_0 \int_1^{b/a} \frac{\mu'(\tau_a/\rho)}{\rho} d\rho \tag{20}$$

Here the prime symbol denotes the derivative of the function $\mu(\tau_a / \rho)$ with respect to its argument. Since $\mu(\tau)$ is a monotonically increasing function of its argument, this derivative is positive at any value of τ_a / ρ. Therefore, it follows from (20) that $dV / d\tau_a > 0$. Then, the maximum possible value of V is attained at $\tau_a = 1$. This value of V is determined from (16) as

$$\frac{V_{\max}}{2a\xi_0} = \int_{a/b}^{1} \frac{\mu(\tau)}{\tau^2} d\tau \text{ s} \tag{21}$$

Since $\mu(\tau) \to \infty$ as $\tau \to 1$, the integral is improper. If this integral is divergent then $V_{\max} \to \infty$ and the solution at sticking always exists. If the integral is convergent then $V_{\max} < \infty$ and the solution at sticking does not exist for $V > V_{\max}$.

It is necessary to make some assumptions concerning the behaviour of the function $\mu(\tau)$ as $\tau \to 1$ to study the behaviour of the integral in (21). The power-law model is a reasonable assumption. In this case,

$$\mu\left(\frac{\tau_{rz}}{\tau_s}\right) = A\left(1-\frac{\tau_{rz}}{\tau_s}\right)^{-\alpha} + o\left[\left(1-\frac{\tau_{rz}}{\tau_s}\right)^{-\alpha}\right] \tag{22}$$

As $\tau_{rz} \to \tau_s$. Here $A > 0$ and $\alpha > 0$. Using the definition for τ from (9) equation (22) can be rewritten as

$$\mu(\tau) = A(1-\tau)^{-\alpha} + o\left[(1-\tau)^{-\alpha}\right] \tag{23}$$

As $\tau \to 1$. Substituting (23) into (21) shows that the integral is convergent if $0 < \alpha < 1$ and is divergent if $1 \le \alpha < \infty$. Therefore, the regime of sliding at the inner friction surface occurs at $V > V_{\max}$ if

$$0 < \alpha < 1 \tag{24}$$

This solution at sliding is independent of the value of V.

Equations (3), (4) and (23) combine to give

$$\eta\left(\frac{\xi_{rz}}{\xi_0}\right) = 1 - A^{1/\alpha}\left(\frac{\zeta_{rz}}{\zeta_0}\right)^{-1/\alpha} + o\left[\left(\frac{\xi_{rz}}{\xi_0}\right)^{-1/\alpha}\right] \tag{25}$$

As $\xi_{rz} \to \infty$. It follows from (5), (11), (13) and (23) that the exact asymptotic representation of the shear strain rate ξ_{rz} and, therefore, the quadratic invariant of the strain rate tensor in the vicinity of the friction surface $\rho = 1$ is given by

$$\xi_{rz} = \xi_e = \xi_0 A(\rho-1)^{-\alpha} + o\left[(\rho-1)^{-\alpha}\right] \tag{26}$$

As $\rho \to 1$ if $0 < \alpha \le 1$.

The shear rate dependent viscosity is defined by the following ratio:

$$v = \frac{\tau_{rz}}{2\xi_{rz}} \tag{27}$$

Since $\tau_{rz} \to \tau_s < \infty$ as $\xi_{rz} \to \infty$, it follows from (27) that

$$v = O\left(\xi_{rz}^{-1}\right) \tag{28}$$

As $\xi_{rz} \to \infty$. Thus $v \to 0$ as $\xi_{rz} \to \infty$. This is a property of shear thinning fluids. A typical assumption concerning the shear rate dependent viscosity is (Chapman et al., 1997; Haines et al., 2013)

$$v = O\left(\xi_{rz}^{n-1}\right) \tag{29}$$

where $0 < n \le 1$. In this case, (17) is satisfied. Therefore, the mathematical solutions for the models satisfying (28) and (29) are qualitatively different. However, it is unlikely that any direct experiment is capable of distinguishing between (28) and (29). On the other hand, the solution found can be used for interpreting data from the sliding cylinder rheometer proposed in Tsai and Soong (1985). Using this indirect method there should be possible to find which dependence, (28) or (29), is more appropriate for a given fluid.

5 Conclusions

A closed form solution for quasi-static flow of non-Newtonian fluids between two co-axial cylinders has been derived. From this solution, the following conclusions can be drawn.

1. The transition between the regimes of sticking and sliding is controlled by the constitutive equation without using any slip boundary condition if the strain rate dependent viscosity satisfies (28).
2. Under certain conditions, no solution at sticking exists if the strain rate dependent viscosity satisfies (28).
3. The solution at sliding is singular and the quadratic invariant of strain rate tensor satisfies (26) in the vicinity of the friction surface.
4. The solution found can be used for interpreting data from the sliding cylinder rheometer proposed in Tsai and Soong (1985).
5. Similar solutions can be found for interpreting data from other rheometers, which will be the subject of a subsequent investigation.

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