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Nonlinear Dynamics of Extremely Short Solitons of the Generalized Reduced Maxwell–Bloch System

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Abstract—The effect distant quantum states have on two allocated atomic levels during the propagation of extremely short electromagnetic pulses in a medium is considered without using the slowly varying envelope approximation. A generalization of the reduced Maxwell–Bloch system is obtained that is integrable via inverse scattering. Its soliton and breather solutions are determined and studied.

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INTRODUCTION

One trend in the development of laser physics is the creation of optical pulses of ever shorter durations under laboratory conditions [1–3]. The approximation of slowly varying envelopes (SVEs) already cannot be applied to pulses whose duration is of the order of one or two periods of electromagnetic oscillations. More precisely, we cannot even speak of a pulse envelope. It is customary in this case to talk about extremely short pulses (ESPs).

The first time the SVE approximation was shown to be inapplicable was most likely in [4], where an alternative approach to describing the phenomenon of self-induced transparency in a two-level medium was described. Instead of the SVE approximation, the authors proposed using that of unidirectional propagation [4], in which the concentration n of two-level atoms is assumed to be small. As a result, they arrived at the so-called system of reduced Maxwell–Bloch (RMB) equations. This system turned out to be integrable by means of the inverse scattering problem (MISP) [5–7].

By the beginning of the 1990s, the need had arisen to develop theoretical means for the nonlinear optics of ESPs in connection with experimental achievements in generating them. The authors of [8, 9] proposed using the sudden excitation approximation to describe the propagation of a low-period pulse in a two-level medium, based on the condition

$$\omega_0\tau_p \ll 1, \tag{1}$$

where ω_0 is the frequency of a given quantum transition and τ_p is the duration of a pulse. The concentration of atoms is in this case not assumed to be small. By eliminating material variables, the familiar Gordon sine equation was obtained for the time integral of electric field E of a pulse, which is also integrable within the MISP [6].

Since the ESP spectrum is quite broad, many quantum transitions can interact with it. As a result, the two-level medium approximation becomes inapplicable. The quantum transitions from the two relevant levels into overlying quantum states were considered in [10, 11]. These states were approximated by two additional levels with order numbers 3 and 4. The optical transparency approximation was used, which assumes fulfillment of the conditions

$$\omega_{31}\tau_p \sim \omega_{42}\tau_p \gg 1, \tag{2}$$

where ω_{31} and ω_{42} are the frequencies of transitions $1 \leftrightarrow 3$ and $2 \to 4$, respectively.

In [10, 11], condition (1) was imposed on the frequency of the allocated transition between states with order numbers 1 and 2. The sudden excitation approximation was then applied. A generalized Gordon sine equation was thus obtained for the time integral of the field of a pulse, which turned out to be integrable using the MISP [11].

The aim of this work is to seek an integrable generalization of the RMB system in studying the propagation of ESPs in a multilevel medium of low atomic concentration without imposing condition (1).

GENERALIZED REDUCED MAXWELL–BLOCH SYSTEM

As in [10, 11], we shall consider a four-level medium. Assuming a low concentration of atoms [4], we write the wave equation in reduced form

$$\frac{\partial E}{\partial z} + \frac{1}{c}\frac{\partial E}{\partial t} = -\frac{2\pi}{c}\frac{\partial P}{\partial t}, \tag{3}$$

where z is the axis of propagation of a pulse; c is the speed of light in vacuum; t is time; P is the polarization response of the medium, determined through elements $\rho_{\mu\nu}$ of density matrix $\hat{\rho}$ of an atom and dipole moments $d_{\mu\nu}$ corresponding to allowable transitions

$$P = n[d_{21}(\rho_{21} + \rho_{21}^*) + d_{31}(\rho_{31} + \rho_{31}^*) + d_{42}(\rho_{42} + \rho_{42}^*)]. \quad (4)$$

The material conditions in this case have the form

$$\frac{\partial \rho_{ml}}{\partial t} = -i\omega_{ml}\rho_{ml} + \frac{i}{\hbar}E\left[\hat{d},\hat{\rho}\right]_{ml} \quad (5)$$

$(m,l = 1,2,3,4)$, where ω_{ml} is the frequency of transition $m \leftrightarrow l$ and $\hat{d}$ represents the dipole moment matrix that allocates allowable quantum transitions:

$$\hat{d} = \begin{pmatrix} 0 & 0 & d_{42} & 0 \\ 0 & 0 & 0 & d_{31} \\ d_{42} & 0 & 0 & d_{21} \\ 0 & d_{31} & d_{21} & 0 \end{pmatrix}. \quad (6)$$

System (3)–(6) determines the initial physical model.

Because of condition (2), we ignore the left parts in Eqs. (5) for elements ρ_{31}, ρ_{42}, ρ_{32}, ρ_{43} and ρ_{41} of the density matrix; i.e. we write $\partial\rho_{31}/\partial t = \partial\rho_{42}/\partial t = \partial\rho_{32}/\partial t = \partial\rho_{43}/\partial t = \partial\rho_{41}/\partial t = 0$. Assuming the states are not populated before the action of a pulse, we then obtain approximately the expressions [11]

$$\rho_{31} = \frac{d_{31}E}{\hbar\omega_{31}}\rho_{11}, \quad \rho_{42} = \frac{d_{42}E}{\hbar\omega_{42}}\rho_{22}, \quad (7)$$

$$\rho_{32} = \frac{d_{31}E\rho_{21}^*}{\hbar\omega_{32}} \approx \frac{d_{31}E}{\hbar\omega_{31}}\rho_{21}^*, \qquad \rho_{41} = \frac{d_{42}E\rho_{21}}{\hbar\omega_{41}} \approx \frac{d_{42}E}{\hbar\omega_{42}}\rho_{21}. \quad (8)$$

In expressions (8), we use approximate equalities $\omega_{32} \approx \omega_{31}$ and $\omega_{41} \approx \omega_{42}$.

Substituting (8) into (5) for ρ_{21}, we obtain

$$\frac{\partial \rho_{21}}{\partial t} = -i\left[\omega_{21} + \left(\frac{d_{31}^2}{\omega_{31}} - \frac{d_{42}^2}{\omega_{42}}\right)\frac{E^2}{\hbar^2}\right]\rho_{21} + i\frac{d_{21}}{\hbar}E(\rho_{11} - \rho_{22}). \quad (9)$$

After substituting expressions (7) and (4) into Eq. (5) for ρ_{11}, we have

$$P = n\left[d_{21}\left(\rho_{21} + \rho_{21}^*\right) + \frac{2E}{\hbar}\left(\frac{d_{31}^2}{\omega_{31}}\rho_{11} + \frac{d_{42}^2}{\omega_{42}}\rho_{22}\right)\right], \quad (10)$$

$$\frac{\partial \rho_{11}}{\partial t} = -\frac{\partial \rho_{22}}{\partial t} = i\frac{d_{21}}{\hbar}E\left(\rho_{21} - \rho_{21}^*\right). \quad (11)$$

It follows that in the accepted approximation, $\rho_{11} + \rho_{22} = 1$. Introducing real Bloch variables $W = (\rho_{22} - \rho_{11})/2$, $U = \left(\rho_{21} + \rho_{21}^*\right)/2$ and $V = -i\left(\rho_{21}^* - \rho_{21}\right)/2$, from (9)–(11) and (3), we obtain the system

$$\frac{\partial U}{\partial \tau} = -\left(\omega_0 + \beta\Omega^2\right)V, \quad (12)$$

$$\frac{\partial V}{\partial \tau} = \left(\omega_0 + \beta\Omega^2\right)U + \Omega W, \quad (13)$$

$$\frac{\partial W}{\partial \tau} = -\Omega V, \quad (14)$$

$$\frac{\partial \Omega}{\partial z} = -\alpha\frac{\partial}{\partial \tau}(U - 2\beta\Omega W). \quad (15)$$

Here we write $\tau = t - z/v_0$, $\omega_0 \equiv \omega_{21}$, and $\Omega = 2d_{21}E/\hbar$; constants α, β and v_0 are expressed through the physical parameters of the given problem.

Inserting $\beta = 0$ into Eqs. (12)–(15), we obtain the familiar RMB system, which is valid for a system of two-level atoms. Equations (12)–(15) are referred to as the generalized RMB system (GMRB). It should be emphasized that the dipole moments of allowed quantum transitions can be in different quantitative relations to one another. The terms in Eqs. (12), (13), and (15), which consider the deviation from the two-level medium model, therefore cannot be considered as merely small corrections to the RMB system.

SOLITON COLLISIONS AND BREATHER SOLUTIONS

Systems of equations equivalent to modified RMB equations (MRMB) were considered in [12–15]:

$$\frac{\partial U_0}{\partial T} = -2\omega_0\sqrt{1 - \varepsilon^2\Omega_0^2}V_0, \quad (16)$$

$$\frac{\partial V_0}{\partial T} = 2\omega_0\sqrt{1 - \varepsilon^2\Omega_0^2}U_0 + \Omega_0 W_0, \quad (17)$$

$$\frac{\partial W_0}{\partial T} = -\Omega_0 V_0, \quad (18)$$

$$\frac{\partial \Omega_0}{\partial Z} = -\frac{1}{\omega_0}\frac{\partial U_0}{\partial T}. \quad (19)$$

These equations are integrable using the MISP and describe the nonlinear dynamics of two-component electromagnetic and acoustic ESPs in anisotropic two-level media. Different types of soliton solutions to MRMB equations were studied in detail in [13–15].

GRMB system (12)–(15) also proves to be integrable using the MISP [16] and is associated with MRMB equations (16)–(19) when condition $\varepsilon = \sqrt{-\beta/\omega_0}$ is met by replacing dependent and independent variables $(T, Z, \Omega_0, U_0, V_0, W_0) \to (\tau, z, \Omega, U, V, W)$, where

$$d\tau = \left(1 + \sqrt{1 - \varepsilon^2 \Omega_0^2(T,Z)}\right) dT + 2\varepsilon^2 W_0 dZ,$$
$$dz = \frac{dZ}{\omega_0 \alpha}, \quad \Omega(\tau, z) = \frac{\Omega_0(T,Z)}{1 + \sqrt{1 - \varepsilon^2 \Omega_0^2(T,Z)}}, \tag{20}$$
$$W(\tau, z) = W_0(T,Z), \quad U(\tau, z) = V_0(T,Z),$$
$$V(\tau, z) = -U_0(T,Z).$$

This relation allows us to obtain solutions to GRMB system (12)–(15) by using known solutions to the MRMB equations. Let us consider some different cases:

$$\text{Case} \quad \beta < -\frac{1}{4\omega_0}.$$

Changing the variables in (20) does not allow us to obtain a nonsingular solution to the GMRB system if we use the one-soliton solution to the GRMB equations. However, the breather solution to the MRMB equations produces nonsingular solutions. Without loss of generality, variable Ω_0 of this solution is written in the form

$$\Omega_0 = \frac{1}{i\sigma} \frac{\partial}{\partial T} \log \left| \frac{s_-}{s_+} \right|, \tag{21}$$

where

$$s_\pm = \nu_I \left[r_+ \exp(-\theta_R) + r_- \exp(\theta_R) \right] \pm i\nu_R \left[r_+ \exp(-i\theta_I) + r_- \exp(i\theta_I) \right],$$
$$r_\pm = \varepsilon(\nu_I - i\nu_R) \pm i\sigma,$$
$$\theta_R = 2\nu_R \left[T - \frac{W_0^{(0)} \left(\nu_R^2 + \nu_I^2 + \omega_0^2\right) Z}{\nu_R^4 + 2\left(\nu_I^2 + \omega_0^2\right)\nu_R^2 + \left(\nu_I^2 - \omega_0^2\right)^2} \right],$$
$$\theta_I = 2\nu_I \left[T + \frac{W_0^{(0)} \left(\nu_R^2 + \nu_I^2 - \omega_0^2\right) Z}{\nu_R^4 + 2\left(\nu_I^2 + \omega_0^2\right)\nu_R^2 + \left(\nu_I^2 - \omega_0^2\right)^2} \right],$$

$\sigma = \sqrt{1 + 4\beta\omega_0}/2$, ν_R, ν_I and $W_0^{(0)}$ are real constants.

Substituting expression (21) into relations (20) yields an implicit definition of variable Ω of the breather solution to the GMRB system (12)–(15). Parameters ν_R and ν_I in this case set the duration and carrier frequency of the breather. If the carrier frequency is high, the resulting breather is similar to that of the RMB system. As the carrier frequency falls, a peaked oscillation arises in the center of the breather of the GRMB system, and its amplitude greatly exceeds that of neighboring oscillations.

$$\text{Case} \quad -\frac{1}{4\omega_0} < \beta < 0.$$

Here, the expression for variable Ω_0 of a single-soliton solution to the MRMB equations, which allows us to obtain a nonsingular solution to the GMRB system, has the form

$$\Omega_0 = \pm 2\sqrt{A} \frac{\cosh\theta}{A\cosh^2\theta + \varepsilon^2}, \tag{22}$$

where $A = 1/(4\nu^2) - \varepsilon^2\left(1 + \omega_0^2/\nu^2\right)$, $\theta = 2\nu\left(T - W_0^{(0)} Z/\left(\nu^2 + \omega_0^2\right)\right)$, and ν is a real constant that meets the condition $|\nu| < |\sigma/\varepsilon|$.

Substituting expression (22) into formulas (20), we obtain an implicit definition of variable Ω of the one-soliton solution to GRMB system (12)–(15). This solution is stationary, and parameter ν sets the duration of a soliton. In contrast to the RMB system, the amplitude of a soliton is not inversely proportional to its duration, and it grows without limited when $|\nu| \to |\sigma/\varepsilon|$.

Without loss of generality, the expression for variable Ω_0 of the two-soliton solution to the MRMB equations has the form

$$\Omega_0 = \frac{1}{\sigma} \frac{\partial}{\partial T} \left(\arctan \frac{\nu_+ \sinh\theta_-}{\nu_- \sinh\theta_+} + \arctan \frac{\nu_+[\eta_- \sinh\theta_- - 2\nu_-\varepsilon\sigma\cosh\theta_-]}{\nu_-[\eta_+ \sinh\theta_+ - 2\nu_+\varepsilon\sigma\cosh\theta_+]} \right), \tag{23}$$

where $\nu_\pm = (\nu_1 \pm \nu_2)/2$, $\theta_\pm = (\theta_1 \pm \theta_2)/2$, and $\eta_\pm = \sigma^2 \pm \nu_1\nu_2\varepsilon^2$; ν_1 and ν_2 are real constants such that $|\nu_{1,2}| < |\sigma/\varepsilon|$, $\theta_{1,2} = 2\nu_{1,2}\left(T - W_0^{(0)} Z/\left(\nu_{1,2}^2 + \omega_0^2\right)\right)$.

Substituting expression (23) into formulas (20) yields an implicit definition of variable Ω of the two-soliton solution to GRMB system (12)–(15). This solution describes the elastic collision of stationary solitons considered above. Interesting features arise in a collision between solitons of opposite polarities $(\nu_1\nu_2 > 0)$. If the absolute value of parameter ν_1 or ν_2 is close to $\left|\sigma/\sqrt{2}\varepsilon\right|$, a short-lived pulse of high amplitude arises whose dynamics is similar to that of a killer wave [17] (see Fig. 1). Note that in the RMB system, a collision between solitons of opposite polarities produces a pulse whose amplitude is equal to the sum of those of the colliding solitons.

The expression for variable Ω_0 of the breather solution to the MRMB equations has the form

$$\Omega_0 = \frac{1}{\sigma} \frac{\partial}{\partial T} \left(\arctan \frac{\nu_R \sin\theta_I}{\nu_I \cosh\theta_R} + \arctan \frac{\nu_R[\tilde{\eta}_- \sin\theta_I - 2\nu_I\varepsilon\sigma\cos\theta_I]}{\nu_I[\tilde{\eta}_+ \cosh\theta_R - 2\nu_R\varepsilon\sigma\sinh\theta_R]} \right), \tag{24}$$

where $\tilde{\eta}_\pm = \sigma^2 \pm \left(\nu_R^2 + \nu_I^2\right)\varepsilon^2$. Substituting expression (24) into formulas (20), we obtain an implicit definition of variable Ω of the breather solution to GRMB system (12)–(15). As in the previous case, peaked

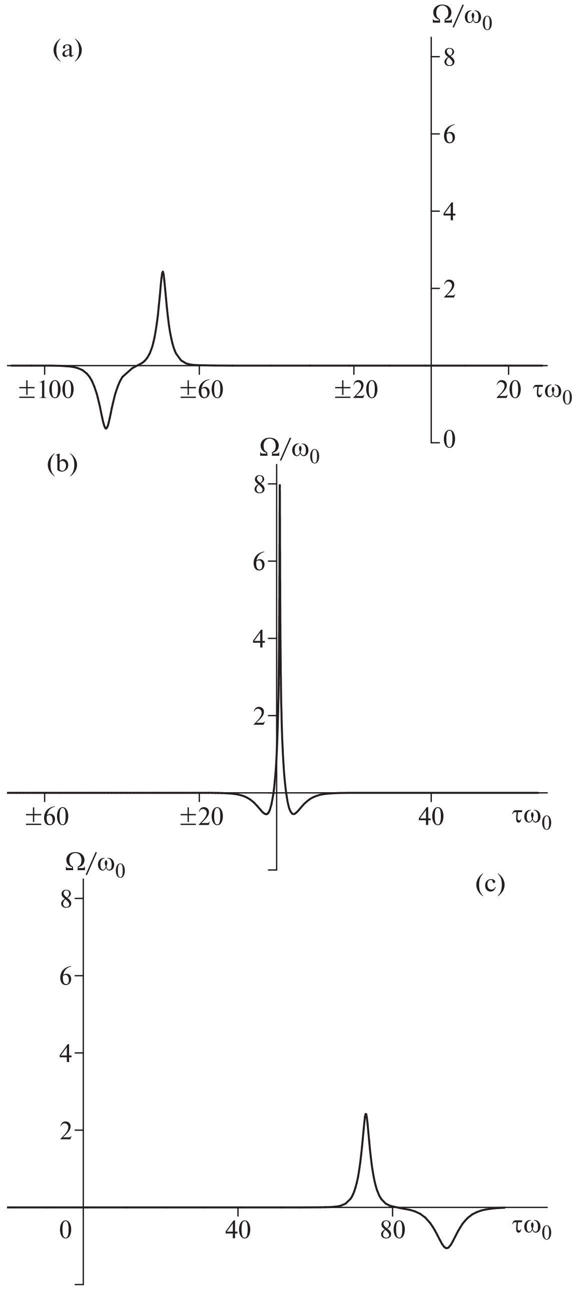

Fig. 1. Profiles of variable Ω of a two-soliton solution with parameters $\beta = -1/8\omega_0$, $W_0^{(0)} = -1/2$, $\nu_1 = 0.65\omega_0$, $\nu_2 = 0.5\omega_0$ and $Z = 90\omega_0$ (a), $Z = -0.95\omega_0$ (b), $Z = -90\omega_0$ (c).

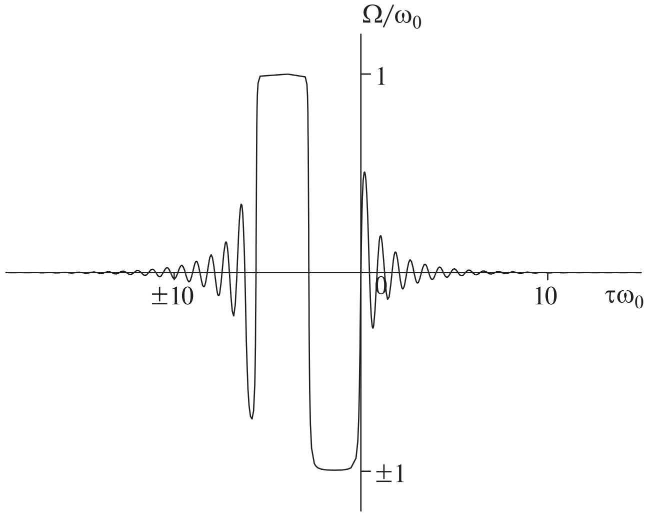

Fig. 2. Profiles of variable Ω of the breather solution with parameters $\beta = 1/\omega_0$, $W_0^{(0)} = -1/2$, $Z = 0$, $\nu_R = 0.5\omega_0$, $\nu_I = 8\omega_0$.

oscillations can arise if the carrier frequency of the breather is low enough.

Case $\beta > 0$.

Here the expression for variable Ω_0 of the one-soliton solution to the MRMB equations is written in the form

$$\Omega_0 = \frac{1}{\sigma}\frac{\partial}{\partial T}\arctan\frac{2\sigma^2\exp(\theta)}{\sigma^2[1-\exp(2\theta)]-\nu^2\varepsilon^2}. \qquad (25)$$

Substituting (25) into (20) yields an implicit definition of variable Ω of the one-soliton solution to GMRB system (12)–(15) in the given case. This solution is stationary and nonsingular for any real value of parameter ν. In limit $|\nu| \to \infty$, the duration of a soliton tends to minimum value $\tau_{\min} = \pi|\varepsilon/\sigma|$, and amplitude $|\Omega|$ tends to maximum value $|1/\varepsilon|$. The solution therefore becomes rectangular in this limit. Note that similar solutions were obtained for generalizations of the Gordon sine equation in [10, 11, 16].

The two-soliton solution to GRMB system (12)–(15) describes the elastic collision between solitons considered above. The collision between rectangular solitons in this case occurs in the form of overflowing [11].

Without loss of generality, the expression for variable Ω_0 of the breather solution of the MRMB equations is written in the form

$$\Omega_0 = \frac{1}{\sigma}\frac{\partial}{\partial T}\left(\arctan\frac{\nu_R q_-}{\nu_I p_+} - \arctan\frac{\nu_R q_+}{\nu_I p_-}\right), \qquad (26)$$

where

$$p_\pm = \sigma + i\varepsilon\nu_I \pm i\varepsilon\nu_R\sin(\theta_I)\exp(-\theta_R) + (\sigma - i\varepsilon\nu_I)\exp(-2\theta_R),$$

$$q_\pm = p_\pm - \sigma[1 \mp 2\cos(\theta_I)\exp(-\theta_R) + \exp(-2\theta_R)].$$

Substituting (26) into (20), we obtain an implicit definition of variable Ω of the breather solotion to GMRB system (12)–(15).

The shape of the two oscillations in the center of the breather of the MISP system becomes rectangular if $|\nu_I| > |\sigma/\varepsilon|$ and $\nu_R \to 0$. Their duration and amplitude are approximately equal to those of rectangular solitons; i.e., τ_{min} and $|1/\varepsilon|$, respectively (see Fig. 2).

CONCLUSIONS

A physical generalization of the RMB system was obtained that considered transitions to remote quantum states from two allocated stationary levels. The resulting system of GRMB equations (12)–(15) turned out to be integrable according to the inverse scattering problem. Using the GMRB system, we can consider situations in which the approximation of sudden excitations is not valid, assuming condition (1) is satisfied. This extends the spectral range of ESPs, the dynamics of which is described using the GMRB system.

The soliton and breather solutions studied in this work have features that discriminate them from the corresponding solutions of the RMB system. This largely concerns the collision of solitons of different polarities. In contrast to the collision of solitons of the RMB system, a short-lived pulse of high amplitude whose dynamics are similar to that of a killer wave can arise here.

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