Differential Invariants of Third Order Evolutionary Non-Linear PDEs

Elena N. Kushner
Moscow State Technical University of Civil Aviation
Moscow, Russia
ekushner@ro.ru

Alexei G. Kushner
Institute of Control Sciences of RAS, Lomonosov Moscow State University, Bauman Moscow State Technical University, Moscow Pedagogical State University
Moscow, Russia
kushner@physics.msu.ru

Alexey V. Samohin
Institute of Control Sciences of RAS
Moscow, Russia
samohinalexey@gmail.com

***Abstract*—For the third order evolution differential equations, the algebra of differential invariants with respect to the pseudogroup of point transformations is constructed. Such equations arise in the study of filtration processes (the Rapoport – Leas equation), the theory of nonlinear waves (the Korteweg – de Vries equation).**

***Index Terms*—jets, algebra of differential invariants, point transformations**

I. Introduction

Consider the following problem: describe regular orbits of the class of differential equations of third order

$$u_t = A(u)_x + B(u)_{xx} + C(u)_{xxx} \tag{1}$$

with respect to point transformations. Here $u = u(t,x)$ is unknown function, A, B and C are given function of class C^∞. We assume that these functions are defined in the same interval.

Equations (1) describe a wide class of nonlinear processes. For example, at $C = 0$, equations (1) arise in the theory of nonlinear filtering [1,2] and are called the generalized Rapoport – Leas equations. Finite-dimensional dynamics and attractors of such equations were constructed in [3]. Differential invariants of Rapoport – Leas equations were found in [4].

If

$$A(u) = -3u^2, \quad B(u) = 0, \quad C(u) = -u$$

then we get the classical Korteweg – de Vries equation [5]

$$u_t + 6uu_x + u_{xxx} = 0. \tag{2}$$

If

$$A(u) = -3u^2, \quad B(u) = \mu u, \quad C(u) = -u$$

then we get the Korteweg – de Vries equation with a dissipative member:

$$u_t + 6uu_x + u_{xxx} = \mu u_{xx}. \tag{3}$$

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By this reason we call equations (1) by *generalized Korteweg – de Vries equations*.

In this article, point transformations that preserve the class of equations (1) are found. It turned out that such transformations form a Lie group. A field of rational differential invariants has been found. This field separates the regular orbits, which makes it possible to classify the generalized Korteweg – de Vries equations.

II. Admissible transformations

Equation (1) can be rewritten in the following form:

$$\begin{aligned} u_t =& a(u)u_x + b(u)u_{xx} + b'(u)u_x^2 + c(u)u_{xxx} + \\ & 3c'(u)u_xu_{xx} + c''(u)u_x^3, \end{aligned} \tag{4}$$

where

$$a(u) = A'(u), \quad b(u) = B'(u), \quad c(u) = C'(u).$$

Thus, equation (4) is uniquely determined by three functions a, b, c of one variable u and therefore it can be considered as a parametrically defined curve in the space $\mathbb{R}^3$ with coordinates a, b, c.

Equation (4) defines the hypersurface

$$\mathcal{E} = \mathcal{E}_{(a,b,c)} = \{F(a,b,c) = 0\}$$

in the space of 3-jets $J^3(\mathbb{R}^2)$. Here

$$\begin{aligned} F(a,b,c) =& u_{1,0} - a(u_{0,0})u_{0,1} - b(u_{0,0})u_{0,2} - b'(u_{0,0})u_{0,1}^2 - \\ & c(u_{0,0})u_{0,3} - 3c'(u_{0,0})u_{0,1}u_{0,2} - c''(u_{0,0})u_{0,1}^3 \end{aligned}$$

and $t, x, u_{0,0}, u_{1,0}, \ldots, u_{0,3}$ are canonical coordinates on $J^3(\mathbb{R}^2)$ (see [6]).

Consider another equation $\widetilde{\mathcal{E}} = \mathcal{E}_{(\widetilde{a},\widetilde{b},\widetilde{c})}$ of type (4).

Equations $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ are equivalent with respect to point transformations if there exist a point transformation $\varphi : J^0(\mathbb{R}^2) \to J^0(\mathbb{R}^2)$ such that

$$\varphi^{(3)}(\widetilde{\mathcal{E}}) = \mathcal{E},$$

where $\varphi^{(3)}$ is a prolongation of the point transformation φ to the space $J^3(\mathbb{R}^2)$. It means that

$$\left(\varphi^{(3)}\right)^* (F(a,b,c)) = \lambda F(\widetilde{a}, \widetilde{b}, \widetilde{c}), \tag{5}$$

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where λ is a function on $J^3(\mathbb{R}^2)$.

Using Sophus Lie's approach we consider infinitesimal point transformations only. Let X be a vector field on the space $J^0(\mathbb{R}^2)$ and let φ_τ be a translation along trajectories of X from $\tau = 0$ to τ. Then φ_0 is a identiacl transformation. We get

$$\left(\varphi_\tau^{(3)}\right)^* (F(a,b,c)) = \lambda_\tau F(\widetilde{a}_\tau, \widetilde{b}_\tau, \widetilde{c}_\tau),$$

where λ_τ is a one-parameter family of functions on $J^3(\mathbb{R}^2)$ and $\widetilde{a}_\tau, \widetilde{b}_\tau, \widetilde{c}_\tau$ are one-parameter families of functions. Note that $\lambda_0 = 1$, $\widetilde{a}_0 = a, \widetilde{b}_0 = b, \widetilde{c}_0 = b$.

Differentiating both sides of the last equality with respect to τ at $\tau = 0$ and restricting the resulting equality to the equation $\mathcal{E}$ we get

$$X^{(3)}(F(a,b,c))\Big|_{\mathcal{E}} = -\alpha(u_{0,0})u_{0,1} - \beta(u_{0,0})u_{0,2} - \beta'(u_{0,0})u_{0,1}^2 - \gamma(u_{0,0})u_{0,3} - 3\gamma'(u_{0,0})u_{0,1}u_{0,2} - \gamma''(u_{0,0})u_{0,1}^3.$$

Here $X^{(3)}$ is the prolongation of the vector field X to $J^3(\mathbb{R}^2)$,

$$\alpha = \frac{da_\tau}{d\tau}\Big|_{\tau=0}, \quad \beta = \frac{db_\tau}{d\tau}\Big|_{\tau=0}, \quad \gamma = \frac{dc_\tau}{d\tau}\Big|_{\tau=0}.$$

The last formula gives a system of partial differential equations for coefficients of the vector field X. Solving it we obtain that the vector field X is a linear combination of the following vector fields:

$$\partial_t, \quad t\partial_t + \frac{1}{3}x\partial_x, \quad \partial_x, \quad x\partial_x, \quad t\partial_x, \quad \partial_{u_{0,0}}, \quad u_{0,0}\partial_{u_{0,0}}.$$

So, we get the following theorem:

Theorem 1: Admissible infinitesimal transformations of generalized Korteweg – de Vries equations form a 7-parametric Lie group.

III. Actions of admissible transformations on the coefficients of equations

Show how the Lie group of admissible transformations acts on the functions a, b, c.

Introduce the space $\mathbb{R}^4$ with coordinates u, a, b, c and the space $\mathbb{R}$ with a coordinate u and define the trivial bundle

$$\pi : \mathbb{R}^4 \to \mathbb{R}, \quad \pi : (u,a,b,c) \mapsto u.$$

A restriction of the Lie group of admissible transformations to this bundle form a five-dimensional Lie group, which we denote by G_π.

The corresponding Lie algebra $\mathfrak{g}_\pi$ is generated by the following vector fields:

$$\begin{aligned}
Y_1 &= \partial_u, \\
Y_2 &= u\partial_u, \\
Y_3 &= \partial_a, \\
Y_4 &= 2a\partial_a + b\partial_b, \\
Y_5 &= a\partial_a + 2b\partial_b + 3c\partial_c.
\end{aligned}$$

Let $J^k(\pi)$ be a k-jets of sections of the bundle π with canonical coordinates u, $a = a_0$, $b = b_0$, $c = c_0$, a_1, b_1, c_1, $\ldots$, a_k, b_k, c_k. Prolongations of the vector fields $Y_1, \ldots, Y_5$ into space $J^k(\pi)$ are the following:

$$\begin{aligned}
Y_1^{(k)} &= \partial_u, \\
Y_2^{(k)} &= u\partial_u - \sum_{i=1}^{k} i(a_i\partial_{a_i} + b_i\partial_{b_i} + c_i\partial_{a_i}), \\
Y_3^{(k)} &= \partial_a, \\
Y_4^{(k)} &= \sum_{i=0}^{k} (2a_i\partial_{a_i} + b_i\partial_{b_i}), \\
Y_5^{(k)} &= \sum_{i=0}^{k} (a_i\partial_{a_i} + 2b_i\partial_{b_i} + 3c_i\partial_{c_i}).
\end{aligned}$$

IV. Differential invariants

A function J on the space $J^k(\pi)$ is called a *differential invariant* of order $\leq k$ of the Lie group G_π (and the generalized Korteweg – de Vries equation) if it is constant on the orbits of the Lie group $G_\pi^{(k)}$. Here $G_\pi^{(k)}$ is a prolongation of the Lie group G_π to the space $J^k(\pi)$.

A differential invariant J satisfies the system of five linear differential equations

$$Y_i^{(k)}(J) = 0 \quad (i = 1, \ldots, 5). \tag{6}$$

A point $\theta \in J^k(\pi)$ is called *regular* if the tangent vectors $Y_{1,\theta}^{(k)}, \ldots, Y_{5,\theta}^{(k)}$ are linearly independent.

An orbit O_σ of the point $\sigma \in J^k(\pi)$ is called *regular* if each its point is regular.

Differential invariants form an algebra with respect to the operations of addition and multiplication, that is, if J_1 and J_2 are differential invariants, then their sum $J_1 + J_2$ and the product $J_1 J_2$ are also differential invariants.

The differential invariants $J_1, \ldots, J_s$ of order at most k are called *basic* if they are functionally independent and any other differential invariant of order at most k is their function. In this case, the number s is called the dimension of the algebra of differential invariants of order at most k.

A dimension of the algebra of differential invariants of order at most k is equal to the codimension of the regular orbit of the Lie group $G_\pi^{(k)}$.

In our case this dimension is equal to $3k - 1$, $k \geq 1$.

Among all differential invariants, one can distinguish rational invariants. Such invariants form a field and, according to the Lie – Tresse theorem [7] , they separate regular orbits.

Theorem 2: The field of rational differential invariants of generalized Korteweg – de Vries equations is generated by invariants

$$J_{k,1} = \frac{b_k b_0^{2k-1}}{a_1^k c_0^k},$$
$$J_{k,2} = \frac{c_k b_0^{2k}}{a_1^k c_0^{k+1}},$$
$$J_{k,3} = \frac{a_{k+1} b_0^{2k}}{a_1^{k+1} c_0^k},$$

where $k = 1, 2, \ldots$.

This algebra separates regular orbits of the Lie group $G_\pi^{(k)}$.

Example 1: For equations (2) and (3) all constructed differential invariants $J_{k,1}$, $J_{k,2}$ and $J_{k,3}$ are zero. This means that these equations belong to singular orbits of the Lie group $G_\pi^{(k)}$.

Example 2: For Korteweg – de Vries equation with cubic nonlinearity

$$u_t + u^2 u_x + \eta u_{xxx} = \mu u_{xx}$$

the invariants $J_{1,1}, J_{1,2}, J_{2,2}, J_{2,3}$ are zero and

$$J_{2,1} = \frac{\mu^2}{2\eta u^2}.$$

Example 3: For equation

$$u_t = \exp(u)_x + \exp(u)_{xx} + \exp(u)_{xxx}$$

all invariants are 1.

Example 4: For equation

$$u_t = A(u)_x + A(u)_{xx} + A(u)_{xxx}$$

we have

$$J_{1,1} = J_{1,2} = 1,$$
$$J_{2,1} = J_{2,2} = J_{2,3} = \frac{A'''A'}{(A'')^2},$$
$$J_{3,1} = J_{3,2} = J_{3,3} = \frac{A''''(A')^2}{(A'')^3}$$

and so on.

V. Equations with constant invariants

Consider case when first order differential invariants are constant, i.e.

$$J_{1,1} = \lambda, \quad J_{1,1} = \mu, \tag{7}$$

where λ and μ are nonzero constant.

Find the function A, B, C in equation (1).

Suppose condition (7) hold. Then the function a, b, c satisfy the following system of two differential equations:

$$\begin{cases} b'b - \lambda c a' = 0, \\ c'b^2 - \mu c^2 a' = 0. \end{cases}$$

Solving this system we get that $b(u)$ is arbitrary function and

$$a(u) = \frac{\mu}{C_2(2\lambda - \mu)} b(u)^{2-\frac{\mu}{\lambda}} + C_1,$$
$$c(u) = C_2 b(u)^{\frac{\mu}{\lambda}}.$$

Therefore

$$A(u) = \frac{\mu}{C_2(2\lambda - \mu)} \int B'(u)^{2-\frac{\mu}{\lambda}} du + C_1 u + C_3,$$
$$C(u) = C_2 \int B'(u)^{\frac{\mu}{\lambda}} du + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constant.

VI. Conclusion

Rational differential invariants of point transformations were constructed for generalized Korteweg – de Vries equations. These invariants form a field and they separate regular orbits.

Invariants can be used to solve the problem of equivalence of equations.

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