Optimality condition for infinite horizon optimal control problem

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Let X be a nonempty open convex subset of R^n, U be an arbitrary nonempty set in R^m. Let us consider the functional

$$J(u(\cdot), x_0, t_0, T) := \int_{t_0}^{T} g(x(t), u(t), t)\,\mathrm{d}t,$$

that may be unbounded when $T \to \infty$, subject to the dynamic constraint

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(t_0) = x_0, \tag{1}$$

where $u(t) \in U$ and $x(t) \in X$ exists for all $t \geq t_0$. Such control $u(\cdot)$ and trajectory $x(\cdot)$ are called *admissible*. Functions f and g are differentiable w.r.t. their first argument, x, and together with those partial derivatives are defined and locally bounded, measurable in t for every $(x, u) \in X \times U$, and continuous in (x, u) for almost every $t \in [0, \infty)$.

In addition to the *maximum principle* we find a new form of necessary conditions for the two following concepts of optimality. An admissible control $\hat{u}(\cdot)$ for which the corresponding trajectory $\hat{x}(\cdot)$ exists on $[t_0, +\infty)$ is

overtaking optimal (OO) if for all admissible controls $u(\cdot)$

$$\limsup_{T\to\infty} \left(J(u(\cdot), x_0, t_0, T) - J(\hat{u}(\cdot), x_0, t_0, T)\right) \leq 0,$$

weakly overtaking optimal (WOO) if for all admissible controls $u(\cdot)$

$$\liminf_{T\to\infty} \left(J(u(\cdot), x_0, t_0, T) - J(\hat{u}(\cdot), x_0, t_0, T)\right) \leq 0.$$

Proposition. Let for all $\tau \geq t_0$

$$\lim_{\alpha\to 0} \liminf_{T\to\infty} \left(\frac{J(\hat{u}(\cdot), \hat{x}(\tau) + \alpha\zeta, \tau, T) - J(\hat{u}(\cdot), \hat{x}(\tau), \tau, T)}{\alpha} - \langle \hat{J}_x(\tau, T), \zeta \rangle \right) \geq 0,$$

with all perturbations of the initial conditions, $x(\tau) = \hat{x}(\tau) + \alpha\zeta$, such that the resulting trajectories are feasible, $x(t) \in X$ in $[\tau, \infty)$, and the following upper limit being finite

$$\limsup_{T\to\infty} \left| \hat{J}_x(\tau, T) \right|,$$

where we denote the derivative of the functional w.r.t. initial condition as

$$\hat{J}_x(\tau, T) := \int_\tau^T K^*(t, \tau) \frac{\partial g}{\partial x}(\hat{x}(t), \hat{u}(t), t) \, \mathrm{d}t.$$

If control $\hat{u}$ is OO, then for all $\tau \in [t_0, \infty)$ and $u \in \hat{U}$

$$\limsup_{T\to\infty} \Big(\mathcal{H}(\hat{x}(\tau), u, \tau, \hat{J}_x(\tau, T), 1) - \mathcal{H}(\hat{x}(\tau), \hat{u}(\tau), \tau, \hat{J}_x(\tau, T), 1) \Big) \le 0.$$

If control $\hat{u}$ is WOO, then for all $\tau \in [t_0, \infty)$ and $u \in \hat{U}$

$$\liminf_{T\to\infty} \Big(\mathcal{H}(\hat{x}(\tau), u, \tau, \hat{J}_x(\tau, T), 1) - \mathcal{H}(\hat{x}(\tau), \hat{u}(\tau), \tau, \hat{J}_x(\tau, T), 1) \Big) \le 0,$$

where we use the *Hamilton-Pontryagin function*

$$\mathcal{H}(x, u, t, \psi, \lambda) = \lambda \, g(x, u, t) + \langle \psi, f(x, u, t) \rangle,$$

brackets $\langle \cdot, \cdot \rangle$ denote scalar product of two vectors, and $\hat{U} = \hat{U}(\hat{x}(\tau), \tau)$ is the set of control values $u(\tau)$ of all feasible pairs $(u(\cdot), x(\cdot))$ satisfying, for some scalar λ and vector ψ_0 such that $(\lambda, \psi_0) \neq 0$, the *maximum condition*:

$$\mathcal{H}(x(t), v, t, \psi(t), \lambda) - \mathcal{H}(x(t), u(t), t, \psi(t), \lambda) \le 0, \quad \forall v \in U,$$

the *state equation* (1) with $x_0 = \hat{x}(\tau)$ and $t_0 = \tau$, and the *adjoint equation*:

$$-\dot{\psi}(t) = \frac{\partial \mathcal{H}}{\partial x}(x(t), u(t), t, \psi(t), \lambda), \quad \psi(\tau) = \psi_0.$$

Corollary. Let, in addition to the conditions of the Proposition, there exists a number $\beta(\tau) > 0$ such that for all $x(\tau) \in X$ satisfying the inequality $|x(\tau) - \hat{x}(\tau)| < \beta(\tau)$, the initial value problem (1) with $u = \hat{u}$ and the initial condition $x(t_0) = \hat{x}(\tau)$ at $t_0 = \tau$ has an admissible solution solution, i.e. $x(t) \in X$ for all $t \ge \tau$. Then in the Proposition $\hat{U} = U$.

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