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# A Simple Description of the Turbulent Transport in a Stratified Shear Flow as Applied to the Description of Thermohydrodynamics of Inland Water Bodies

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**Abstract**—A way to parameterize the turbulent Prandtl number is proposed based on the model of turbulent transport in a stratified fluid, which allows for the two-sided transformation of the kinetic and potential energies of turbulent fluctuations. Numerical experiments aimed at studying the influence of the proposed parameterization on characteristic features of thermohydrodynamic processes in inland water bodies have been carried out.

*Keywords:* turbulence, numerical simulation, inland water body, stable stratification, turbulent Prandtl number, gradient Richardson number

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## INTRODUCTION

By now, it has been established [1–4] that inland water bodies play an important part in processes of the interaction between the atmosphere and active layer of the land surface. At the same time, it is well known that processes of small-scale turbulent mixing in the near-surface layer of the ocean and inland water bodies play the key role both in the development of global climate models [5–8] and in the creation of regional weather forecast models [9, 10]. Present-day global oceanic models do not resolve small-scale processes of turbulent mixing in the near-surface layer of the ocean. For this reason, these processes are taken into account due to the parameterization of turbulent exchange coefficients—turbulent viscosity and diffusion, which relate turbulent fluxes of mass, momentum, buoyancy, etc., with characteristics of average large-scale fields of velocity, density (temperature), and other parameters. Values of these coefficients significantly depend on the state of the water–air interface, density stratification, and other factors. In most present-day numerical models, the description of turbulent mixing processes in the ocean and inland water bodies requires solving partial differential equations for the mean flow; the turbulent transfer terms (Reynolds stresses) appearing in these equations are determined according to the assumed hypotheses of turbulence closure, e.g., gradient ones allowing one to relate Reynolds stresses to gradients of the average velocity (or any other scalar transported substance [11]). One-dimensional (along the vertical) models are most widely used in modeling turbulent processes in inland water bodies, which is related to their computational simplicity. Such models can also include the so-called $k$ [12, 13] and $k$–$\varepsilon$ [14, 15] schemes, in which evolutionary equations for the kinetic energy of turbulent fluctuations are considered ($k = \frac{1}{2}\langle u_i' u_i' \rangle$, where $u_i'$ are turbulent fluctuations of the i-th velocity component), or an additional equation for the rate of its (kinetic energy density) dissipation ($\varepsilon$). Note that the introduction of an additional equation for $\varepsilon$ is related to the fact that the expressions for coefficients of turbulent viscosity $K_m$ and turbulent heat conduction $K_h$, as well as dissipation rate $\varepsilon$, in the $k$ model with a single equation for the kinetic energy of turbulent fluctuations (see, e.g., [16]) include an unknown quantity—the external linear turbulence scale $L$. In particular, according to the Kolmogorov–Prandtl hypothesis [12], $K_h = c_\mu \sqrt{k} L$ and $\varepsilon = \frac{Ck^{3/2}}{L}$; the determination of $L$ in the case of complex flows is difficult. Solving equations in the

$k$–$\varepsilon$ model at large Reynolds numbers under conditions of local isotropic turbulence allows one to determine the distribution of linear turbulence scales $L = \frac{k^{\frac{3}{2}}}{\varepsilon}$. It is important to note that the calculation of heat transfer within the framework of the standard $k$–$\varepsilon$ model assumes that the turbulent Prandtl number relating the coefficients of turbulent viscosity and diffusion $(\mathrm{Pr}_T = K_m / K_h)$ is constant.

Applicability of the $k$–$\varepsilon$ scheme can be extended using additional algebraic relationships taking into account, e.g., the action of buoyancy forces, turbulence anisotropy, etc. In recent years, more complicated models also appeared, allowing one to take into account, e.g., the difference in the evolution of different components of Reynolds stresses (e.g., [17]). This, in turn, requires turning to more complicated schemes for the closure of higher order turbulent Reynolds stresses. However, equations obtained in this process allow one to take into account and analyze a series of interesting features appearing in turbulent flows in the presence of, e.g., buoyancy forces, horizontal motions of inland water bodies, anisotropy, etc.

The small-scale turbulence in oceans, seas, and inland water bodies is affected as a rule by two factors—density stratification and velocity shear, which must be taken into account when constructing its statistical theory. As was shown in [18], taking into account the stratification of density depending, e.g., only on temperature, leads to the appearance of the so-called dynamic (or flux) Richardson number $\mathrm{Ri}_f = -g \frac{\langle \rho' u_i' \rangle}{\langle u_i' u_j' \rangle V_{0z}}$ (here, $\langle \rho' u_i' \rangle$ is the mass flux, $\langle u_i' u_j' \rangle$ is the momentum flux, and $V_{0z}$ is the velocity shear) in the energy balance equations. The number characterizes the role of Archimedean forces and averaged shear motion in the balance of turbulent energy. In particular, it was shown in [18] that undamped turbulence is possible only at $\mathrm{Ri}_f < \mathrm{Ri}_{f\mathrm{cr}} < 1$ or, with allowance for the definition of coefficients of turbulent viscosity and turbulent diffusion, the condition for the existence of undamped turbulence is reduced to $\mathrm{Ri} < \mathrm{Pr}_T$, where $\mathrm{Ri} = \frac{N^2}{V_{0z}^2}$, ($N$ is the Brunt–Väisälä frequency) is the gradient Richardson number. However, there are some reasons to believe that the Prandtl number $\mathrm{Pr}_T$ does not remain constant and, with a change in the gradient Richardson number Ri, can be so large that the dynamic Richardson number $\mathrm{Ri}_f$ remains lower than the critical value and turbulence exists [18]. In particular, experimental measurements in the atmosphere, upper ocean layer, and laboratory [18–21] demonstrate that the Prandtl number can take large values even under strongly stable stratification, i.e., the coefficient of turbulent heat diffusion turns out to be considerably lower than the turbulent viscosity coefficient; therefore, turbulence can also exist at large values of the gradient Richardson number. It follows from the above that the question about the dependence $\mathrm{Pr}_T(\mathrm{Ri})$ seems to be important for studying the possibility to sustain the turbulence by weak shear in a stratified fluid which are characteristic both for the ocean and for inland water bodies.

There is a sufficient number of works in which features of semi-empirical models of turbulent closure [11–14] for stratified shear flows and, in particular, question about the existence of turbulence at large Richardson numbers are studied. Numerous empirical dependences $\mathrm{Pr}_T(\mathrm{Ri})$ were proposed (for example, [22–25]); they characterize mainly the asymptotic behavior of the dependence $\mathrm{Pr}_T(\mathrm{Ri})$, but do not allow one to sufficiently accurately establish the dependence shape at intermediate Ri values characterizing the transition from weak to strong turbulent stratified shear flows of the fluid.

In this work, we propose a parameterization of the Prandtl number obtained based on the modified theory of turbulent closure in stratified shear flows. The theory was developed by one of the authors in 1987[1] [28]. The procedure of obtaining the corresponding equations for means is similar to that used in the kinetic theory of gases [29, 30]. It includes a solution of the equation for the single-point distribution function $f$—the so-called kinetic equation; Reynolds stresses are calculated by the known distribution function. The proposed approach made it possible to take into account some important but usually neglected effects, e.g., the dependence of vertical turbulence anisotropy on stratification; nongradient correction to the traditional expression of the turbulent mass flow; and inverse transition from potential energy of turbulent fluctuations to kinetic energy, which eliminates restrictions on the existence of turbulence at large Richardson numbers.

We verify the dependence of the Prandtl number within the framework of numerical simulation of the thermohydrodynamic regime of inland water bodies with the use of the three-dimensional thermohydrostatic model [31, 32] verified earlier [33] using a more widely known one-dimensional (along the vertical) LAKE model by a modification of the $k$–$\varepsilon$ scheme, allowing one to take into account particularities of turbulent mixing in stratified water bodies. It is shown

[1] Later, in works by S.S. Zilitinkevich's group [26, 27], the theory of turbulent closure was proposed. It is based on balance equations for the kinetic and potential energy of turbulence, turbulent momentum fluxes, potential temperature, and relaxation equation for the turbulent time scale. This model also allows one to remove restrictions on the existence of turbulence at large Reynolds numbers.

that a similar parameterization of the Prandtl number can also be obtained within the framework of the modified $k$–ε scheme taking into account the mutual transformation of kinetic and potential turbulence energies.

## FUNDAMENTAL PRINCIPLES OF THE MODEL OF TURBULENT TRANSPORT IN A STRATIFIED SHEAR FLOW

Let us briefly turn our attention to main results of the model [28]. The main problem arising in calculations of average values of hydrodynamic quantities is related to an adequate approximation of the so-called collision integral (summands related to pressure fluctuations and fluctuation components of viscous forces). As is shown in [28], to obtain general expressions for turbulent flows of the momentum, density, energy, and other hydrodynamic quantities, it is possible to solve the equation of single-point distribution function $f$ under some assumptions. After finding the distribution function, Reynolds stresses are calculated by formulas of the probability theory. For the turbulent fluxes of momentum $\left\langle u_i'u_j'\right\rangle$ and mass $\left\langle \rho' u_i'\right\rangle$, as well as for fluxes of kinetic energy $\left\langle \sum_{j=1}^{3} u_i'^2 u_j'\right\rangle$ and density variance $\left\langle \sum_{j=1}^{3} \rho'^2 u_j'\right\rangle$, the following expressions were obtained:

$$\left\langle u_i'u_j'\right\rangle = V^2\delta_{ij} - LV\left(\frac{\partial\langle u_i\rangle}{\partial x_j} + \frac{\partial\langle u_j\rangle}{\partial x_i}\right), \quad (1)$$

$$\left\langle \rho' u_i'\right\rangle = -LV\left(\frac{\partial\langle\rho\rangle}{\partial x_i} + g_i\frac{\left\langle\rho'^2\right\rangle}{V^2\rho_0} - \frac{g\beta_i}{V^2\rho_0}\right), \quad (2)$$

$$\left\langle \sum_{j=1}^{3} u_j'^2 u_i'\right\rangle = -5LV\frac{\partial V^2}{\partial x_i}, \quad \left\langle \rho'^2 u_i'\right\rangle = -LV\frac{\partial\left\langle\rho'^2\right\rangle}{\partial x_i}. \quad (3)$$

Here, $L$ is the characteristic external scale of turbulence, $V$ is the characteristic scale of the velocity, $\vec{g}$ is the acceleration of gravity, and $\beta_i$ are components of the vector $\vec{\beta}$ characterizing pressure fluctuations in the stratified fluid (see below).

Note that the expression for the mass flux (2) includes an additional summand $g_i\frac{\left\langle\rho'^2\right\rangle}{V^2\rho_0} - \frac{g\beta_i}{V^2\rho_0}$, which leads to some significant differences from results obtained within the framework of usual gradient models [34]. For a statistically homogeneous field of density fluctuations, components of the vector $\vec{\beta}$ have the form $\beta_x = \beta_y = 0$ and $\beta_z = \left\langle\rho'^2\right\rangle R$, where $R$ is the anisotropy parameter:

$$R = \begin{cases} 1, & L_z \ll L_r, \\ \approx \left(\dfrac{L_r}{L_z}\right)^2, & L_z \gg L_r, \end{cases}$$

where $L_z, L_r$ are the vertical and horizontal scales of the density field correlation, respectively.

The expressions presented above allow one to obtain a closed model of a turbulent flow in a stratified fluid in the form of equations for averages: velocity $\langle\vec{u}\rangle$ and density $\langle\rho\rangle$, as well as kinetic energy of turbulence $k$ and variance of density pulsations $\left\langle\rho'^2\right\rangle$:

$$\frac{\partial\langle u_i\rangle}{\partial t} + \langle u_j\rangle\frac{\partial\langle u_i\rangle}{\partial x_j} + \frac{1}{\rho_0}\frac{\partial\langle p\rangle}{\partial x_i} + g_i\frac{\langle\rho\rangle-\rho_0}{\rho_0} = \frac{\partial}{\partial x_j}\left(L\sqrt{k}\left(\frac{\partial\langle u_i\rangle}{\partial x_j} + \frac{\partial\langle u_j\rangle}{\partial x_i}\right)\right), \quad (4.1)$$

$$\frac{\partial\langle\rho\rangle}{\partial t} + \langle u_i\rangle\frac{\partial\langle\rho\rangle}{\partial x_i} = 2\frac{\partial}{\partial x_i}L\sqrt{k}\left(\frac{\partial\langle\rho\rangle}{\partial x_i} + \frac{3}{2k\rho_0}\left(g_i\left\langle\rho'^2\right\rangle + g\beta_i\right)\right), \quad (4.2)$$

$$\frac{\partial k}{\partial t} + \langle u_i\rangle\frac{\partial k}{\partial x_i} - L\sqrt{k}\left(\frac{\partial\langle u_i\rangle}{\partial x_j} + \frac{\partial\langle u_j\rangle}{\partial x_i}\right)^2 - \frac{g}{\rho_0}L\sqrt{k} \times\left(\frac{\partial\langle\rho\rangle}{\partial z} + \frac{3g}{2k\rho_0}\left(\left\langle\rho'^2\right\rangle + \beta_z\right)\right) + \frac{Ck^{3/2}}{L} = \frac{5}{3}\frac{\partial}{\partial x_i}\left(L\sqrt{k}\frac{\partial k}{\partial x_i}\right), \quad (4.3)$$

$$\frac{\partial\left\langle\rho'^2\right\rangle}{\partial t} + \langle u_i\rangle\frac{\partial\left\langle\rho'^2\right\rangle}{\partial x_i} - 2\frac{\partial\rho}{\partial x_i}L\sqrt{k}\left(\frac{\partial\langle\rho\rangle}{\partial x_i} + \left(g_i\left\langle\rho'^2\right\rangle - g\beta_i\right)\frac{3}{2k\rho_0}\right) + \frac{Dk^{1/2}}{L}\left\langle\rho'^2\right\rangle = \frac{\partial}{\partial x_i}L\sqrt{k}\frac{\partial\left\langle\rho'^2\right\rangle}{\partial x_i}. \quad (4.4)$$

Note that dissipative and diffusive processes in Eqs. (4.3) and (4.4) are taken into account using Kolmogorov hypotheses. The turbulence energy dissipation rate ε and turbulent diffusion rate $\varepsilon_D$ are determined by the expressions $\varepsilon = \frac{Ck^{3/2}}{L}$ and $\varepsilon_D = \frac{Dk^{1/2}}{L}\left\langle\rho'^2\right\rangle$, where $C \sim D = 0.09$ are empirical constants.

Based on the proposed model, some examples of the interaction between turbulence and shear flows in a stratified liquid were considered; in particular, in [28], the evolution of homogeneous turbulence in the field of a flow with a constant velocity shear was studied.

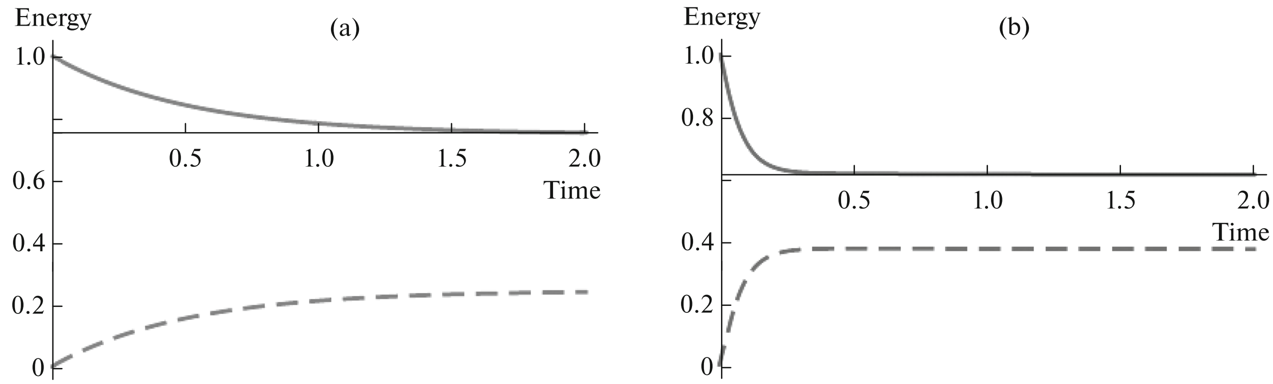


**Fig. 1.** Dimensionless kinetic energy $\bar{k} = k/k_x$, $k_x = (V_{0z})^2 L^2/C$ (the solid line) and potential energy $\bar{\Pi} = \Pi/\Pi_x$, $\Pi_x = (V_{0z})^2 L^2/C$ (the dashed line) of turbulence as functions of dimensionless time $\bar{t} = t/t_x$, $t_x = 1/V_{0z}\sqrt{C}$ at different values of the Richardson number Ri = (a) 0.5 and (b) 5. The anisotropy parameter $R = 0.5$ in both cases.

We turn our attention to the last problem in more detail because, within the framework of this formulation, one can successively analyze the influence of mutual transformation of the kinetic and potential energies on the turbulence evolution at large Richardson numbers and calculate the dependence of the turbulent Prandtl number on the Richardson number.

We briefly mention results of [28]. System (4) under the abovementioned assumptions is reduced to two equations for the kinetic and potential turbulence energies $k$ and $\Pi = \dfrac{\langle \rho'^2 \rangle g^2}{2N^2\rho_0^2}$ and has the form

$$\frac{dk}{dt} = V_{0z}^2 L\sqrt{k} - N^2 L\sqrt{k}\left(1 - \frac{3\Pi}{k}(1-R)\right) - \frac{Ck^{3/2}}{L}, \quad (5.1)$$

$$\frac{d\Pi}{dt} = N^2 L\sqrt{k}\left(1 - \frac{3\Pi}{k}(1-R)\right) - \frac{Dk^{1/2}\Pi}{L}. \quad (5.2)$$

First and foremost, an analysis of system (5) shows that the system has stationary solutions for the kinetic and potential energies:

$$k = k_2 = \frac{V_{0z}^2 L^2}{2C} f(\mathrm{Ri}), \qquad \Pi = \Pi_2 = \frac{V_{0z}^2 L^2}{C} - k_2, \quad (6)$$

which are attained during the time $t_1 \sim 1/V_{0z}\sqrt{C}$ (see Fig. 1), where $f(\mathrm{Ri}) = 1 - (4-3R)\mathrm{Ri} + \left(1 - \mathrm{Ri}(4-3R)^2 + 12(1-R)\mathrm{Ri}\right)^{1/2}$ is a function of the Richardson number.

It is clear from system (5) that the gradient summand $\left(-N^2 L\sqrt{k}\right)$ in the expression for the buoyancy flux describes the transition from kinetic energy to potential energy (stable stratification damps the turbulence). The additional summand $\left(\dfrac{3\Pi N^2 L(1-R)}{\sqrt{k}}\right) \sim \langle \rho'^2 \rangle$ related to the nongradient addition in the mass flux has a sign opposite to the gradient sign and describes the reverse transition from potential energy to kinetic energy due to the work of the buoyancy force. Here, if the Richardson number is large (Ri $\gg$ 1) (see Fig. 1), the fraction of turbulent fluctuation energy transiting from kinetic to potential and back is balanced the work of the buoyancy force during a rather short time $t_2 \sim \left(\dfrac{\sqrt{k}}{N^2 L}\right) \ll t_1$; $\left(\dfrac{t_2}{t_1} = \mathrm{Ri}^{-1}\right)$. This means that, at large Richardson numbers, a relationship $k = 3\Pi(1-R)$ between the kinetic and potential turbulence energies is reached during a rather short time; the further evolution of turbulence does not depend on the Richardson number (see [28]).

The results presented above allow one to calculate the turbulent Prandtl number in the steady state. By definition,

$$\mathrm{Pr}_T = \frac{\langle u_i' u_j' \rangle \rho_{0_z}}{\langle \rho' u_i' \rangle V_{0_z}} = \left(1 - \frac{3\Pi}{k}(1-R)\right)^{-1};$$

from system (6), we obtain an increasing dependence of the Prandtl number on the Richardson number:

$$\mathrm{Pr}_T(\mathrm{Ri}) = \frac{(4-3R)\mathrm{Ri} + 1 + \left(((4-3R)\mathrm{Ri}+1)^2 - 4\mathrm{Ri}\right)^{\frac{1}{2}}}{2}. \quad (7)$$

It is important to emphasize the certain universality of the dependence $\mathrm{Pr}_T(\mathrm{Ri})$, which is determined by the quantity $\frac{3\Pi}{k}(1-R)$ related to the transition of a portion of potential energy to kinetic energy, at $\mathrm{Ri} \gg 1$:

$$\mathrm{Pr}_T(\mathrm{Ri}) = (4-3R)\,\mathrm{Ri}. \quad (8)$$

Note that expressions (7) and (8) also preserve their form within the framework of the $k$–ε model, when the expression $K_m = c_\mu\sqrt{k}L$ is used for the coefficient of eddy viscosity and, correspondingly, for the turbulence scale, $L = C\frac{k^{\frac{3}{2}}}{\varepsilon}$. Using the traditional approximation for the diffusion rate of turbulent fluctuations of a scalar quantity (see, e.g., [16]), one can find the dissipation rate of potential energy of turbulent fluctuations by the expression $\varepsilon_\Pi = R_1\varepsilon\frac{\Pi}{k}$ at $R_1 = 1$ (which means that the time scales of pulsations of the velocity and scalar quantity are equal). Then, expressions (6) are transformed to the form

$$\frac{\varepsilon^2}{k^2} = \frac{V_{0z}^2 C}{2} f(\mathrm{Ri}), \quad (*.1)$$

$$\frac{\Pi}{k} = \frac{2(1-f(\mathrm{Ri}))}{f(\mathrm{Ri})}. \quad (*.2)$$

Since the Prandtl number parameterization obtained above (see (7) and (8)) depends only on the ratio $\Pi/k$, it follows from (*.2) that it is also valid for the extended $k$–ε model.

Therefore, it is clear from (7) and (8) that the condition of turbulence sustainment in a stratified fluid $\mathrm{Ri} \ll \mathrm{Pr}_T$ is satisfied at any Ri; i.e., there is no generation threshold for the Richardson number. Note that the asymptotic dependence (7), (8) agrees with results in [27, 35], where parameterizations with an unbounded growth of the Prandtl number were presented. The comparison with the EFB model of S.S. Zilitinkevich's group is presented in Fig. 2.

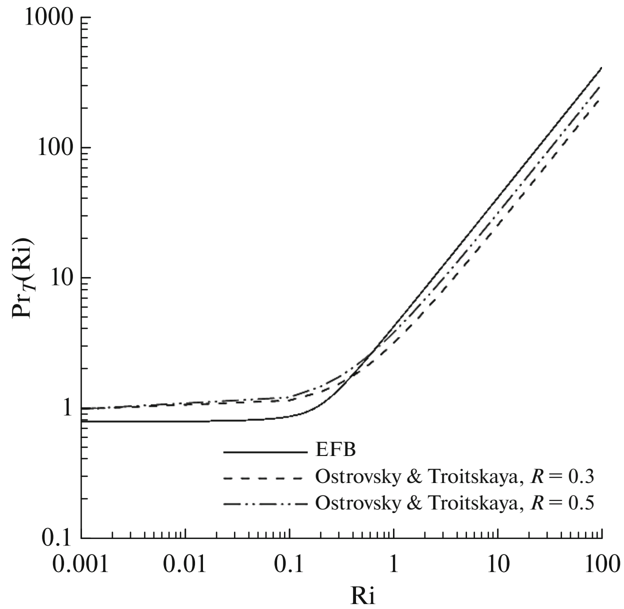


**Fig. 2.** Dependence of the Prandtl number on the Richardson number in two different models.

The next section demonstrates the result of using the obtained parameterization of the turbulent Prandtl number (7) in idealized calculations of thermohydrodynamics of inland water bodies.

## APPLICATION OF THE PROPOSED PARAMETERIZATION IN THE MODEL OF THE THERMODYNAMIC REGIME OF AN IDEALIZED INLAND WATER BODY

The effect of the turbulent Prandtl number parameterization on processes of mixing in inland water bodies in this work was estimated using the RANS (Reynolds-Averaged Navier–Stokes) three-dimensional hydrostatic model developed in the Research Computing Center, Moscow State University, based on the common code combining both the RANS and the DNS (Direct Numerical Simulation) and LES (Large-Eddy Simulation) approaches for the calculation of geophysical turbulent flows at high spatial and temporal resolution (see, e.g., [31–33]).

The numerical model includes equations of hydrodynamics in a stratified turbulent rotating fluid layer in the shallow-water approximation, as well as the equation for heat transfer with allowance for horizontal and vertical diffusion:

$$\frac{\partial u}{\partial t} = -A(u) + D_H(u,\lambda_m) + D_z(u, K_m+\nu) - g\frac{\partial\eta}{\partial x} - \frac{g}{\rho_0}\frac{\partial}{\partial x}\eta\int_z^\eta \rho dz' + fv, \quad (9.1)$$

$$\frac{\partial v}{\partial t} = -A(v) + D_H(v,\lambda_m) + D_z(v, K_m+\nu) - g\frac{\partial\eta}{\partial y} - \frac{g}{\rho_0}\frac{\partial}{\partial y}\int_z^\eta \rho dz' - fu, \quad (9.2)$$

$$\nabla\cdot\mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (9.3)$$

$$\frac{\partial T}{\partial t} = -A(T) + D_H(T,\lambda_h) + D_z(T, K_h+\chi'), \quad (9.4)$$

$$\rho = \rho(T), \quad (9.5)$$

$$\frac{\partial\eta}{\partial t} = w. \quad (9.6)$$

Here, $\mathbf{u} = (u, v, w)$ is the velocity vector; η is the free surface deviation from the equilibrium state; $f$ is the Coriolis parameter; $T$ is the temperature; ρ is the density; $K_m$ ($\lambda_m$) and $K_h$ ($\lambda_h$) are coefficients of vertical (horizontal) turbulent viscosity and temperature conductivity, respectively; $\nu$ and $\chi'$ are coefficients of molecular viscosity and temperature conductivity; $z$ is the vertical coordinate passing from the water body bottom $z = -H(x, y)$ to the surface; and $t$ is time. In addition, $A(q)$ is the advection operator,

$$A(q) = \frac{\partial uq}{\partial x} + \frac{\partial vq}{\partial y} + \frac{\partial wq}{\partial z},$$

and $D_H(q,\lambda)$ and $D_z(q,K)$ are operators of horizontal and vertical diffusion with coefficients $\lambda$ and $K$, respectively:

$$D_H(q,\lambda) = \frac{\partial}{\partial x}\lambda\frac{\partial q}{\partial x} + \frac{\partial}{\partial y}\lambda\frac{\partial q}{\partial y},$$

$$D_z(q,K) = \frac{\partial}{\partial z}K\frac{\partial q}{\partial z}.$$

Processes of vertical turbulent mixing (calculation of the coefficients $K_m$ and $K_h$) were described using the two-equation model, the so-called standard $k$–$\varepsilon$ model (see, e.g., [22]), which involves prognostic equations for the kinetic energy of turbulence and rate of its dissipation:

$$\frac{\partial k}{\partial t} = \frac{\partial}{\partial z}\left(\frac{K_m}{\delta_k} + \nu\right)\frac{\partial k}{\partial z} + P + B - \varepsilon, \qquad (10.1)$$

$$\frac{\partial \varepsilon}{\partial t} = \frac{\partial}{\partial z}\left(\frac{K_m}{\delta_\varepsilon} + \nu\right)\frac{\partial \varepsilon}{\partial z} + \frac{\varepsilon}{k}\left(C_{1\varepsilon}P - C_{2\varepsilon}\varepsilon + C_{3\varepsilon}B\right), \qquad (10.2)$$

$$K_m = C_\varepsilon\frac{k^2}{\varepsilon}, \qquad (10.3)$$

$$K_h = C_{\varepsilon,T}\frac{k^2}{\varepsilon} = \frac{C_{\varepsilon,T}}{C_\varepsilon}K_m = \mathrm{Pr}_T^{-1}K_m. \qquad (10.4)$$

Here, the summand $P$ corresponds to turbulence energy production by the velocity shear and $B$ describes production or consumption by the action of buoyancy forces; $\delta_k$, $\delta_\varepsilon$ are the turbulent Schmidt numbers for the turbulent kinetic energy and dissipation rate, respectively; $C_{1\varepsilon}$, $C_{2\varepsilon}$, and $C_{3\varepsilon}$ are empirical constants; and $C_\varepsilon$ and $C_{\varepsilon,T}$ are the stability functions for the momentum and scalar quantities. In the standard $k$–$\varepsilon$ model, they are assumed to be constant.

Note that, for a homogeneous and steady turbulent flow, using the expressions for sources of turbulence energy $P$ and $G$, it follows from (10.2) that

$$C_{1\varepsilon}V_{0z}^2 - C_{3\varepsilon}N^2\left(1 - \frac{3\Pi}{k}(1 - R)\right) = \frac{C_{2\varepsilon}}{C}\frac{\varepsilon^2}{k^2}. \qquad (***)$$

Taking into account formulas (**.1) for $\varepsilon^2/k^2$ and (**.2) for $\Pi/k$, which express these quantities in terms of the Richardson number Ri, we find that constants of the $k$–$\varepsilon$ model are not independent but connected by the relation depending on Ri:

$$C_{1\varepsilon} - C_{3\varepsilon}\mathrm{Ri}\left(1 - \frac{6(1 - f(\mathrm{Ri}))}{f(\mathrm{Ri})}(1 - R)\right) = \frac{C_{2\varepsilon}}{C2}f(\mathrm{Ri}).$$

In particular, taking into account the asymptotic behavior of the function $f(\mathrm{Ri})$ as $\mathrm{Ri} \to \infty$, we obtain a relationship between the constants in the form

$$C_{1\varepsilon}(4 - 3R) - C_{3\varepsilon} = 3C_{2\varepsilon}(1 - R).$$

This feature was taken into account when choosing constants of the $k$–$\varepsilon$ model used in calculations.

As for numerical methods used in the calculations, the system of equations is spatially discretized using conservative second-order accurate finite-difference schemes and integrated in time with by semi-implicit approximation [20].

To estimate the effect of turbulent Prandtl number parameterization on the description of turbulent mixing processes, we carried out in this model two types of numerical experiments. In both cases, idealized water bodies with a rectangular cross section and parameters characteristic of real lakes and water reservoirs were considered: depth of 10 m, surface temperature of 20°C with the initial gradient $\partial T/\partial z = 1.5$°C/m, and Brunt–Väisälä frequency (buoyancy frequency) $N = 4 \times 10^{-2}$ s$^{-1}$. Within the framework of the first experiment, the calculations by the standard $k$–$\varepsilon$ scheme were compared; in this scheme, the Prandtl number was assumed to be constant: $\mathrm{Pr}_{T_0} = 1.25$. Such a value agrees with estimates of $\mathrm{Pr}_{T_0}$ at $\mathrm{Ri} \approx 0$ according to data of laboratory investigations and direct numerical simulation [36, 37]; as a rule, it is used in calculations of circulation in inland water bodies with neutral (or close to neutral) stratification (see, e.g., [38]). In the second series of numerical experiments, calculations with the use of the modified $k$–$\varepsilon$ were compared with the standard model at the Prandtl number $\mathrm{Pr}_{T_0} = 1$, which, according to parameterization (7) presented above, corresponds to the case $\mathrm{Ri} \to 0$. The influence of the magnitude of the momentum flux on the surface and rotational forces on mixing processes in both closures was also considered. In the first experiment, wind forcing was assumed to be one-dimensional: components of the friction velocity are $u^* = 0.00316$ m/s and $v^* = 0$ m/s and the value of the anisotropy parameter $R$ in the formula for $\mathrm{Pr}_T(\mathrm{Ri})$ is 0.5. In the second experiment, forcing was increased: $u^* = v^* = 0.00316$ m/s; values of the parameter $R$ were equal to 0.2 and 0.7.

The following presents calculation results for the main characteristics of the thermohydrodynamic regime of mixing in an idealized inland water body: the vertical distribution of temperature and kinetic energy in the first experiment (Fig. 3); the same characteris-

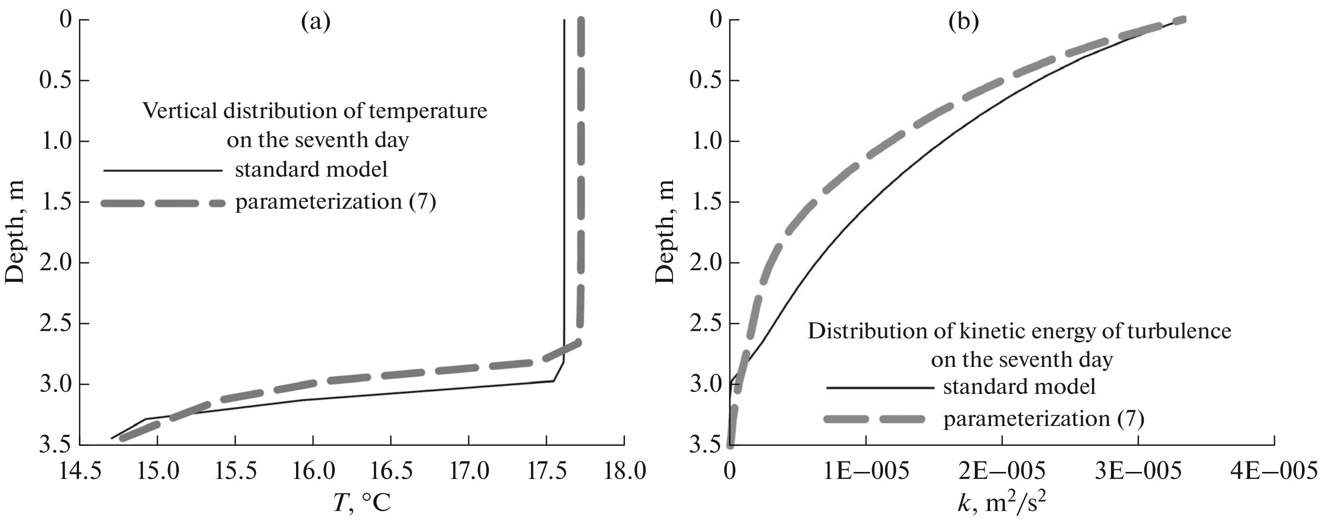


**Fig. 3.** Vertical distribution of (a) temperature, °C, and (b) kinetic energy of turbulence, $m^2/s^2$, in the first experiment.

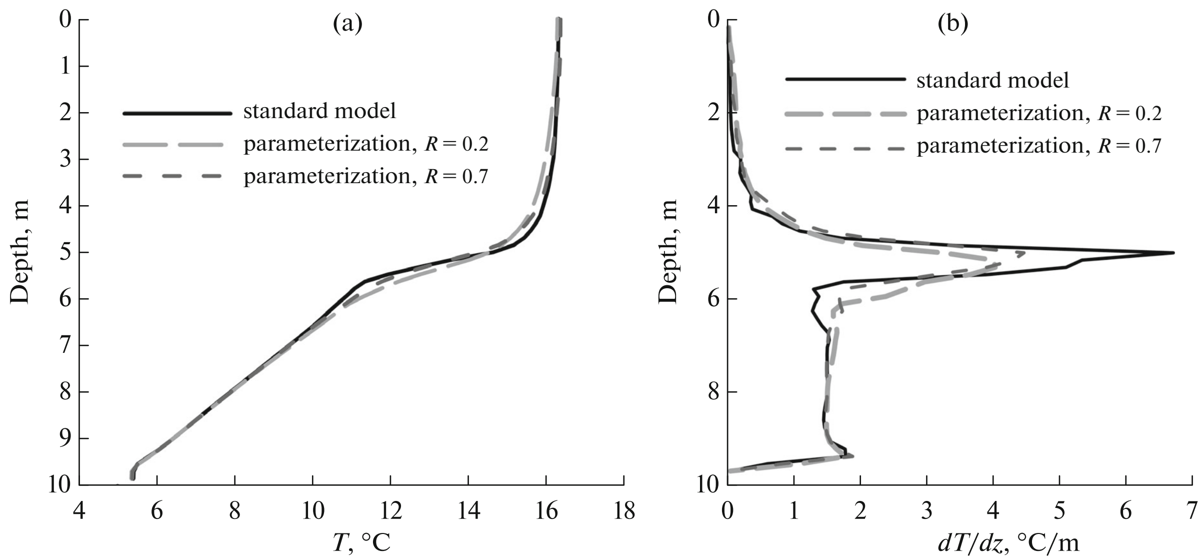


**Fig. 4.** Vertical distribution of (a) temperature and (b) temperature gradient in the second experiment.

tics, as well as their gradients (Figs. 4, 5), in the second experiment; and profiles of variation in the Richardson number in the process of mixing (Fig. 6).

Results of calculations with parameterization demonstrate that the main characteristics of mixing—vertical distribution of temperature and profile of the heat conduction coefficient—are sensitive to the parameterization $\mathrm{Pr}_T(\mathrm{Ri})$. In the process of the numerical experiment, the Richardson number Ri varies considerably in the range from ~0.001 to ~100 and reaches the value of 10 as early as at a depth of about 2.5–3 m. The kinetic energy varies smoothly over the whole depth of the mixing region. It is important to emphasize that taking into account the parameterization leads to smoothing all sharp changes in vertical distributions of turbulent kinetic energy, temperature, and thickness of the transition layer. The results are related to the features of this parameterization from which the existence of turbulence follows at $\mathrm{Ri} \gg 1$.

A change in input calculation data (forcing and anisotropy parameter) also has an effect on characteristic parameters of mixing. However, these changes are expectable and are not of a fundamental nature (the changing quantities are the formation time, depth, and thickness of the temperature thermocline) for this model. A change in values of the friction velocity has an effect mainly on the time of the mixed layer formation

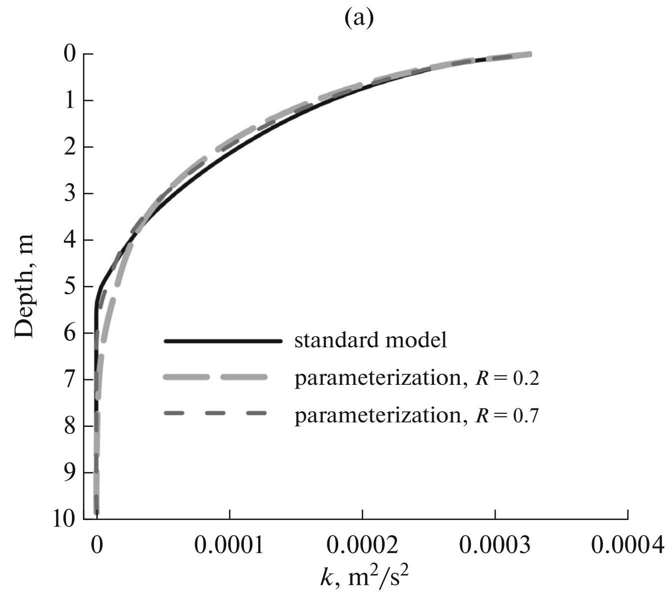


**Fig. 5.** Vertical distribution of (a) turbulent kinetic energy and (b) turbulent kinetic energy gradient in the second experiment.

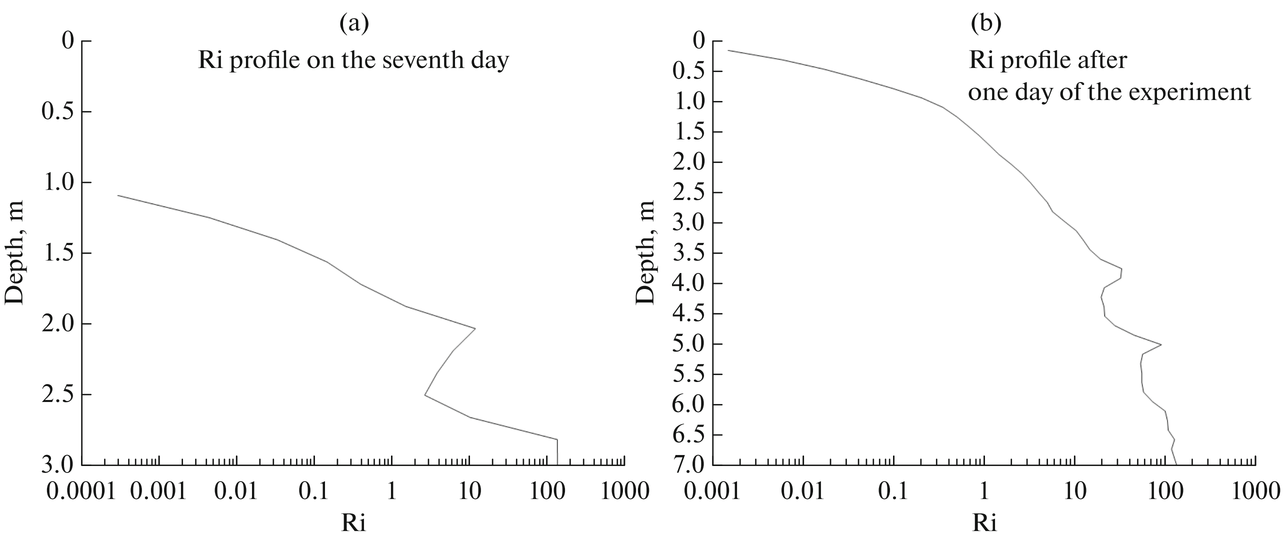


**Fig. 6.** Profile of the Richardson number (a) in the first experiment on the seventh day and (b) in the second experiment after one day.

(in particular, processes of mixing occur faster with an increase in $u$*; also see calculations in [20]).

It should be noted that the sensitivity of the numerical scheme to the parameterization $\mathrm{Pr}_T(\mathrm{Ri})$ (especially in the region of large gradients) seems to be important in problems of reproducing concentrations of biochemical admixtures in inland water bodies and describing the gas exchange with the atmosphere (see [39]). In this work, we restricted ourselves to the consideration of a thermally stratified medium in the absence of passive admixtures; however, in models of inland water bodies including the transport of biochemical components, turbulent Schmidt numbers describing the similarity between processes of turbulent diffusion of the scalar and turbulent transport of the momentum are, as a rule, supposed to be equal to the turbulent Prandtl number (see, e.g., [40]). Thus, one can expect that using the parameterization $\mathrm{Pr}_T(\mathrm{Ri})$ can have an effect on the transport of biochemical admixtures in small inland water bodies, in particular, through the thermocline. In turn, measurement data [41] demonstrate that seasonal changes in the extinction coefficient of penetrating shortwave radiation for water bodies in northern latitudes are directly related to the vertical distribution of concentrations of organic substances having an effect on transmission capacity of the fluid. Results of the numerical simulation on reproducing the seasonal variability and, in particular, the time of ice cover formation [41], are strongly sensitive to parameterization of these processes and, therefore, require a detailed study with the use of complete models, including a description of radiative processes and the transport of biochemical admixtures.

## CONCLUSIONS

In this work, a turbulent Prandtl number parameterization based on the turbulent closure model [28] is proposed. It takes into account the two-sided transformation of kinetic and potential energies of turbulent fluctuations. One distinctive feature of the proposed parameterization is that it contains the ratio of the potential energy of turbulent fluctuations $\Pi$ to the kinetic energy $k$; the ratio, in turn, determines the nongradient addition in the expression for the average mass flux. This parameterization is in good qualitative agreement with the model obtained relatively recently by S.S. Zilitinkevich and colleagues [26, 27]. The parameterization is introduced into the formula for the coefficient of turbulent heat conduction of the $k$–$\varepsilon$ model to correctly take into account the stable stratification in calculations of the thermohydrodynamic regime of inland water bodies.

Results of calculations allow one to conclude that the description of the vertical mixing in inland water bodies, even in the idealized formulation, is sensitive to the parameterization of the turbulent Prandtl number. It is important to emphasize that taking into account the parameterization leads to smoothing all sharp changes in vertical distributions of turbulent kinetic energy, temperature, and thickness of the transition layer. This is related to features of this parameterization, which is valid in a wide range of Richardson numbers (in particular, at large values $\mathrm{Ri} \gg 1$), while calculations by the standard model with a constant value of the Prandtl number assume the existence of a critical value of the dynamic (flux) Richardson number, in the neighborhood of which some hypotheses of the semiempirical theory of turbulence lose their meaning [18].

The result that was obtained suggests the expediency of carrying out additional laboratory and field measurements of corresponding coefficients of turbulent viscosity and diffusion with the aim of choosing justified parameterizations of the turbulent Prandtl number in calculations of the seasonal and interannual dynamics of stratified inland water bodies.

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## REFERENCES

1. Persson, T., Evaporation and Heat-Flux Aggregation in Heterogeneous Boreal Landscape, *D. Phil. Dissertation*, Uppsala: Uppsala University, 2004.
2. Eerola, K., Rontu, L., Kourzeneva, E., Pour, H.K., and Duguay, C., "Impact of partly ice-free Lake Ladoga on temperature and cloudiness in an anticyclonic winter situation – a case study using a limited area model," *Tellus A,* 2014, vol. 66, pp. 1–17.
3. Stepanenko, V.M., and Mikushin, D.N., "Numerical simulation of mesoscale dynamics of the atmosphere and pollutant transfer over a hydrologically inhomogeneous surface," *Vych. Tekhnol.,* 2008, vol. 13, pp. 104–110.
4. Stepanenko, V.M., Miranda, P.M., and Lykosov, V.N., "Numerical simulation of mesoscale interaction of the atmosphere with a hydrologically inhomogeneous land," *Vych. Tekhnol.,* 2006, vol. 11, pp. 118–127.
5. Tsuang, B.-J., Tu, C.-J. and Arpe, K., *Lake Parameterization for Climate Models,* Hamburg: Max Planck Institute for Meteorology, 2001.
6. Mironov, D., Terzhevik, A., Beyrich, F., Golosov, S., Haise, E., Kirillin, G., Kourzeneva, E., Ritter, B., and Schneider, N., "Parameterization of lakes in numerical weather prediction: description of a lake model, single-column tests, and implementation into the limited-area NWP model," *Boundary-Layer Meteorol.,* 2006, Spec. Issue.
7. Long, Z., et al., "Northern lake impacts on local seasonal climate," *J. Hydrometeorol.,* 2007, vol. 8 (4), 881–896.
8. Ljungemyr, P., Gustafsson, N., and Omstedt, A., "Parameterization of lake thermodynamics in a high-resolution weather forecasting model," *Tellus A,* 1996, vol. 48, pp. 608–621.
9. Mackay, M., "Modeling the regional climate impact of boreal lakes," *Geophys. Res. Abstr.,* 2006, vol. 8, pp. 05405.
10. Hostetler, S.W., Bates, G.T., and Giorgi, F., "Interactive coupling of a lake thermal model with a regional climate model," J. Geophys. Res. 1993, vol. 98D, pp. 5045–5057.
11. Monin, A.S. and Yaglom, A.M., *Statisticheskaya gidromekhanika. Chast' 1* (Statistical Hydromechanics. Part 1), Golitsyn, G.S., Ed., Moscow: Nauka, 1965, p. 333.
12. Monin, A.S. and Yaglom, A.M., *Statisticheskaya gidromekhanika. Chast' 1* (Statistical Hydromechanics. Part 1), Golitsyn, G.S., Ed., Moscow: Nauka, 1965, pp. 318–325.
13. Stepanenko, V.M., Numerical modeling of interactions between the atmosphere and land water bodies, *Doctoral (Math.-Phys.) Dissertation,* Moscow, 2007, pp. 27–28.
14. Mellor, C.L. and Yamada, T., "A hierarchy of turbulence closure models for planetary boundary layers," *J. Atmos. Sci.,* 1974, vol. 31, pp. 1791–1806.
15. Lykossov, V.N., "On the problem of closure of turbulent boundary layer models using equations for the kinetic energy of turbulence and the rate of its dissipa-

tion," *Izv. Akad. Nauk SSSR, Fiz. Atmos. Okeana,* 1992, vol. 28, pp. 696–704.

16. Rodi, W., in *Prediction Methods for Turbulent Flows,* Kollmann, W., Ed., London: Hemisphere, 1980.
17. Yakovenko, S.N., "Budget of equations for Reynolds stresses in the region of turbulence arising from the breaking of internal waves," *Vestnik Novosib. Gos. Univ. Ser.: Fiz.,* 2012, vol. 7, no. 4, pp. 87–95.
18. Monin, A.S. and Yaglom, A.M., *Statisticheskaya gidromekhanika. Chast' 1* (Statistical Hydromechanics. Part 1), Golitsyn, G.S., Ed., Moscow: Nauka, 1965, pp. 338–437.
19. Kondo, B. and Curry, J.A., "Heat and momentum transfer under strong stability in the atmospheric surface layer," *J. Atmos. Sci.,* 2000, vol. 57, pp. 1417–1423.
20. Strang, E.J. and Fernando, H.J., "Vertical mixing and transports through a stratified shear layer," *J. Phys. Oceanogr.,* 2001, vol. 31, pp. 2026–2048.
21. Stretch, D.D., Rot, J.W. et al., in *14th Australasian Fluid Mech. Conf. (Adelaide, Australia, 10–14 December 2001),* pp. 625–628.
22. Kantha, L. and Clayson, S., "An improved mixed layer model for geophysical applications," *J. Geophys. Res.,* 1994, vol. 99, no. C12, pp. 25235–25266.
23. Mellor, G. and Yamada, T., "Development of a turbulence closure model for geophysical problems," *Rev. Geophys. Space Phys.,* 1982, vol. 20, no. 4, pp. 851–875.
24. Baas, P., de Roode, S., and Lenderink, G., "The scaling behaviour of a turbulent kinetic energy closure model for stably stratified conditions," *Boundary-Layer Meteorol.,* 2008, vol. 127, pp. 17–36.
25. U. Schumann and T. Gerz, "Turbulent mixing in stably stratified shear flows," *J. Appl. Meteorol.,* 1996, vol. 34, pp. 33–48.
26. Zilitinkevich, S.S., Elperin, T., Kleeorin, N., and Rogachevskii, I., "Energy- and Flux-Budget (EFB) turbulence closure models for stably-stratified flows. Part I: Steady-state, homogeneous regimes," *Boundary-Layer Meteorol.,* 2007, vol. 125, pp. 167–191.
27. Zilitinkevich, S.S., Elperin, T., Kleeorin, N., Rogachevskii, I., and Esau, I., "A hierarchy of Energy- and Flux-Budget (EFB) turbulence closure models for stably-stratified geophysical flow," *Boundary-Layer Meteorol.,* 2013, vol. 146, pp. 341–373.
28. Ostrovskii, L.A. and Troitskaya, Yu.I., "Model of turbulent transfer and the dynamics of turbulence in a stratified shear flux," *Izv. Akad. Nauk SSSR, Fiz. Atmos. Okeana,* 1987, no. 3, pp. 101–104.
29. Lundgren, T.S., "Distribution functions in the statistical theory of turbulence," *Phys. Fluids,* 1967, vol. 10, p. 969.
30. Vlasov, A.A., *Statisticheskie funktsii raspredeleniya* (Statistical Distribution Functions), Moscow: Nauka, 1966.
31. Mortikov, E.V., "Numerical simulation of the motion of an ice keel in stratified flow," *Izv., Atmos. Oceanic Phys.,* 2016, vol. 52, pp. 108–115.
32. Mortikov, E.V., Glazunov, A.V. and Lykosov, V.N., "Numerical study of plane Couette flow: turbulence statistics and the structure of pressure-strain correlations," *Russ. J. Num. Anal. Math. Modell.,* 2019, vol. 34, no. 2, pp. 1–14.
33. Gladskikh, D.S., Stepanenko, V.M., and Mortikov, E.M., "On the influence of the horizontal dimensions of inland water bodies on the thickness of the upper mixed layer," *Vodn. Resur.*, 2021 (in press).
34. Monin, A.S. and Ozmidov, R.V., *Okeanskaya turbulentnost'* (Ocean Turbulence) Leningrad: Gidrometeoizdat, 1981, pp. 26–29, 237–240.
35. Sukoriansky, S., Galperin, B., and Perov, V., "A quasi-normal scale elimination model of turbulence and its application to stably stratified flows," *Nonlin. Proc. Geophys.,* 2006, vol. 13, pp. 9–22.
36. Li, D., "Turbulent Prandtl number in the atmospheric boundary layer – where are we now?," *Atmos. Res.,* 2019, vol. 216, no. 1, pp. 86–105.
37. Kays, W., "Turbulent Prandtl number – where are we?," *J. Heat Transfer,* 1994, vol. 116, pp. 284–295.
38. Goudsmit, G.-H., Burchard, H. Peeters, F., and Wuest, A., "Application of k-eps turbulence models to enclosed basins: The role of internal seiches," *J. Geophys. Res.,* 2002, vol. 107, no. C12, p. 3230.
39. Karpowicz, M., and Ejsmont-Karabin, J., "Effect of metalimnetic gradient on phytoplankton and zooplankton (Rotifera, Crucstacea) communities in different trophic conditions," *Environ. Monit. Assess.,* 2017, vol. 189, no. 8, p. 367.
40. Heiskanen, J.J., Mammarella, I., Ojala, A., Stepanenko, V., Erkkilä, K.-M., Miettinen, H., Sandstrom, H., Eugster, W., Leppäranta, M., Järvinen, H., Vesala, T., and Nordbo, A., "Effects of water clarity on lake stratification and lake-atmosphere heat exchange," *J. Geophys. Res.: Atmos.,* 2015, vol. 120, no. 5, pp. 7412–7428.
41. Stepanenko, V., Mammarella, I., Ojala, A., Miettinen, H., Lykosov, V., and Vesala, T., "LAKE 2.0: a model for temperature, methane, carbon dioxide and oxygen dynamics in lakes," *Geosci. Model Dev.,* 2016, vol. 9, no. 5, pp. 1977–2006.

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