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GEOMETRIC FORMULATION OF NON-AUTONOMOUS MECHANICS

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We address classical and quantum mechanics in a general setting of arbitrary time-dependent transformations. Classical non-relativistic mechanics is formulated as a particular field theory on smooth fiber bundles over a time axis $\mathbb{R}$. Connections on these bundles describe reference frames. Quantum non-autonomous mechanics is phrased in geometric terms of Banach and Hilbert bundles and connections on these bundles. A quantization scheme speaking this language is geometric quantization.

Keywords: Non-relativistic mechanics; Lagrangian mechanics; Hamiltonian mechanics; quantum mechanics; reference frame.

1. Introduction

The technique of symplectic manifolds is well known to provide the adequate Hamiltonian formulation of autonomous mechanics [25, 42]. Its realistic example is a mechanical system whose configuration space is a manifold M and whose phase space is the cotangent bundle T^*M of M provided with the canonical symplectic form $\Omega_M = dp_i \wedge dq^i$, written with respect to the holonomic coordinates $(q^i, p_i = \dot{q}_i)$ on T^*M. Any autonomous Hamiltonian system locally is of this type.

However, this geometric formulation of autonomous mechanics is not extended to mechanics under time-dependent transformations because the symplectic form Ω_M fails to be invariant under these transformations. As a palliative variant, one has developed time-dependent mechanics on a configuration space $Q = \mathbb{R} \times M$ where $\mathbb{R}$ is the time axis [5, 23]. Its phase space $\mathbb{R} \times T^*M$ is provided with the pull-back presymplectic form $\mathrm{pr}_2^*\Omega_M = dp_i \wedge dq^i$. However, this presymplectic form also is broken by time-dependent transformations.

We address non-relativistic mechanics in a case of arbitrary time-dependent transformations [20, 22, 26]. Its configuration space is a fiber bundle $Q \to \mathbb{R}$

endowed with bundle coordinates (t, q^i), where t is the standard Cartesian coordinate on the time axis $\mathbb{R}$ with transition functions $t' = t + \text{const}$. Its velocity space is the first-order jet manifold J^1Q of sections of $Q \to \mathbb{R}$ coordinated by (t, q^i, q^i_t). A phase space is the vertical cotangent bundle V^*Q of $Q \to \mathbb{R}$ [22, 32].

This formulation of non-relativistic mechanics is similar to that of classical field theory on fiber bundles over a base of dimension > 1 [21, 35]. A difference between mechanics and field theory however lies in the fact that connections on bundles over $\mathbb{R}$ are flat, and they fail to be dynamic variables, but describe reference frames.

Note that relativistic mechanics is adequately formulated as particular classical string theory of one-dimensional submanifolds [21, 22, 37, 38].

In Sec. 6, non-autonomous integrable Hamiltonian systems and mechanics with time-dependent parameters are considered.

2. Non-Autonomous Dynamic Equations

Let us start with the notion of a reference frame in non-relativistic mechanics. A fiber bundle $Q \to \mathbb{R}$ is always trivial. By the well known theorem [21, 27], there is one-to-one correspondence between the connections

$$\Gamma = \partial_t + \Gamma^i \partial_i, \tag{1}$$

on $Q \to \mathbb{R}$ and the atlases of local constant trivializations of $Q \to \mathbb{R}$ with time-independent transition functions $q^i \to q'^i(q^j)$ so that $\Gamma = \partial_t$ with respect to an associated atlas. This fact leads to definition of a reference frame in non-relativistic mechanics as a connection Γ on a configuration space $Q \to \mathbb{R}$ [22, 31, 32]. The corresponding covariant differential

$$D^\Gamma : J^1Q \ni \partial_t + q^i_t \partial_i \to (q^i_t - \Gamma^i)\partial_i \in VQ$$

determines the relative velocity $(q^i_t - \Gamma^i)\partial_i$ with respect to a reference frame Γ.

Equations of motion of non-relativistic mechanics usually are first and second-order dynamic equations [20, 22, 26]. A first-order dynamic equation on a fiber bundle $Q \to \mathbb{R}$ is a kernel of the covariant differential $D^\Gamma = (q^i_t - \Gamma^i)\partial_i$ of some connection Γ (1) on $Q \to \mathbb{R}$. Second-order dynamic equations

$$q^i_{tt} = \xi^i(t, q^j, q^j_t), \quad \xi = \partial_t + q^i_t \partial_i + \xi^i \partial^t_i, \tag{2}$$

on $Q \to \mathbb{R}$ are conventionally defined as holonomic connections ξ on a jet bundle $J^1Q \to \mathbb{R}$. These equations also are represented by connections

$$\gamma = dq^\lambda \otimes (\partial_\lambda + \gamma^i_\lambda \partial^t_i),$$

on an affine jet bundle $J^1Q \to Q$ and, due to the canonical imbedding $J^1Q \to TQ$, they are equivalent to geodesic equations on the tangent bundle TQ of Q [22, 28].

One says that the second-order dynamic equation (2) is a free motion equation if there exists a reference frame $(t, \overline{q}^i)$ on Q such that this equation reads $\overline{q}^i_{tt} = 0$.

Relative to an arbitrary frame (t, q^i), a free motion equation takes a form

$$q^i_{tt} = d_t\Gamma^i + \partial_j\Gamma^i(q^j_t - \Gamma^j) - \frac{\partial q^i}{\partial\overline{q}^m}\frac{\partial\overline{q}^m}{\partial q^j\partial q^k}(q^j_t - \Gamma^j)(q^k_t - \Gamma^k), \quad \Gamma^i = \partial_t q^i(t, \overline{q}^j).$$

Its right-hand side is treated as an inertial force. One can show that a free motion equation on a fiber bundle $Q \to \mathbb{R}$ exists if and only if Q is a toroidal cylinder.

To consider a relative acceleration with respect to a reference frame Γ, one should prolong a connection Γ on a configuration space $Q \to \mathbb{R}$ to a holonomic connection ξ_Γ on a jet bundle $J^1Q \to \mathbb{R}$. Given a second-order dynamic equation ξ, one can treat the vertical vector field $a_\Gamma = \xi - \xi_\Gamma = (\xi^i - \xi^i_\Gamma)\partial^t_i$ on $J^1Q \to Q$ as a relative acceleration with respect to a frame Γ. Then the second-order dynamic equation (2) can be written in a covariant form $q^i_{tt} - \xi^i_\Gamma = a_\Gamma$ [22].

3. Lagrangian Non-Autonomous Mechanics

Lagrangian mechanics is formulated in the framework of Lagrangian formalism on fiber bundles [21, 22, 40]. We restrict our consideration to first-order Lagrangian theory on a fiber bundle $Q \to \mathbb{R}$ which is the case of non-relativistic mechanics.

A first-order Lagrangian is defined as a density

$$L = \mathcal{L}dt, \quad \mathcal{L} : J^1Q \to \mathbb{R}, \tag{3}$$

on a velocity space J^1Q. There is the decomposition

$$dL = \delta L - d_H H_L, \tag{4}$$

where we have the second-order Lagrange operator

$$\delta L = (\partial_i\mathcal{L} - d_t\partial^t_i\mathcal{L})dq^i \wedge dt \tag{5}$$

and the Poincaré–Cartan form

$$H_L = \partial^t_i\mathcal{L}dq^i - (q^i_t\partial^t_i\mathcal{L} - \mathcal{L})dt. \tag{6}$$

A kernel of the Lagrange operator (5) provides a second-order Lagrange equation

$$(\partial_i - d_t\partial^t_i)\mathcal{L} = 0. \tag{7}$$

Every first-order Lagrangian L (3) yields the Legendre map

$$\widehat{L} : J^1Q \underset{Q}{\to} V^*Q, \quad p_i \circ \widehat{L} = \pi_i = \partial^t_i\mathcal{L}, \tag{8}$$

where (t, q^i, p_i) are holonomic coordinates on the vertical cotangent bundle V^*Q of $Q \to \mathbb{R}$. A Lagrangian L is called hyperregular if $\widehat{L}$ (8) is a diffeomorphism and almost regular if a Lagrangian constraint space $N_L = \widehat{L}(J^1Q)$ is a closed imbedded subbundle of the Legendre bundle $\pi_\Pi : V^*Q \to Q$ and the Legendre map $\widehat{L} : J^1Q \to N_L$ is a fibered manifold with connected fibers.

Besides the Lagrange equation (7), the Cartan equation also is considered in Lagrangian mechanics. It is readily observed that the Poincaré–Cartan form H_L

(6) also is a Poincaré–Cartan form of a first-order Lagrangian

$$\widetilde{L} = \widehat{h}_0(H_L) = (\mathcal{L} + (q^i_{(t)} - q^i_t)\pi_i)dt, \quad \widehat{h}_0(dq^i) = q^i_{(t)}dt,$$

on a repeated jet manifold J^1J^1Y [21, 22]. The Lagrange operator for $\widetilde{L}$ reads

$$\delta\widetilde{L} = [(\partial_i\mathcal{L} - \widehat{d}_t\pi_i + \partial_i\pi_j(q^j_{(t)} - q^j_t))dq^i + \partial^t_i\pi_j(q^j_{(t)} - q^j_t)dq^i_t] \wedge dt.$$

Its kernel Ker $\delta\overline{L} \subset J^1J^1Q$ defines a first-order Cartan equation

$$\partial^t_i\pi_j(q^j_{(t)} - q^j_t) = 0, \quad \partial_i\mathcal{L} - \widehat{d}_t\pi_i + \partial_i\pi_j(q^j_{(t)} - q^j_t) = 0, \tag{9}$$

on J^1Q. A key point is that the Cartan equation (9), but not the Lagrange one (7) is associated to a Hamilton equation in Hamiltonian mechanics.

The Poincaré–Cartan form H_L (6) yields a homogeneous Legendre map $\widehat{H}_L : J^1Q \to T^*Q$. Given holonomic coordinates (t, q^i, p_0, p_i) on T^*Q, it reads

$$(p_0, p_i) \circ \widehat{H}_L = (\mathcal{L} - q^i_t\pi_i, \pi_i).$$

We have a one-dimensional affine bundle $\zeta : T^*Q \to V^*Q$ over the vertical cotangent bundle V^*Q, and the Legendre map $\widehat{L}$ (8) is the composition of morphisms $\widehat{L} = \zeta \circ \widehat{H}_L$. In comparison with a phase space V^*Q of non-relativistic mechanics, the cotangent bundle T^*Q is its homogeneous phase space.

In accordance with the first Noether theorem, Lagrangian conservation laws in Lagrangian mechanics can be defined [22, 30]. Let $u = u^t\partial_t + u^i\partial_i$, $u^t = 0, 1$, be a vector field on a fiber bundle $Q \to \mathbb{R}$. The Lie derivative $\mathbf{L}_{J^1u}L$ of a Lagrangian L along the jet prolongation J^1u of u onto J^1Q fulfils the first variational formula

$$\mathbf{L}_{J^1u}L = u_V\rfloor\delta L + d_H(u\rfloor H_L), \tag{10}$$

which results from the decomposition (4). A vector field u is called a symmetry of a Lagrangian L if the Lie derivative $\mathbf{L}_{J^1u}L$ vanishes. In this case, the first variational formula (10) leads to a weak conservation law

$$0 \approx d_t\mathfrak{T}_u, \quad \mathfrak{T}_u = u\rfloor H_L = (u^i - u^tq^i_t)\pi_i + u^t\mathcal{L}, \tag{11}$$

of a symmetry current $\mathfrak{T}_u$ along a vector field u.

For instance, if $u^t = 1$, we have a reference frame $u = \Gamma$, and the symmetry current (11) is an energy function

$$E_\Gamma = -\mathfrak{T}_\Gamma = \pi_i(q^i_t - \Gamma^i) - \mathcal{L}$$

relative to a reference frame Γ [6, 22, 32].

4. Hamiltonian Non-Autonomous Mechanics

A phase space V^*Q of Hamiltonian non-autonomous mechanics is provided with the canonical Poisson structure

$$\{f, g\}_V = \partial^i f\partial_i g - \partial^i g\partial_i f, \quad f, g \in C^\infty(V^*Q), \tag{12}$$

such that $\zeta^*\{f,g\}_V = \{\zeta^* f, \zeta^* g\}_T$, where $\{f,g\}_T$ is the Poisson bracket for the canonical symplectic structure Ω_Q on the cotangent bundle T^*Q of Q.

However, Hamiltonian mechanics is not familiar Poisson Hamiltonian theory on a Poisson manifold V^*Q because all Hamiltonian vector fields on V^*Q are vertical. Hamiltonian mechanics on V^*Q is formulated as particular (polysymplectic) Hamiltonian formalism on fiber bundles [11, 21, 22]. Its Hamiltonian is a global section

$$h : V^*Q \to T^*Q, \quad p_0 \circ h = \mathcal{H}(t, q^j, p_j), \tag{13}$$

of an affine bundle $T^*Q \to V^*Q$. The pull-back $(-h)^*\Xi$ of the canonical Liouville form $\Xi = p_\mu dq^\mu$ on T^*Q with respect to this section is a Hamiltonian one-form

$$H = (-h)^*\Xi = p_k dq^k - \mathcal{H} dt \tag{14}$$

on V^*Q [22, 32]. This is the well-known invariant of Poincaré–Cartan [1].

For instance, any connection Γ (1) on $Q \to \mathbb{R}$ defines the global section $h_\Gamma = p_i \Gamma^i$ (13) of an affine bundle $T^*Q \to V^*Q$ and the corresponding Hamiltonian form

$$H_\Gamma = p_k dq^k - \mathcal{H}_\Gamma dt = p_k dq^k - p_i \Gamma^i dt. \tag{15}$$

Furthermore, given a connection Γ, any Hamiltonian form (14) admits a splitting

$$H = H_\Gamma - \mathcal{E}_\Gamma dt, \quad \mathcal{E}_\Gamma = \mathcal{H} - \mathcal{H}_\Gamma = \mathcal{H} - p_i \Gamma^i, \tag{16}$$

where $\mathcal{E}_\Gamma$ is called the Hamiltonian function on V^*Q relative to a frame Γ.

Given the Hamiltonian form H (14), there exists a unique connection

$$\gamma_H = \partial_t + \partial^k \mathcal{H} \partial_k - \partial_k \mathcal{H} \partial^k,$$

on $V^*Q \to \mathbb{R}$ such that $\gamma_H \rfloor dH = 0$. It yields a first-order Hamilton equation

$$q^k_t = \partial^k \mathcal{H}, \quad p_{tk} = -\partial_k \mathcal{H} \tag{17}$$

on $V^*Q \to \mathbb{R}$, where $(t, q^k, p_k, q^k_t, p_{tk})$ are the adapted coordinates on $J^1 V^*Q$.

Herewith, a non-autonomous Hamiltonian system $(\mathcal{H}, V^*Q)$ is associated to the homogeneous autonomous Hamiltonian system with a Hamiltonian $\mathcal{H}^* = p_0 + \mathcal{H}$ on the cotangent bundle T^*Q so that the Hamilton equation (17) on V^*Q is equivalent to an autonomous Hamilton equation on T^*Q [4, 22, 29].

Moreover, the Hamilton equation (17) on V^*Q also is equivalent to the Lagrange equation of a Lagrangian

$$L_H = h_0(H) = (p_i q^i_t - \mathcal{H}) dt \tag{18}$$

on the jet manifold $J^1 V^*Q$ of $V^*Q \to \mathbb{R}$ [22, 30, 32]. As a consequence, Hamiltonian conservation laws can be formulated as the Lagrangian ones. In particular, any integral of motion F of the Hamilton equation (17) is a conserved current of the Lagrangian (18), and *vice versa*. It obeys the evolution equation

$$\mathbf{L}_{\gamma_H} F = \partial_t F + \{\mathcal{H}, F\}_V = 0 \tag{19}$$

and, equivalently, the homogeneous evolution equation

$$\zeta^*(\mathbf{L}_{\gamma_H} F) = \{\mathcal{H}^*, \zeta^* F\}_T = 0. \tag{20}$$

In particular, let $\mathcal{E}_\Gamma$ (16) be a Hamiltonian function relative to a reference frame Γ. Given bundle coordinates adapted to Γ, its evolution equation (19) takes a form

$$\mathbf{L}_{\gamma_H} \mathcal{E}_\Gamma = \partial_t \mathcal{E}_\Gamma = \partial_t \mathcal{H}.$$

It follows that, a Hamiltonian function $\mathcal{E}_\Gamma$ relative to a reference frame Γ is an integral of motion if and only if a Hamiltonian, written with respect to Γ, is time-independent. One can think of $\mathcal{E}_\Gamma$ as being an energy function relative to a reference frame Γ [6, 22, 30, 32]. Indeed, if $\mathcal{E}_\Gamma$ is an integral of motion, it is a conserved symmetry current of the canonical lift onto V^*Q of the vector field $-\Gamma$ (1) on Q.

Lagrangian and Hamiltonian formulations of non-autonomous mechanics fail to be equivalent. The relations between Lagrangian and Hamiltonian formalisms are based on the facts that: (i) every first-order Lagrangian L (3) on a velocity space J^1Q induces the Legendre map (8) of this velocity space to a phase space V^*Q, (ii) every Hamiltonian form H (14) on a phase space V^*Q yields a Hamiltonian map

$$\widehat{H} : V^*Q \to J^1Q, \quad q^i_t \circ \widehat{H} = \partial^i \mathcal{H},$$

of this phase space to a velocity space J^1Q.

Given a Lagrangian L, the Hamiltonian form H (14) is said to be associated with L if H satisfies the relations

$$\widehat{L} \circ \widehat{H} \circ \widehat{L} = \widehat{L}, \quad \widehat{H}^* L_H = \widehat{H}^* L, \tag{21}$$

where L_H is the Lagrangian (18).

For instance, let L be a hyperregular Lagrangian. It follows from the relations (21) that, in this case, $\widehat{H} = \widehat{L}^{-1}$ and there exists a unique Hamiltonian form

$$H = p_k dq^k - \mathcal{H} dt, \quad \mathcal{H} = p_i \widehat{L}^{-1i} - \mathcal{L}(t, q^j, \widehat{L}^{-1j}), \tag{22}$$

associated with L. Let s be a solution of the Lagrange equation (7) for a Lagrangian L. A direct computation shows that $\widehat{L} \circ J^1 s$ is a solution of the Hamilton equation (17) for the Hamiltonian form H (22). Conversely, if r is a solution of the Hamilton equation (17) for the Hamiltonian form H (22), then $s = \pi_\Pi \circ r$ is a solution of the Lagrange equation (7) for L. It follows that, in the case of hyperregular Lagrangians, Hamiltonian formalism is equivalent to Lagrangian one.

If a Lagrangian is not hyperregular, an associated Hamiltonian form need not exist or it is not unique. Comprehensive relations between Lagrangian and Hamiltonian systems are established in the case of almost regular Lagrangians [22, 29, 32].

5. Quantum Non-Autonomous Mechanics

Quantum non-autonomous mechanics is phrased in geometric terms of Banach and Hilbert manifolds and Hilbert and C^*-algebra bundles. Quantization schemes speaking this language are instantwise and geometric quantizations [12, 18, 22].

A definition of smooth Banach and Hilbert manifolds follows that of the finite-dimensional ones, but Banach manifolds are not locally compact, and they need not be paracompact [18, 24, 41]. It is essential that Hilbert manifolds satisfy the inverse function theorem and, therefore, locally trivial Hilbert bundles are defined. However, the following fact leads to the non-equivalence of Schrödinger and Heisenberg quantization. Let E be a Hilbert space and B some C^*-algebra of bounded operators in E. There is a topological obstruction to the existence of associated Hilbert and C^*-algebra bundles $\mathcal{E}$ and $\mathcal{B}$ with typical fibers E and B, respectively. First, transition functions of $\mathcal{E}$ define those of $\mathcal{B}$, but the latter are not continuous in general. Second, transition functions of $\mathcal{B}$ need not give rise to those of $\mathcal{E}$.

One also meets a problem of the definition of connections on C^*-algebra bundles. It comes from the fact that a C^*-algebra need not admit non-zero bounded derivations. An unbounded derivation of a C^*-algebra A obeying certain conditions is an infinitesimal generator of a strongly (but not uniformly) continuous one-parameter group of automorphisms of A [3, 18, 22]. Therefore, one must introduce a connection on a C^*-algebra bundle in terms of parallel transport operators, but not their infinitesimal generators [2, 18]. Moreover, a representation of A need not imply a unitary representation of its strongly continuous one-parameter group of automorphisms. In contrast, connections on a Hilbert bundle over a smooth manifold can be defined as first-order differential operators on a module of its sections [18, 22].

In particular, this is the case of instantwise quantization describing evolution of quantum systems in terms of Hilbert bundles over $\mathbb{R}$ [13, 18, 22, 33]. Namely, let us consider a Hilbert bundle $\mathfrak{E} \to \mathbb{R}$ with a typical fiber E and a connection ∇_t on a $C^\infty(\mathbb{R})$-module $\mathfrak{E}(\mathbb{R})$ of smooth sections of $\mathfrak{E} \to \mathbb{R}$. It obeys the Leibniz rule

$$\nabla_t(f\psi) = \partial_t f\psi + f\nabla_t\psi, \quad \psi \in \mathfrak{E}(\mathbb{R}), \ \ f \in C^\infty(\mathbb{R}).$$

Given a trivialization $\mathfrak{E} = \mathbb{R} \times E$, the connection ∇_t reads

$$\nabla_t\psi = (\partial_t + i\mathbf{H}(t))\psi, \tag{23}$$

where $\mathbf{H}(t)$ are bounded self-adjoint operators in E for all $t \in \mathbb{R}$. A section ψ of $\mathfrak{E} \to \mathbb{R}$ is an integral section of the connection ∇_t (23) if it obeys the equation

$$\nabla_t\psi(t) = (\partial_t + i\mathbf{H}(t))\psi(t) = 0. \tag{24}$$

One can think of this equation as being the Schrödinger equation.

The most of quantum models come from canonical quantization of classical mechanical systems by means of replacement of a Poisson bracket $\{f, f'\}$ of smooth functions with a bracket $[\widehat{f}, \widehat{f}']$ of Hermitian operators in a Hilbert space in accordance with Dirac's condition $[\widehat{f}, \widehat{f}'] = -i\widehat{\{f, f'\}}$. Canonical quantization of Hamiltonian non-autonomous mechanics on a configuration space $Q \to \mathbb{R}$ is geometric quantization [12, 13, 18, 22]. A key point is that, in this case, the evolution equation (19) is not reduced to the Poisson bracket on a phase space V^*Q, but is expressed as (20) in the Poisson bracket on the homogeneous phase space T^*Q. Therefore, the compatible geometric quantization both of the symplectic cotangent bundle T^*Q and the Poisson vertical cotangent bundle V^*Q of Q is required.

Note that geometric quantization of Poisson manifolds is formulated in terms of contravariant connections [42]. Though there is one-to-one correspondence between the Poisson structures on a manifold and its symplectic foliations, this quantization of a Poisson manifold need not imply quantization of its symplectic leaves [43]. Geometric quantization of symplectic foliations disposes of this problem [13, 18, 22, 34]. A quantum algebra of a symplectic foliation also is that of an associated Poisson manifold whose restriction to each symplectic leaf is its quantum algebra.

Namely, the standard prequantization of the cotangent bundle T^*Q yields the compatible prequantization of a Poisson manifold V^*Q. However, polarization of T^*Q need not induce any polarization of V^*Q, unless it contains the vertical cotangent bundle of a fiber bundle $T^*Q \to V^*Q$ spanned by vectors ∂^0. A unique canonical real polarization of T^*Q, satisfying this condition, is the vertical tangent bundle of $T^*Q \to Q$. The associated quantum algebra $\mathcal{A}_T$ consists of functions on T^*Q which are affine in momenta p_μ. This polarization of T^*Q yields polarization of a Poisson manifold V^*Q such that the corresponding quantum algebra $\mathcal{A}_V$ consists of functions on V^*Q which are affine in momenta p_i, i.e. $\mathcal{A}_V$ is a subalgebra of $\mathcal{A}_T$. After metaplectic correction, we obtain compatible Schrödinger representations

$$\widehat{f}\rho = \left(-ia^\lambda\partial_\lambda - \frac{i}{2}\partial_\lambda a^\lambda - b\right)\rho, \quad f = a^\lambda(q^\mu)p_\lambda + b(q^\mu) \in \mathcal{A}_T, \tag{25}$$

$$\widehat{f}\rho = \left(-ia^k\partial_k - \frac{i}{2}\partial_k a^k - b\right)\rho, \quad f = a^k(q^\mu)p_k + b(q^\mu) \in \mathcal{A}_V, \tag{26}$$

of $\mathcal{A}_T$ and $\mathcal{A}_V$ in the space $\mathcal{D}_{1/2}(Q)$ of complex half-densities ρ on Q.

The Schrödinger quantization (26) of V^*Q provides instantwise quantization of non-autonomous mechanics [22]. Indeed, a glance at the Poisson bracket (12) shows that the Poisson algebra $C^\infty(V^*Q)$ is a Lie algebra over the ring $C^\infty(\mathbb{R})$ of functions of time, where algebraic operations in fact are instantwise operations depending on time as a parameter. One can show that the Schrödinger quantization (26) of a Poisson manifold V^*Q yields geometric quantization of its symplectic fibers V_t^*Q, $t \in \mathbb{R}$, such that the quantum algebra $\mathcal{A}_t$ of V_t^*Q consists of elements $f \in \mathcal{A}_V$ restricted to V_t^*Q. Bearing in mind that $\rho \in \mathcal{D}_{1/2}[Q]$ are fiberwise half-densities on $Q \to \mathbb{R}$, let us choose a carrier space of the Schrödinger representation (26) of $\mathcal{A}_V$ which consists of complex half-densities ρ on Q such that ρ on Q_t for any $t \in \mathbb{R}$ is of compact support. It is a pre-Hilbert $C^\infty(\mathbb{R})$-module $\mathfrak{E}_R$ which also is a carrier space for the quantum algebra $\mathcal{A}_T$, but its action in $\mathfrak{E}_R$ is not instantwise.

Let us turn to quantization of an evolution equation. Since Eq. (19) is not reduced to a Poisson bracket, quantization of a Poisson manifold V^*Q fails to provide quantization of this evolution equation. Therefore, we quantize the equivalent homogeneous evolution equation (20) on a symplectic manifold T^*Q. The Schrödinger representation (25) of a Lie algebra $\mathcal{A}_T$ is extended to its enveloping algebra, and defines the quantization $\widehat{\mathcal{H}}^*$ of a homogeneous Hamiltonian $\mathcal{H}^*$.

Moreover, since $\widehat{p}_0 = -i\partial_t$, an operator $i\widehat{\mathcal{H}}^*$ obeys the Leibniz rule

$$i\widehat{\mathcal{H}}^*(r\rho) = \partial_t r\rho + r(i\widehat{\mathcal{H}}^*\rho), \quad r \in C^\infty(\mathbb{R}), \ \rho \in \mathfrak{E}_R.$$

Thus, it is a connection on a $C^\infty(\mathbb{R})$-module $\mathfrak{E}_R$. Then a quantum constraint

$$i\widehat{\mathcal{H}}^*\rho = 0, \quad \rho \in \mathfrak{E}_R, \tag{27}$$

is the Schrödinger equation (24) in quantum non-autonomous mechanics.

This quantization depends on a reference frame as follows. In accordance with the Schrödinger representation (25), a homogeneous Hamiltonian $\mathcal{H}^* = p_0 + \mathcal{H}$ is quantized as a Hamilton operator

$$\widehat{\mathcal{H}}^* = \widehat{p}_0 + \widehat{\mathcal{H}} = -i\partial_t + \widehat{\mathcal{H}}. \tag{28}$$

A problem is that the decomposition $\mathcal{H}^* = p_0 + \mathcal{H}$ and the corresponding splitting (28) of a Hamilton operator $\widehat{\mathcal{H}}^*$ are ill defined. At the same time, any reference frame Γ yields the decomposition

$$\mathcal{H}^* = (p_0 + \mathcal{H}_\Gamma) + (\mathcal{H} - \mathcal{H}_\Gamma) = \mathcal{H}^*_\Gamma + \mathcal{E}_\Gamma,$$

where $\mathcal{H}_\Gamma$ is the Hamiltonian (15) and $\mathcal{E}_\Gamma$ (16) is an energy function relative to a reference frame Γ. Accordingly, we obtain the splitting of a Hamilton operator

$$\widehat{\mathcal{H}}^* = \widehat{\mathcal{H}}^*_\Gamma + \widehat{\mathcal{E}}_\Gamma, \quad \widehat{\mathcal{H}}^*_\Gamma = -i\partial_t - i\Gamma^k\partial_k - \frac{i}{2}\partial_k\Gamma^k$$

and $\widehat{\mathcal{E}}_\Gamma$ is the operator of energy relative to a reference frame Γ [22, 30]. Given a reference frame Γ, the energy function $\mathcal{E}_\Gamma$ is quantized as $\widehat{\mathcal{E}}_\Gamma = \widehat{\mathcal{H}}^* - \widehat{\mathcal{H}}^*_\Gamma$. As a consequence, the Schrödinger equation (27) reads

$$(\widehat{\mathcal{H}}_\Gamma + \widehat{\mathcal{E}}_\Gamma)\rho = -i\left(\partial_t + \Gamma^k\partial_k + \frac{1}{2}\partial_k\Gamma^k\right)\rho + \widehat{\mathcal{E}}_\Gamma\rho = 0.$$

6. Outcomes

The Liouville–Arnold theorem for completely integrable systems and the Mishchenko–Fomenko theorem for the superintegrable ones state the existence of action-angle coordinates around a compact invariant submanifold of a Hamiltonian integrable system. These theorems have been generalized to the case of non-compact invariant submanifolds [8–10, 16, 18, 22, 36]. In particular, this is the case of non-autonomous completely integrable and superintegrable systems [14, 18, 22, 39]. Geometric quantization of completely integrable and superintegrable Hamiltonian systems with respect to action-angle variables has been considered [7, 15, 18, 19, 22].

At present, quantum systems with classical parameters attract special attention in connection with holonomic quantum computation. These parameters can be seen as sections of some smooth fiber bundle $\Sigma \to \mathbb{R}$. Then a configuration space of a mechanical system with time-dependent parameters is a composite fiber bundle $Q \to \Sigma \to \mathbb{R}$ [13, 18, 22, 33]. The corresponding total velocity and phase spaces are the first-order jet manifold J^1Q and the vertical cotangent bundle V^*Q of the configuration bundle $Q \to \mathbb{R}$, respectively. However, since parameters are classical, a phase space of a quantum system with time-dependent parameters is the vertical cotangent bundle $V^*_\Sigma Q$ of a fiber bundle $Q \to \Sigma$. We apply to $V^*_\Sigma Q \to \Sigma$ the technique of leafwise geometric quantization [13, 18, 22].

Geometric Berry's phase factor is a phenomenon peculiar to quantum systems with classical parameters. It is characterized by a holonomy operator driving a carrier Hilbert space over a parameter manifold. A problem lies in separation of a geometric phase factor from an evolution operator without using an adiabatic assumption. Therefore, we address the Berry phase phenomena in completely integrable systems. A reason is that, being constant under an internal dynamic evolution, action variables of a completely integrable system are driven only by a perturbation holonomy operator without any adiabatic approximation [17, 18, 22].

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