On parametric resonance in the pendulum with impulse excitation

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We consider for example a pendulum with variable length described by the following ordinary differential equations obtained from the law of momentum alteration, see [2],

$$\dot{\varphi} = \frac{s}{(l(t)/l_0)^2}, \quad \dot{s} = -\beta s - (2\pi/T_0)^2 (l(t)/l_0) \sin(\varphi), \tag{1}$$

where s is the specific sector velocity, β is a small damping coefficient, T_0 is the period of small oscillations of the pendulum when its length is constant, $l(t) \equiv l_0$. The relative length of the pendulum can be discontinuous like in Example in [2]:

$$\frac{l(t)}{l_0} = \begin{cases} 1+\varepsilon, & t \in \left[nT, \frac{T}{2} + nT\right), \\ 1-\varepsilon, & t \in \left[\frac{T}{2} + nT, T + nT\right), \end{cases} \tag{2}$$

where $l_0 > 0$ is the mean lenght of the pendulum, $|\varepsilon| < 1$ is the relative amplitude of excitation, and $n = 0, 1, 2, 3, \ldots$. So, the angular velocity $\dot{\varphi}$ changes abruptly at $t = nT/2$. Such behavior of the system is also known as *impulse motion*.

In order to find domains of parametric resonance (instability of an equilibrium $\varphi = 0$, $s = 0$) of the pendulum we write system (1) linearized about the equilibrium, for one period of excitations in the form $\dot{x}(t) = \mathbf{J}(t)\, x(t)$, where vector $x(t)$ corresponds to $(\varphi(t), s(t))'$ and $\mathbf{J}(t)$ is the Jacobian matrix of the original system at the equilibrium position $(\varphi, s)' = (0, 0)'$. Since $\mathbf{J}(t)$ is piecewise continuous and integrable we can apply *Folquet stability analysis*, see e.g. [1]. Solution of matrix differential equation $\dot{\mathbf{X}}(t) = \mathbf{J}(t)\, \mathbf{X}(t)$, where initial value $\mathbf{X}(0) = \mathbf{I}$ is the identity matrix, yields *monodromy matrix* as $\mathbf{F} = \mathbf{X}(T)$. Due to (2) the linearized system has piecewise constant Jacobian matrix: $\mathbf{J}(t) = \mathbf{J}_+$ if $t \in [0, T/2)$ and $\mathbf{J}(t) = \mathbf{J}_-$ if $t \in [T/2, T)$, where

$$\mathbf{J}_+ = \begin{pmatrix} 0 & (1+\varepsilon)^{-2} \\ -(1+\varepsilon)(2\pi/T_0)^2 & -\beta \end{pmatrix}, \quad \mathbf{J}_- = \begin{pmatrix} 0 & (1-\varepsilon)^{-2} \\ -(1-\varepsilon)(2\pi/T_0)^2 & -\beta \end{pmatrix}. \tag{3}$$

We have simple expression of the monodromy matrix via matrix exponents

$$\mathbf{F} = \exp\left(\frac{T}{2}\,\mathbf{J}_-\right) \cdot \exp\left(\frac{T}{2}\,\mathbf{J}_+\right). \tag{4}$$

Floquet multipliers ρ_1 and ρ_2, eigenvalues of $\mathbf{F}$, determine the stability of the solution $x = (0,0)'$ of the linearized system, and can be obtained from characteristic polynomial:

$$\rho^2 - \operatorname{tr}(\mathbf{F})\,\rho + \det(\mathbf{F}) = 0. \tag{5}$$

Stability conditions ($|\rho_1| \le 1$ and $|\rho_2| \le 1$) written in case of real roots as $\rho \in [-1,1]$ and in case of complex conjugate roots as $\rho_1\rho_2 \le 1$, with the use of (5) and Vieta's formula $\rho_1\rho_2 = \det(\mathbf{F})$, take the form

$$|\operatorname{tr}(\mathbf{F})| \le 1 + \det(\mathbf{F}) \quad \text{and} \quad \det(\mathbf{F}) \le 1, \tag{6}$$

where for asymptotic stability all inequalities should be strict. For instability, it is sufficient that at least one of the eigenvalues has absolute value grater than one ($|\rho_1| > 1$ or $|\rho_2| > 1$), so that at least one of the conditions in (6) is violated. Since the determinant of a matrix product equals the product of the determinants, from (4) and (3) we have $\det(\mathbf{F}) = e^{\frac{T}{2}\operatorname{tr}(\mathbf{J}_-)} e^{\frac{T}{2}\operatorname{tr}(\mathbf{J}_+)} = e^{-\beta T}$, where we take into account that $\det(\exp(\frac{T}{2}\,\mathbf{J}_\pm)) = e^{\frac{T}{2}\operatorname{tr}(\mathbf{J}_\pm)}$. The same expression, $\det(\mathbf{F}) = e^{-\beta T}$, can be obtained by Liouville's formula for any piecewise continuous integrable T-periodic modulation function, see [1]. So with positive damping coefficient, $\beta > 0$, stability can only be lost when the first condition in (6) is violated, i.e. when

$$|\operatorname{tr}(\mathbf{F})| > 1 + e^{-\beta T}. \tag{7}$$

We calculate exact instability domains in parameter space $(T/T_0, \varepsilon, \beta)$ via (7) using (3)–(4) and compare them with approximations derived in [2].

References

[1] V. A. Yakubovich and V. M. Starzhinskii, Parametric Resonance in Linear Systems. Nauka, Moscow, 1987.

[2] A.P. Seyranian, The swing: parametric resonance, Journal of Applied Mathematics and Mechanics 68 (2004) 757–764.