Research Article

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A version of Hake's theorem for Kurzweil–Henstock integral in terms of variational measure

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Abstract: We introduce the notion of variational measure with respect to a derivation basis in a topological measure space and consider a Kurzweil–Henstock-type integral related to this basis. We prove a version of Hake's theorem in terms of a variational measure.

Keywords: Topological measure space, derivation basis, Kurzweil–Henstock integral, variational measure, Hake property

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A classical Hake theorem in the theory of integration (see for example [10, Lemma 3.1, Chapter VIII]) states that, in contrast to the Lebesgue integral, the Perron integral on a compact interval is equivalent to the improper Perron integral. As the Perron integral on the real line is known to be equivalent to the Kurzweil–Henstock integral (see [9]), the same property is true for the latter integral. The general idea of computing the improper integral as a limit of the integral over increasing families $\{A_\alpha\}$ of sets can be realized in the multidimensional case in several different ways depending on the type of integral and on what family $\{A_\alpha\}$ is chosen to generalize the compact intervals of the one-dimensional construction. This gives rise to various types of the Hake property. A version of this property for certain Kurzweil–Henstock-type integrals in $\mathbb{R}^n$ was studied in [3, 5, 8].

A generalized Hake theorem in terms of the limit of an integral over increasing families of sets for a Kurzweil–Henstock-type integral on a topological space with respect to an abstract derivation basis was considered in [15]. Another version of the Hake theorem in terms of so-called variational measures generated by an indefinite integral was proved in [11, 12] for the Kurzweil–Henstock integral in $\mathbb{R}^n$ and in a metric space, respectively.

In this paper, we obtain a generalization of the latter results to our case of a Kurzweil–Henstock-type integral on a topological space and show that the conditions for the Hake property in terms of increasing families of sets as in [15] and in terms of variational measures as in [11, 12] are in fact equivalent.

The ambient set X in this paper is a Hausdorff topological space with an outer regular Borel measure μ on it. For any set $E \in X$ we use the notation $\operatorname{int} E$, $\overline{E}$ and ∂E for the interior, the closure and the boundary of E, respectively. The notation $\operatorname{int}_L(E)$ will mean the interior of $E \subset L$ with respect to the topology in L induced by the topology of the space X.

We use the following version of the general definition of a derivation basis (see [9, 17, 18]): a *derivation basis* (or simply a *basis*) $\mathcal{B}$ in $(X, \mathcal{M}, \mu)$ is a filter base on the product space $\mathcal{I} \times X$, where $\mathcal{I}$ is a family of closed subsets of X having finite positive measure μ and called *generalized intervals* or $\mathcal{B}$-*intervals*. That is, $\mathcal{B}$ is

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