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Verification of the iterative procedure for solving the elastoplastic Kirsch problem on the Lame problem

Verification of the iterative procedure for solving the elastoplastic Kirsch problem on the Lame problem

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Abstract. An iterative procedure for solving the elastoplastic Lame problem was proposed to further develop an iterative solution of the elastoplastic Kirsch problem. A ring under an external pressure is considered at which a certain plastic zone arises in the ring. The proposed iterative procedure for solving the elastoplastic Lame problem consists in an explicit analytical representation for stresses in the plastic region and the iterative technique for an elastic analytical solution. Furthermore, the plastic zone radius is beforehand unknown. The convergence of the iterative procedure for the elastoplastic Lame problem is shown in comparison with well-known analytical solution to this problem in elastoplastic formulation. A numerical-analytical iterative solution of the Lame problem in elastoplastic formulation is given for various external pressure and internal radius of the ring. Shows the convergence of numerical-analytical iterative solutions the problem in comparison with numerical solutions in elastoplastic formulation.

1. Introduction

The solution of elastoplastic problem at a plane stress state are obtained in a limited number of cases. In particular, there are solutions to the elastoplastic Lame problem in various formulations [1-4]. There are also analytical approaches for solving an elastoplastic ring with additional conditions [5-10]. However, Kirsch's elastoplastic problem still has no analytical solution, despite a number of approximate analytical and numerical approaches to its solution [11-14].

In the Kirsch problem, at a given tension in a plate exceeding half of the ultimate yield stress, plastic zones do occur that do not fully cover the hole contour. This significantly complicates its solution in comparison with the situation of the two-axis tension state, in which the plastic zone completely covers the hole.

This work presents a semi-analytical approach to solving problem of this type [15], which consists in an explicit analytical representation for tensions in the plastic area and an iterative procedure of numerical solution in the elastic area at a previously unknown boundary of the plastic zone. The estimation of the convergence rate of the iteration method with the exact solution at an extended combination of possible parameters of the external pressure and the internal radius of the ring is given.

2. Methods

2.1. Analytical Solution of an Elastoplastic Lame Problem by the Elastoplastic Solution Iteration Method

The iterative procedure for solving the elastoplastic Lame problem for the radial (σ_r) and circumferential (σ_θ) components of tensions of a plane stress state in a ring with an internal radius a and an external radius b (b>a), being under the action of uniform external pressure $P_0 > \frac{\sigma_Y}{2}$ (σ_Y - the ultimate yield stress) is constructed on the basis of the elastic solution of this problem:

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \end{Bmatrix} = \pm \frac{A}{r^2} + B, \ A = \left(P_0 - P_1\right)r_1^2 b^2 \left(b^2 - r_1^2\right)^{-1}, B = P_1 r_1^2 - P_0 b^2 \left(b^2 - r_1^2\right)^{-1} \tag{1}$$

where r is radial coordinate with the beginning in the center of the ring; in initial approximation: $r_1 = a$, $P_1 = 0$. Having calculated with the help of terms (1) the equivalent stress:

$$\sigma_{eq} = \left(\sigma_r^2 + \sigma_\theta^2 - \sigma_r \sigma_\theta\right)^{1/2} \tag{2}$$

and equating eq. (2) to the ultimate yield stress σ_Y, we obtain an approximate expression for the radius of the plastic zone r_T:

$$r_T^4 = 3A^2 \left(\sigma_Y^2 - B^2\right)^{-1} \tag{3}$$

Figure 1 shows the stages of the iteration procedure: (a) - initial calculation scheme for the ring loaded on the outer contour, (b) - intermediate calculation scheme for the elastic problem, in which $r_1 = r_T$, $P_1 = \sigma_r^p$ solutions obtained at the previous stage (shown by a bar line in figure 1 a).
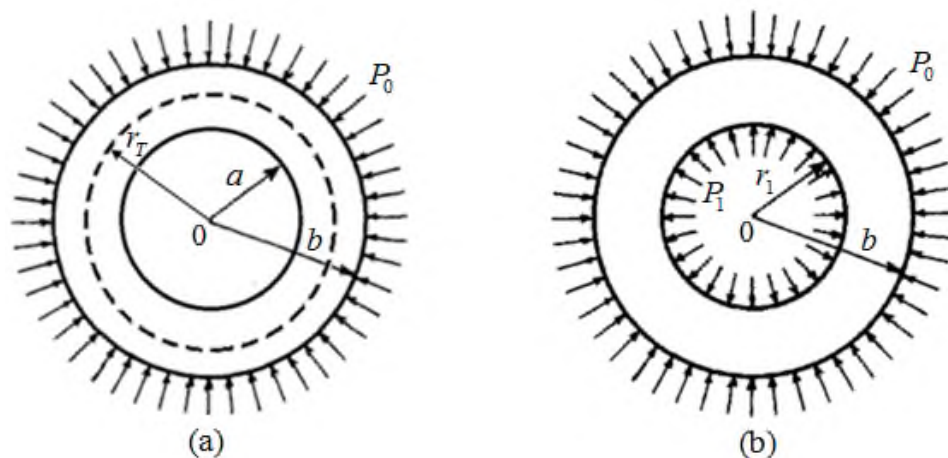

Figure 1. Calculation diagrams by stages of iteration procedure.

For the second iteration the elastic ring with internal boundary passing on a circle of radius r_T, along which the pressure P_1 equal to a radial component of stress σ_r^p in a plastic zone is set is considered. Expressions for circumferential and radial component of tensions in plastic zone can be deduced form as:

$$\sigma_r^p = -\frac{2\sin\psi}{\sqrt{3}}\sigma_Y, \ \sigma_\theta^p = -\left(\frac{\sin\psi}{\sqrt{3}} + \cos\psi\right)\sigma_Y, \ r = a\frac{\sqrt[4]{3}}{\sqrt{2}}\frac{\exp\left(\psi\sqrt{3}/2\right)}{\sqrt{\sin(\pi/3-\psi)}} \tag{4}$$

where ψ is a certain function of the radial coordinate r. Having set in it $r=r_T$, we find $\psi=\psi_T$ and further - pressure value P_1.

Using the found values P_1 and $r_1 = r_T$ using formula (3), we determine the second approximation for the plastic zone radius. Subsequent approximations are made in the same way as described.

2.2. Numerical and analytical iteration solution of the Lame elastoplastic problem
Let us now consider the solution of the Lame problem in an elastoplastic formulation by a numerical-analytical approach. Calculation domain with to using finite element method (FEM) is a ring with an internal radius a and an external radius b. Since the model is axisymmetric, this allows us to consider a quarter of the ring (figure 2).
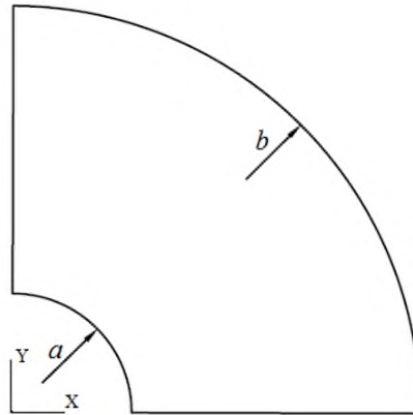

Figure 2. Calculation domain for FEM solution.

All material properties are elastic for the iteration method. Once the geometry is defined it can be meshed. The problem will be solved with the following boundary conditions: an external pressure P_0 acting in the plane of the ring, the border of the hole is free, the sides of the model bases can only be moved along their axes (for axisymetrical cases).

After the model is solved in elastic formulation we have the radius $r_1 = r_T$ in a first approximation. Then taking into account the radius r_1 and equations (1), (2), (3), (4), we obtain the radius $r_2 = r_T$ by a numerical solution. This procedure will continue until the change criterion for radius r_T is met.

3. Results and discussion
As an example, for the analytical iteration method, let us determine the parameter ψ, the plastic zone radius r_T (for each iteration) and pressure P_1 for an steel ring with a $=1.5$ mm, b $=10$ mm, $\sigma_Y = 250$ MPa at a pressure $P_0 = 0.9\sigma_Y$. Results of calculation for 6 iterations are summarized in table 1.

Table 1. Analytical iteration solution of the elastoplastic Lame problem.

Iteration	r_T, mm	$\psi(r_T)$	$\dfrac{P_1}{\sigma_Y}$
1	3.03	0.509	0.563
2	4.067	0.665	0.712
3	4.295	0.69	0.735
4	4.307	0.692	0.736
5	4.296	0.69	0.735
6	4.309	0.691	0.736

It can be seen from table 1 that starting from the 4th iteration, the values of the plastic zone radius stabilize in the vicinity of 4.3 mm. This value coincides with the value of the plastic zone radius found analytically. The numerical solution with these parameters in elastoplastic formulation gives the same result.

Numerical-analytical solution of the elastoplastic problem by the elastic solution method was carried out by means of finite element software ANSYS. It was carried out in 2D space dimension. Mesh was chosen as quadrilateral with adequate minimum element dimensions (the approximate size of element length is 0.02 mm). The material of the steel ring is perfectly plastic with the following

elastic properties: σ_Y=250 MPa, E =210 GPa, ν =0.3, where E is the Young's modulus and ν is the Poisson's coefficient of the material of the ring.

Thus, the iteration elastic solution with an initial elastic setting in Ansys for 5 iterations is summarized in table 2.

Table 2. An iterative solution of the elastoplastic Lame problem using FEM.

Iteration	r_T, mm	$\psi(r_T)$	$\dfrac{P_1}{\sigma_Y}$
1	3.03	0.509	0.563
2	4.08	0.664	0.711
3	4.288	0.689	0.734
4	4.305	0.691	0.736
5	4.305	0.691	0.736

It can be seen that in the first approximation the plastic zone radius coincides with the analytical value obtained from the elastic solution. Further approximations were also close to analytical results. Since the 4th iteration, the value of the plastic zone radius coincided with the value of the plastic zone radius found from the analytical solution of the Lame elastoplastic problem and did not change further.

Additionally, numerical calculations were carried out for various combinations of P_0 and a. The results of calculations are presented in the form of diagrams in figure 3 (solid lines), where also shown (dashed) the results of the precise analytical solution of the corresponding Lamé problems in the elastoplastic formulation by the analytical method.
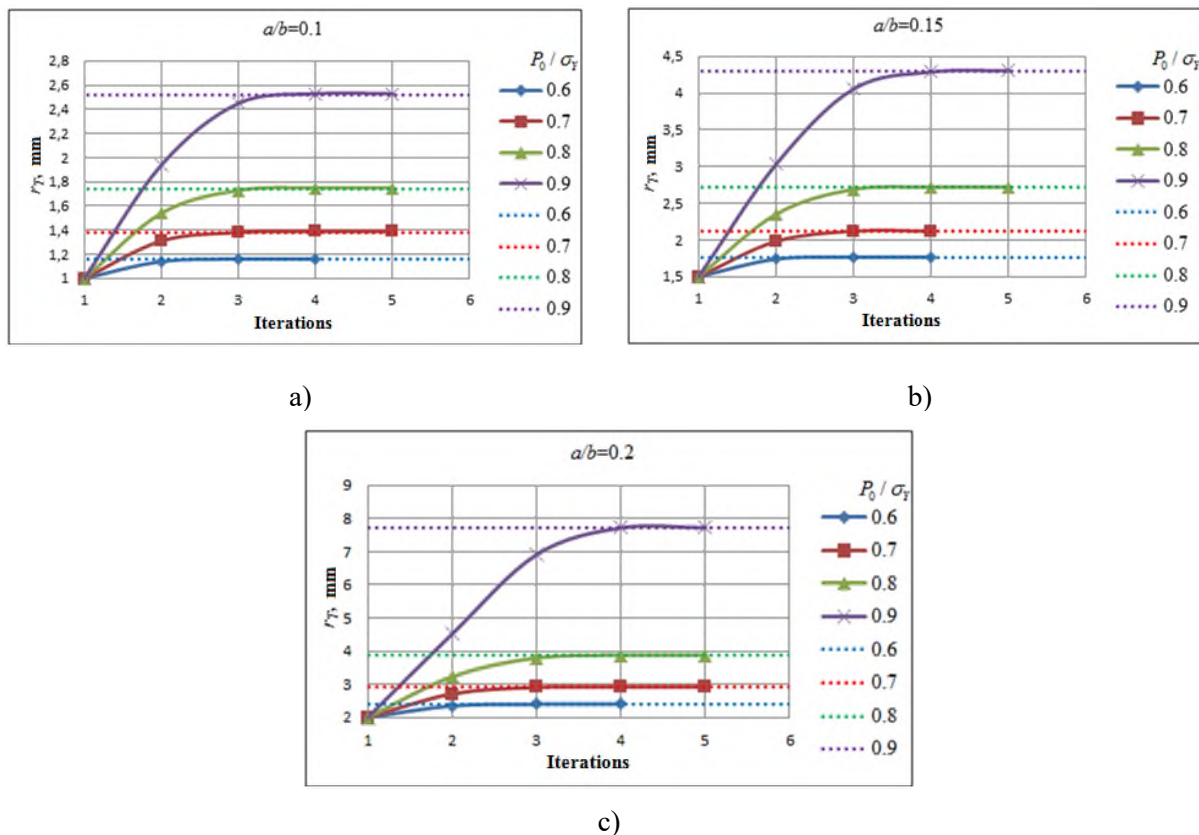

a)

b)

c)

Figure 3. The results of solving the elastoplastic Lame problem for various ratios $\dfrac{P_0}{\sigma_Y}$: a) a =1 mm, b) a =1.5 mm, c) a =2 mm.

As can be seen from the diagrams above, the convergence to accurate analytical solutions for the plastic zone radius r_T comes on the 3rd and 4th iterations, depending on the values of external pressure P_0 and internal radius a. It can also be noted that the maximum deviation of the plastic zone radius r_T does not exceed 0.72% of the value obtained from the accurate solution.

4. Conclusions

In this paper, an iterative procedure for a semi-analytical solution is presented in the elastoplastic formulation of the model Lame problem, where in the plastic region explicit analytical expressions for tension components are used and in the elastic area either an analytical solution (due to the simplicity of the model) or a numerical solution is found. The numerical estimation of the convergence of the iteration procedure at different values of the problem parameters is performed. The boundary of the elastoplastic area and the pressure at it are determined at each stage of the elastic problem solution. Using the numerical method, the procedure was developed for different ratios of external pressure and internal radius of the ring. It is shown that in all cases a small number of iterations is required for the convergence of the procedure for determining the plastic zone boundary, and the deviation of the plastic zone radius r_T is less than 1% of the exact analytical solution.

The proposed approach will be later generalized to implement the semi-analytical solution of the elastoplastic Kirsch problem.

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