ISSN 1064-2307, Journal of Computer and Systems Sciences International, 2022, Vol. 61, No. 2, pp. 240–252. © Pleiades Publishing, Ltd., 2022.
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NAVIGATION SYSTEMS

# Calibration of an Accelerometer Unit with Asymmetric Models of Readings of Sensors

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Received December 12, 2020; revised October 20, 2021; accepted November 29, 2021

**Abstract**—The problem of simultaneous calibration of an accelerometer unit and a nominally high-precision test bench is considered. In contrast to the traditional formulations of calibration problems, it is assumed that the scale factor errors depend on the signs of the accelerometer input signals. A guaranteed approach is applied to the corresponding estimation problem. The optimal calibration design is obtained numerically, and the optimal guaranteed estimates for the required parameters are constructed.

**DOI:** 10.1134/S1064230722020071

## INTRODUCTION

Inertial navigation systems [1, 2] are widely used in engineering. A unit of three accelerometers is one of the main sensors of an inertial navigation system. This unit needs to be calibrated before the navigation system can function. Numerous publications are devoted to the calibration of an accelerometer unit (see, e.g., [3–24]). For a long time, it was assumed that the angular and geometric errors of precision test benches are small enough to be neglected. However, analysis of experiments shows that, in addition to the errors of the unit itself, it is reasonable to include in the estimated parameters possible geometric errors of the nominally high-precision test bench (skewing of the rotation axes, nonhorizontality of the base due to subsidence of the foundation) and its instrumental errors (systematic errors in measuring the rotation angles). In this case, the estimation problem becomes multiparametric, then the choice of the experimental design is not obvious [8]. The use of a guaranteed approach to calibration makes it quite easy to find the optimal angular positions of the test bench and construct the optimal algorithms for calibrating an accelerometer unit (see, e.g., [3, 4, 8, 9]).

However, traditional linear models of readings from an accelerometer unit are not always completely satisfactory. In some cases, it is assumed that a unit's scale factor errors depend on the sign of the signal arriving at the accelerometer input. This paper considers an asymmetric (piecewise linear) model of an accelerometer unit. When the guaranteed approach to parameter estimation is used, the optimal experimental design is determined and the corresponding estimation algorithms are constructed.

## 1. STATEMENT OF THE ESTIMATION PROBLEM

Let us briefly describe a two-degree test bench [8]. We assume that the base of the test bench is motionless relative to the Earth and that due to inaccuracy in the base installation, the outer axis of the test bench deviates from the horizontal plane by a small angle $\kappa$. The outer frame of the test bench can rotate relative to the base around the outer axis; we denote the angle of the corresponding rotation by $\alpha_{tr}$; the inner frame can rotate relative to the outer frame around the inner axis; and we denote the angle of the corresponding rotation by $\beta_{tr}$. The specified angles of rotation of the frames are measured against the noise background:

$$\alpha = \alpha_{tr} + \Delta\alpha + \Delta\alpha_{fl}, \quad \beta = \beta_{tr} + \Delta\beta + \Delta\beta_{fl},$$

where $(\alpha, \beta)$ are the measurement results, $\Delta\alpha$ and $\Delta\beta$ are unknown constants identical for all measurements, and $\Delta\alpha_{fl}$ and $\Delta\beta_{fl}$ are unknown nonparametric (fluctuation) components, which are different for different angle measurements. For high-precision test benches, we further neglect the fluctuation components $\Delta\alpha_{fl}$ and $\Delta\beta_{fl}$, considering them sufficiently small (on the order of 1 arcsec).

At $\alpha_{tr} = 0$, the inner axis is directed (almost) along the geographic vertical; the deviation of the internal axis from the vertical plane formed by the external axis and the geographic vertical, due to the nonhorizontal nature of the base, is denoted by a small angle $\alpha^*$. We assume that the outer and inner axes intersect at point $M^b$, but may not be exactly orthogonal; the small angle of their nonorthogonality will be denoted by $\varepsilon$. We rigidly fix the coordinate system to the inner frame $M^b j$ as follows. The third axis $M^b j_3$ is directed along the internal axis. The $M^b j_1$ axis at $\beta_{tr} = 0$ lies in the plane formed by the external and internal axes, orthogonal to $M^b j_3$ and close to the outer axis. The $M^b j_2$ axis forms a right orthogonal reference system with $M^b j_1$ and $M^b j_3$.

Let us connect the right orthogonal reference system with the accelerometer unit $Mz$, along the axes of which (in the absence of unit errors) the sensitivity axes of accelerometers should be located (point $M$ is the center of the unit). Reference system $Mz$ is called the instrumental reference system. The accelerometer unit is installed on the inner frame so that the axes of the instrumental reference system $Mz$ are directed as accurately as possible along the axes of the reference system $Mf$, which is connected to the inner frame and laid out relative to $M^b j$ in the prescribed manner. Let this rotation be determined by the known orthogonal orientation matrix $Q$: $\chi'' = Q\chi'$, where $\chi'$ and $\chi''$ are the projections of an arbitrary vector on the $M^b j$ and $Mf$ axes, respectively. The installation error $Mz$ along the $Mf$ axes is described by an unknown small rotation vector $\vartheta = (\vartheta_1, \vartheta_2, \vartheta_3)^T$. The error in determining gravity acceleration at the testing point is denoted by $\Delta g$.

The structural model of readings of the accelerometer unit has the form

$$f' = (I_3 + \Gamma) f_z + \Delta f^0 + \Delta f^s, \tag{1.1}$$

where $f' \in \mathbb{R}^3$ are the readings of the accelerometer unit, $I_3 \in \mathbb{R}^{3\times3}$ is the identity matrix, $\Gamma \in \mathbb{R}^{3\times3}$ is the unit error matrix (in which the diagonal elements characterize the scale factor errors, and the off-diagonal elements characterize the angular misalignments of the accelerometers), $f_z \in \mathbb{R}^3$ is the vector of the specific force acting on the sensitive mass of the accelerometer in projections onto the axis of the instrumental reference system (in our static experiments, this specific force is equal to gravity acceleration with the opposite sign), $\Delta f^0 \in \mathbb{R}^3$ are the systematic biases of the unit's readings, and $\Delta f^s \in \mathbb{R}^3$ are the fluctuation errors (the influence of which is significantly reduced by averaging to the level of the residual unmodeled signal of the electromechanical circuit of the accelerometer unit).

Let us assume $\langle f' \rangle$ are the known averaged readings of the accelerometer unit in a certain angular position fixed relative to the Earth, $(\alpha, \beta)$ are the result of measuring the angles of rotation of the outer and inner frames of the test bench, and $\tilde{f}(\alpha,\beta)$ are the unit readings predicted by direct measurement of the angles of rotation of the frames of the test bench (they are known exactly). Then the measurements for the corresponding estimation problem are formed as the normalized difference between the average readings of the accelerometer unit and their predicted values from measurements of the test bench angles. In a linear approximation, the vector of these measurements has the form

$$\begin{aligned} z(\alpha,\beta) &= g^{-1}\{\langle f' \rangle - \tilde{f}(\alpha,\beta)\} \\ &= -g^{-1}\{(I_3 + \Gamma)(I_3 + \hat{\vartheta})Qg^G(\alpha,\beta,q) + \Delta f^0 - Q\tilde{g}^G(\alpha,\beta)\} + \varrho(\alpha,\beta), \end{aligned} \tag{1.2}$$

where $g$ is the model value of gravity acceleration, $g^G(\alpha,\beta,q)$ is the true value of the gravity acceleration vector in projections onto the axis of the reference system $M^b j$ (referenced to the inner frame), $q = (\kappa, \epsilon, \Delta\alpha, \Delta\beta, \alpha^*, \Delta g/g)^T$ is the vector of the error parameters of the test bench, $\tilde{g}^G(\alpha,\beta) = g^G(\alpha,\beta,0)$ is the predicted gravity acceleration vector in projections onto the axes of the reference system $M^b j$, calculated exactly from measurements of the angles, and $\varrho(\alpha,\beta)$ is the vector of the fluctuation components of the measurement errors [3, 8].

After simple but cumbersome calculations, it is easy to show that in a linear approximation

$$g^G(\alpha,\beta,q\ ) = g\left\{-\begin{pmatrix}\sin\alpha\sin\beta\\ \sin\alpha\cos\beta\\ \cos\alpha\end{pmatrix}\left(1-\frac{\Delta g}{g}\right)+\begin{pmatrix}\cos\beta\\ -\sin\beta\\ 0\end{pmatrix}\kappa+\begin{pmatrix}\cos\alpha\cos\beta\\ -\cos\alpha\sin\beta\\ 0\end{pmatrix}\epsilon\right.$$

$$\left.+\begin{pmatrix}\cos\alpha\sin\beta\\ \cos\alpha\cos\beta\\ -\sin\alpha\end{pmatrix}(\Delta\alpha+\alpha^*)+\begin{pmatrix}\sin\alpha\cos\beta\\ -\sin\alpha\sin\beta\\ 0\end{pmatrix}\Delta\beta\right\},\quad \hat{\vartheta}=\begin{pmatrix}0 & \vartheta_3 & -\vartheta_2\\ -\vartheta_3 & 0 & \vartheta_1\\ \vartheta_2 & -\vartheta_1 & 0\end{pmatrix}.$$

Then, from (1.2) we obtain

$$z(\alpha,\beta) = -Q\begin{pmatrix}\cos\beta\\ -\sin\beta\\ 0\end{pmatrix}\kappa - Q\begin{pmatrix}\cos\alpha\sin\beta\\ \cos\alpha\cos\beta\\ -\sin\alpha\end{pmatrix}(\Delta\alpha+\alpha^*) - Q\begin{pmatrix}\cos\alpha\cos\beta\\ -\cos\alpha\sin\beta\\ 0\end{pmatrix}\epsilon \tag{1.3}$$

$$+\left(\Gamma+\hat{\vartheta}-I_3\frac{\Delta g}{g}+QNQ^{-1}\Delta\beta\right)Q\begin{pmatrix}\sin\alpha\sin\beta\\ \sin\alpha\cos\beta\\ \cos\alpha\end{pmatrix}+\frac{\Delta f^0}{g}+\rho,\quad N=\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Let us represent (1.3) in a more standard form:

$$z(\alpha,\beta) = F^{\mathrm{T}}(\alpha,\beta)x + \varrho(\alpha,\beta), \tag{1.4}$$

where

$$F(\alpha,\beta) = \left(\overset{(1)}{F}(\alpha,\beta), \overset{(2)}{F}(\alpha,\beta), \overset{(3)}{F}(\alpha,\beta),\right) = \{F_{dt}(\alpha,\beta)\},\quad d=\overline{1,15},\quad t=1,2,3,$$

(here the indices over $F$ denote the column numbers) has the form

$$F(\alpha,\beta) = \left(Q\begin{pmatrix}-\cos\beta & -\cos\alpha\sin\beta & -\cos\alpha\cos\beta\\ \sin\beta & -\cos\alpha\cos\beta & \cos\alpha\sin\beta\\ 0 & \sin\alpha & 0\end{pmatrix}, \left(\left(Q\begin{pmatrix}\sin\alpha\sin\beta\\ \sin\alpha\cos\beta\\ \cos\alpha\end{pmatrix}\right)\otimes I_3\right)^{\mathrm{T}}, I_3\right)^{\mathrm{T}} \in \mathbb{R}^{15\times 3}$$

(here $\otimes$ is the symbol for the Kronecker product of the matrices), $x=\mathrm{col}(u,v,w)\in\mathbb{R}^{15}$,

$$u=\begin{pmatrix}\kappa\\ \Delta\alpha+\alpha^*\\ \varepsilon\end{pmatrix},\quad v=\begin{pmatrix}\Gamma_{11}-\dfrac{\Delta g}{g}\\ \Gamma_{21}-\vartheta_3+(Q_{11}Q_{22}-Q_{12}Q_{21})\Delta\beta\\ \Gamma_{31}+\vartheta_2+(Q_{11}Q_{32}-Q_{12}Q_{31})\Delta\beta\\ \Gamma_{12}+\vartheta_3-(Q_{11}Q_{22}-Q_{12}Q_{21})\Delta\beta\\ \Gamma_{22}-\dfrac{\Delta g}{g}\\ \Gamma_{32}-\vartheta_1+(Q_{21}Q_{32}-Q_{22}Q_{31})\Delta\beta\\ \Gamma_{13}-\vartheta_2-(Q_{11}Q_{32}-Q_{12}Q_{31})\Delta\beta\\ \Gamma_{23}+\vartheta_1-(Q_{21}Q_{32}-Q_{22}Q_{31})\Delta\beta\\ \Gamma_{33}-\dfrac{\Delta g}{g}\end{pmatrix},\quad w=\begin{pmatrix}\dfrac{\Delta f_1^0}{g}\\ \dfrac{\Delta f_2^0}{g}\\ \dfrac{\Delta f_3^0}{g}\end{pmatrix}.$$

In contrast to the standard approaches, in this study, it is considered that the scale factor error of the $p$th accelerometer $\Gamma_{pp}$ depends on the sign of the accelerometer's input signal $f_p^{\mathrm{inp}}$; i.e., it is asymmetric (piecewise constant) at the input:

$$\Gamma_{pp} = \begin{cases}\Gamma_{pp}^+, & \text{if}\quad f_p^{\mathrm{inp}}>0,\\ \Gamma_{pp}^-, & \text{if}\quad f_p^{\mathrm{inp}}<0,\end{cases}\qquad p=1,2,3$$

(when the input signal is zero, the value of the scale factor is insignificant).

However, since the input signals of accelerometers $f^{\text{inp}} = \text{col}(f_1^{\text{inp}}, f_2^{\text{inp}}, f_3^{\text{inp}})$ are not directly accessible, we construct estimates for them. The following relation holds:

$$f^{\text{inp}}(\alpha,\beta,q) = f^{\text{inp}}(\alpha,\beta,q) + Q\tilde{g}^G(\alpha,\beta) - Q\tilde{g}^G(\alpha,\beta), \tag{1.5}$$

$$-\tilde{g}^G(\alpha,\beta) = g\begin{pmatrix}\sin\alpha\sin\beta\\ \sin\alpha\cos\beta\\ \cos\alpha\end{pmatrix}.$$

It is clear that the input signal is described by formula (1.1), in which the scale factors, systematic biases, and fluctuation noise component should be set to zero:

$$f^{\text{inp}}(\alpha,\beta,q) = f'(\alpha,\beta,q)\Big|_{\substack{\Gamma_{11}=\Gamma_{22}=\Gamma_{33}=0,\\ \Delta f^0=0,\Delta f^S=0}} = gz(\alpha,\beta)\Big|_{\substack{\Gamma_{11}=\Gamma_{22}=\Gamma_{33}=0,\\ \Delta f^0=0,\varrho=0}} -Q\tilde{g}^G(\alpha,\beta).$$

Then, obviously,

$$f^{\text{inp}}(\alpha,\beta,q) = gF^{\mathrm{T}}(\alpha,\beta)x^0 - Q\tilde{g}^G(\alpha,\beta). \tag{1.6}$$

Here

$$x^0 = \begin{pmatrix}u\\ v^0\\ 0_3\end{pmatrix}\in\mathbb{R}^{15},\quad v^0 = \begin{pmatrix}-\dfrac{\Delta g}{g}\\ \Gamma_{21} - \vartheta_3 + (Q_{11}Q_{22} - Q_{12}Q_{21})\Delta\beta\\ \Gamma_{31} + \vartheta_2 + (Q_{11}Q_{32} - Q_{12}Q_{31})\Delta\beta\\ \Gamma_{12} + \vartheta_3 - (Q_{11}Q_{22} - Q_{12}Q_{21})\Delta\beta\\ -\dfrac{\Delta g}{g}\\ \Gamma_{32} - \vartheta_1 + (Q_{21}Q_{32} - Q_{22}Q_{31})\Delta\beta\\ \Gamma_{13} - \vartheta_2 - (Q_{11}Q_{32} - Q_{12}Q_{31})\Delta\beta\\ \Gamma_{23} + \vartheta_1 - (Q_{21}Q_{32} - Q_{22}Q_{31})\Delta\beta\\ -\dfrac{\Delta g}{g}\end{pmatrix},$$

where $0_n$ is the zero column vector of dimension $n$.

We denote by $u_k^{\max}$, $k = 1,2,3$, and $v_s^{0\max}$, $s = \overline{1,9}$ the maximum absolute values of the components $u_k$ and $v_s^0$; we introduce upper bounds for the vector components $F^{\mathrm{T}}(\alpha,\beta)x^0$:

$$s_p^{\max}(\alpha,\beta) = \sum_{k=1}^{3}|\overset{(p)}{F}_k(\alpha,\beta)|u_k^{\max} + \sum_{s=4}^{12}|\overset{(p)}{F}_s(\alpha,\beta)|v_s^{0\max}, \tag{1.7}$$

$$\overset{(p)}{F}(\alpha,\beta) = \text{col}\left(\overset{(p)}{F}_1(\alpha,\beta),\ldots,\overset{(p)}{F}_{15}(\alpha,\beta)\right).$$

We also introduce the notation

$$\text{col}(M_1(\alpha,\beta), M_2(\alpha,\beta), M_3(\alpha,\beta)) = -g^{-1}Q\tilde{g}^G(\alpha,\beta). \tag{1.8}$$

Thus, if

$$|M_p(\alpha,\beta)| > s_p^{\max}(\alpha,\beta), \tag{1.9}$$

then the sign of the input of the $p$th accelerometer is determined. Otherwise, there is no reliable information about the sign of the input signal, and such measurements should be excluded from consideration. We deem admissible the angular positions for which for the $p$th group of measurements condition (1.9) is satisfied.

Thus, the measurement model should be modified such that, while remaining linear in the estimated parameters, it reflects the ambiguity of the scale factors

$$\Gamma_{pp} = \begin{cases} \Gamma^{+}_{pp}, & \text{if} \quad M_p(\alpha,\beta) > s^{\max}_p(\alpha,\beta), \\ \Gamma^{-}_{pp}, & \text{if} \quad M_p(\alpha,\beta) < -s^{\max}_p(\alpha,\beta), \end{cases} \qquad p = 1,2,3.$$

This can be done by introducing extended vectors from $\mathbb{R}^{18}$:

$$\overset{(1)}{\mathcal{F}}(\alpha,\beta) = \mathrm{col}(F_{11}, F_{21}, F_{31}, M^{+}_1(\alpha,\beta), M^{-}_1(\alpha,\beta), 0_2, M_2(\alpha,\beta), 0_3, M_3(\alpha,\beta), 0_3, 1, 0_2),$$

$$\overset{(2)}{\mathcal{F}}(\alpha,\beta) = \mathrm{col}(F_{12}, F_{22}, F_{32}, 0_2, M_1(\alpha,\beta), 0_2, M^{+}_2(\alpha,\beta), M^{-}_2(\alpha,\beta), 0_2, M_3(\alpha,\beta), 0_3, 1, 0),$$

$$\overset{(3)}{\mathcal{F}}(\alpha,\beta) = \mathrm{col}(F_{13}, F_{23}, F_{33}, 0_3, M_1(\alpha,\beta), 0_3, M_2(\alpha,\beta), 0_2, M^{+}_3(\alpha,\beta), M^{-}_3(\alpha,\beta), 0_2, 1), \tag{1.10}$$

in which

$$M^{+}_p(\alpha,\beta) = \begin{cases} M_p(\alpha,\beta), & \text{if} \quad M_p(\alpha,\beta) > s^{\max}_p(\alpha,\beta), \\ 0, & \text{if} \quad M_p(\alpha,\beta) < -s^{\max}_p(\alpha,\beta), \end{cases} \qquad p = 1,2,3,$$

$$M^{-}_p(\alpha,\beta) = \begin{cases} 0, & \text{if} \quad M_p(\alpha,\beta) > s^{\max}_p(\alpha,\beta), \\ M_p(\alpha,\beta), & \text{if} \quad M_p(\alpha,\beta) < -s^{\max}_p(\alpha,\beta), \end{cases} \qquad p = 1,2,3.$$

Then, the measurements for the case of the scale factor errors depending on the sign of the input signal can be represented in the following extended form:

$$\overset{(p)}{z}(\alpha,\beta) = \overset{(p)}{\mathcal{F}}{}^{\mathrm{T}}(\alpha,\beta)\mathcal{X} + \overset{(p)}{\varrho}(\alpha,\beta), \qquad p = 1,2,3, \tag{1.11}$$

where $\mathcal{X} = \mathrm{col}(u, v^{\pm}, w) \in \mathbb{R}^{18}$,

$$v^{\pm} = \begin{pmatrix} \Gamma^{+}_{11} - \dfrac{\Delta g}{g} \\ \Gamma^{-}_{11} - \dfrac{\Delta g}{g} \\ \Gamma_{21} - \vartheta_3 + (Q_{11}Q_{22} - Q_{12}Q_{21})\Delta\beta \\ \Gamma_{31} + \vartheta_2 + (Q_{11}Q_{32} - Q_{12}Q_{31})\Delta\beta \\ \Gamma_{12} + \vartheta_3 - (Q_{11}Q_{22} - Q_{12}Q_{21})\Delta\beta \\ \Gamma^{+}_{22} - \dfrac{\Delta g}{g} \\ \Gamma^{-}_{22} - \dfrac{\Delta g}{g} \\ \Gamma_{32} - \vartheta_1 + (Q_{21}Q_{32} - Q_{22}Q_{31})\Delta\beta \\ \Gamma_{13} - \vartheta_2 - (Q_{11}Q_{32} - Q_{12}Q_{31})\Delta\beta \\ \Gamma_{23} + \vartheta_1 - (Q_{21}Q_{32} - Q_{22}Q_{31})\Delta\beta \\ \Gamma^{+}_{33} - \dfrac{\Delta g}{g} \\ \Gamma^{-}_{33} - \dfrac{\Delta g}{g} \end{pmatrix} \in \mathbb{R}^{12}, \qquad z(\alpha,\beta) = \begin{pmatrix} \overset{(1)}{z}(\alpha,\beta) \\ \overset{(2)}{z}(\alpha,\beta) \\ \overset{(3)}{z}(\alpha,\beta) \end{pmatrix} \in \mathbb{R}^3, \qquad \varrho(\alpha,\beta) = \begin{pmatrix} \overset{(1)}{\varrho}(\alpha,\beta) \\ \overset{(2)}{\varrho}(\alpha,\beta) \\ \overset{(3)}{\varrho}(\alpha,\beta) \end{pmatrix} \in \mathbb{R}^3.$$

In accordance with measurements (1.11), for all possible admissible angular positions from $[0, 2\pi] \times [0, 2\pi]$, it is necessary to estimate the following parameters: (a) test bench errors, (b) scaling factors (up to the error in determining gravity acceleration), (c) misalignment of the accelerometer sensitivity axes, (d) errors in the angles of installation of the accelerometer unit on the faceplate (accurate to the systematic bias in measuring the angle of rotation of the inner frame), and (e) systematic biases of the accelerometer readings.

During calibration, one of two conventions (either of which) is accepted: (A1) matrix $\Gamma$ is lower triangular, and (A2) matrix $\Gamma$ is symmetrical. In case (A1), this corresponds to estimating the following parameters: (a) $\mathscr{X}_1, \mathscr{X}_2, \mathscr{X}_3$; (b) $\mathscr{X}_4, \mathscr{X}_5, \mathscr{X}_9, \mathscr{X}_{10}, \mathscr{X}_{14}, \mathscr{X}_{15}$; (c) $\mathscr{X}_6 + \mathscr{X}_8, \mathscr{X}_7 + \mathscr{X}_{12}, \mathscr{X}_{11} + \mathscr{X}_{13}$; (d) $\mathscr{X}_8, -\mathscr{X}_{12}, \mathscr{X}_{13}$; and (e) $\mathscr{X}_{16}, \mathscr{X}_{17}, \mathscr{X}_{18}$. In case (A2), sets of estimated parameters (a), (b), and (e) are the same as those for case (A1), and sets (c) and (d) are replaced by (c) $1/2(\mathscr{X}_6 + \mathscr{X}_8)$, $1/2(\mathscr{X}_7 + \mathscr{X}_{12})$, $1/2(\mathscr{X}_{11} + \mathscr{X}_{13})$ and (d) $1/2(\mathscr{X}_8 - \mathscr{X}_6)$, $1/2(\mathscr{X}_7 - \mathscr{X}_{12})$, $1/2(\mathscr{X}_{13} - \mathscr{X}_{11})$.

We set the grid for numerical calculations on a $[0,2\pi]\times[0,2\pi]$ square. The set of grid elements for which condition (1.9) is satisfied will be denoted by $\mathscr{G}_p$.

## 2. GUARANTEED ESTIMATION METHOD

Let us briefly describe the guaranteed estimation method [3, 25–32]. We consider three groups of measurements (1.11) and assume that the measurement errors can be arbitrary numbers, bounded by the known value $\sigma$:

$$|\overset{(p)}{\varrho}(\alpha,\beta)| \le \sigma, \quad (\alpha,\beta)\in\mathscr{G}_p, \quad p = 1,2,3. \tag{2.1}$$

We emphasize that model (2.1) is entirely consistent with the problem of calibrating an accelerometer unit, since after obligatory averaging of the accelerometer readings, the measurement errors consist of the sum of the residual signal of the electromechanical circuit of the accelerometer unit and discarded small quadratic terms. These measurement errors have neither a clear parametric model nor a stable spectrum.

It is required to estimate the scalar quantity $l = a^{\mathrm{T}}\mathscr{X}$, where $a \in \mathbb{R}^{18}$ is the given vector. Let us denote by $e^{(\nu)}$ the basis vector from $\mathbb{R}^{18}$ with a unit on the $\nu$th place. Then, to solve the calibration problems, we need to solve 18 tasks. Moreover, in the case of a lower triangular matrix $\Gamma$, the following holds: $a = e^{(\nu)}$, $\nu = \overline{1,5}, \overline{8,10}, \overline{13,18}$ and $a = e^{(6)} + e^{(8)}$, $a = e^{(7)} + e^{(12)}$, $a = e^{(11)} + e^{(13)}$, and $a = -e^{(12)}$; and in the case of symmetric matrix $\Gamma$, the following holds: $a = e^{(\nu)}$, $\nu = \overline{1,5}, 9, 10, \overline{14,18}$ and $a = 1/2(e^{(6)} + e^{(8)})$, $a = 1/2(e^{(7)} + e^{(12)})$, $a = 1/2(e^{(11)} + e^{(13)})$, $a = 1/2(e^{(8)} - e^{(6)})$, $a = 1/2(e^{(7)} - e^{(12)})$, and $a = 1/2(e^{(13)} - e^{(11)})$.

Let us consider linear estimators for $l = a^{\mathrm{T}}\mathscr{X}$ of the form

$$\tilde{l} = \sum_{p=1}^{3} \sum_{(\alpha,\beta)\in\mathscr{G}_p} \overset{(p)}{\Phi}(\alpha,\beta)\, \overset{(p)}{z}(\alpha,\beta), \tag{2.2}$$

where $\overset{(p)}{\Phi}(\alpha,\beta)$ are some weighting functions.

The quantity

$$\max_{\mathscr{X}\in\mathbb{R}^{18},\ |\varrho(\alpha,\beta)|\le\sigma} |\tilde{l} - l|$$

is called the guaranteed estimation error; the brief notation $|\varrho(\alpha,\beta)| \le \sigma$ means fulfillment of condition (2.1). With the selected estimator, this is the maximum value of the estimation error for all possible values of uncertain factors $\mathscr{X}$ and $\varrho$.

We search for the weight coefficients $\overset{(p)}{\Phi}(\alpha,\beta)$ minimizing the guaranteed estimation error:

$$\min_{\Phi(\alpha,\beta)} \max_{\mathscr{X}\in\mathbb{R}^{18},\ |\varrho(\alpha,\beta)|\le\sigma} |\tilde{l} - l|$$

(here $\Phi(\alpha,\beta)$ is understood as the set $\overset{(p)}{\Phi}(\alpha,\beta)$, $p = 1,2,3$). Such a problem is called the problem of the optimal guaranteed estimation.

Thus, to solve the problem of calibration and diagnostics of the test bench, we need to solve 18 individual problems. It can be shown that the use of arbitrary nonlinear estimators in addition to linear estimators of form (2.2) does not lead to a decrease in the guaranteed estimation error [32]; i.e., we can restrict ourselves to linear estimators.

It is easy to verify that

$$\tilde{l} - l = \left\{ \sum_{p=1}^{3} \sum_{(\alpha,\beta)\in\mathcal{G}_p} \overset{(p)}{\mathcal{F}}(\alpha,\beta)\, \overset{(p)}{\Phi}(\alpha,\beta) - a \right\}^{\mathrm{T}} \mathcal{X} + \sum_{p=1}^{3} \sum_{(\alpha,\beta)\in\mathcal{G}_p} \overset{(p)}{\Phi}(\alpha,\beta)\, \overset{(p)}{\varrho}(\alpha,\beta). \tag{2.3}$$

Explicitly calculating the guaranteed estimation error, taking into account the previous formula, we can show that this problem reduces to a variational problem of the form

$$\min_{\Phi(\alpha,\beta)} \sigma \sum_{p=1}^{3} \sum_{(\alpha,\beta)\in\mathcal{G}_p} |\overset{(p)}{\Phi}(\alpha,\beta)| \tag{2.4}$$

subject to

$$\sum_{p=1}^{3} \sum_{(\alpha,\beta)\in\mathcal{G}_p} \overset{(p)}{\mathcal{F}}(\alpha,\beta)\, \overset{(p)}{\Phi}(\alpha,\beta) = a. \tag{2.5}$$

The value of functional (2.4) determines the guaranteed estimation error. Conditions (2.5) are called unbiasedness conditions, because if they are satisfied in the absence of measurement errors, the estimate $\tilde{l}$ coincides with $l$.

The following statement holds [26, 31, 33].

**Statement.** A solution to problem (2.4) and (2.5) exists, and at least one solution contains at most 18 nonzero components.

This means that the solution to the optimal guaranteed estimation problem (for each estimated parameter) from all elements of the grid $\mathcal{G} = \cup_{p=1}^{3} \mathcal{G}_p$ selects not more than 18 required angular positions of the test bench; i.e., it simultaneously delivers the optimal measurement design. This makes the guaranteed approach very effective for solving the accelerometer unit calibration problem and test bench diagnosis.

A nonsmooth variational problem is reduced to the following canonical linear programming problem [26, 31]:

$$\min_{\Phi(\alpha,\beta)} \sigma \sum_{p=1}^{3} \sum_{(\alpha,\beta)\in\mathcal{G}_p} \left( \overset{(p)}{\Phi}{}^{+}(\alpha,\beta) + \overset{(p)}{\Phi}{}^{-}(\alpha,\beta) \right) \tag{2.6}$$

subject to

$$\sum_{p=1}^{3} \sum_{(\alpha,\beta)\in\mathcal{G}_p} \overset{(p)}{\mathcal{F}}(\alpha,\beta) \left( \overset{(p)}{\Phi}{}^{+}(\alpha,\beta) - \overset{(p)}{\Phi}{}^{-}(\alpha,\beta) \right) = a, \quad \overset{(p)}{\Phi}{}^{+}(\alpha,\beta) \geq 0, \quad \overset{(p)}{\Phi}{}^{-}(\alpha,\beta) \geq 0. \tag{2.7}$$

If $\{\overset{(p)}{\Phi}{}_{\circ}^{+}(\alpha,\beta), \overset{(p)}{\Phi}{}_{\circ}^{-}(\alpha,\beta)\}_{(\alpha,\beta)\in\mathcal{G}_p}^{p=1,2,3}$ is its solution, then $\{\overset{(p)}{\Phi}{}_{\circ}(\alpha,\beta)\}_{(\alpha,\beta)\in\mathcal{G}_p}^{p=1,2,3} = \{\overset{(p)}{\Phi}{}_{\circ}^{+}(\alpha,\beta) - \overset{(p)}{\Phi}{}_{\circ}^{-}(\alpha,\beta)\}_{(\alpha,\beta)\in\mathcal{G}_p}^{p=1,2,3}$ is the solution of the original variational problem (2.4) and (2.5). Problem (2.6) and (2.7) can be solved numerically by the simplex method or the interior point method [33–36].

## 3. NUMERICAL SOLUTION

To numerically solve problem (2.4) and (2.5) (or (2.6), (2.7)), we consider the case when the instrumental reference system is installed along the axes of the inner frame, i.e., at $Q = I_3$. Then these expressions for the regression vectors are simplified and take the form

$$\overset{(1)}{\mathcal{F}} = \mathrm{col}(-\cos\beta, -\cos\alpha\sin\beta, -\cos\alpha\cos\beta, s_1^{+}(\alpha,\beta), s_1^{-}(\alpha,\beta), 0_2, \sin\alpha\cos\beta, 0_3, \cos\alpha, 0_3, 1, 0_2),$$

$$\overset{(2)}{\mathcal{F}} = \mathrm{col}(\sin\beta, -\cos\alpha\cos\beta, \cos\alpha\sin\beta, 0_2, \sin\alpha\sin\beta, 0_2, s_2^{+}(\alpha,\beta), s_2^{-}(\alpha,\beta), 0_2, \cos\alpha, 0_3, 1, 0),$$

$$\overset{(3)}{\mathcal{F}} = \mathrm{col}(0, \sin\alpha, 0_4, \sin\alpha\sin\beta, 0_3, \sin\alpha\cos\beta, 0_2, s_3^{+}(\alpha,\beta), s_3^{-}(\alpha,\beta), 0_2, 1), \tag{3.1}$$

where

$$s_1^+(\alpha,\beta) = \begin{cases} \sin\alpha\sin\beta, & \text{if} \quad \sin\alpha\sin\beta > s_1^{\max}(\alpha,\beta), \\ 0, & \text{if} \quad \sin\alpha\sin\beta < -s_1^{\max}(\alpha,\beta), \end{cases}$$

$$s_1^-(\alpha,\beta) = \begin{cases} 0, & \text{if} \quad \sin\alpha\sin\beta > s_1^{\max}(\alpha,\beta), \\ \sin\alpha\sin\beta, & \text{if} \quad \sin\alpha\sin\beta < -s_1^{\max}(\alpha,\beta), \end{cases}$$

$$s_2^+(\alpha,\beta) = \begin{cases} \sin\alpha\cos\beta, & \text{if} \quad \sin\alpha\cos\beta > s_2^{\max}(\alpha,\beta), \\ 0, & \text{if} \quad \sin\alpha\cos\beta < -s_2^{\max}(\alpha,\beta), \end{cases}$$

$$s_2^-(\alpha,\beta) = \begin{cases} 0, & \text{if} \quad \sin\alpha\cos\beta > s_2^{\max}(\alpha,\beta), \\ \sin\alpha\cos\beta, & \text{if} \quad \sin\alpha\cos\beta < -s_2^{\max}(\alpha,\beta), \end{cases}$$

$$s_3^+(\alpha,\beta) = \begin{cases} \cos\alpha, & \text{if} \quad \cos\alpha > s_3^{\max}(\alpha,\beta), \\ 0, & \text{if} \quad \cos\alpha < -s_3^{\max}(\alpha,\beta), \end{cases}$$

$$s_3^-(\alpha,\beta) = \begin{cases} 0, & \text{if} \quad \cos\alpha > s_3^{\max}(\alpha,\beta), \\ \cos\alpha, & \text{if} \quad \cos\alpha < -s_3^{\max}(\alpha,\beta). \end{cases}$$

Here $s_p^{\max}(\alpha,\beta)$, $p = 1,2,3$, are determined by formula (1.7), in which $v_s^{0\max}$ is the maximum value of the $s$th element of the vector

$$v^0 = \mathrm{col}\left(-\frac{\Delta g}{g}, \Gamma_{21} - \vartheta_3 + \Delta\beta, \Gamma_{31} + \vartheta_2, \Gamma_{12} + \vartheta_3 - \Delta\beta, -\frac{\Delta g}{g}, \Gamma_{32} - \vartheta_1, \Gamma_{13} - \vartheta_2, \Gamma_{23} + \vartheta_1, -\frac{\Delta g}{g}\right),$$

and

$$\overset{(1)}{F}(\alpha,\beta) = \mathrm{col}(-\cos\beta, -\cos\alpha\sin\beta, -\cos\alpha\cos\beta, \sin\alpha\sin\beta, 0, 0, \sin\alpha\cos\beta, 0, 0, \cos\alpha, 0, 0, 1, 0, 0),$$

$$\overset{(2)}{F}(\alpha,\beta) = \mathrm{col}(\sin\beta, -\cos\alpha\cos\beta, \cos\alpha\sin\beta, 0, \sin\alpha\sin\beta, 0, 0, \sin\alpha\cos\beta, 0, 0, \cos\alpha, 0, 0, 1, 0),$$

$$\overset{(3)}{F}(\alpha,\beta) = \mathrm{col}(0, \sin\alpha, 0, 0, 0, \sin\alpha\sin\beta, 0, 0, \sin\alpha\cos\beta, 0, 0, \cos\alpha, 0, 0, 1).$$

When calculating on the initial square of all values $(\alpha,\beta)$, a grid with a step of $1°$ on both angles was taken. With some roughness, it was assumed that $u_k^{\max} = 3\times 10^{-3}$, $k = 1,2,3$, and $v_s^{0\max} = 10^{-2}$, $s = \overline{4,12}$. The calculations were performed using the IBM ILOG CPLEX Optimization Studio package. Two numerical methods were used: the interior point method and the simplex method, which both yielded the same results.

The figure shows the range of allowable grid values $\mathcal{G}_1$, corresponding to the accepted values $u_k^{\max}$ and $v_s^{0\max}$, and the optimal positions of the angles for estimating the component $\mathcal{X}_4$. Domains $\mathcal{G}_2$ and $\mathcal{G}_3$ are very close to $\mathcal{G}_1$. The calculation results are presented in Tables 1 and 2, where the following values are indicated: in the first column, the estimated parameters; in the second column, the values of the test bench's pairs of angles of rotation (in degrees) $(\alpha,\beta)$ and the numbers of the accelerometers used to construct the corresponding estimate; in the third column, the values of the weight coefficients of the estimator; and in the fourth column, the optimal guaranteed accuracies (the values of the functionals).

## 4. SUBOPTIMAL ANGULAR POSITIONS

Let us denote the optimal angular positions for the $p$th measurement group $\overset{(p)}{z}(\alpha,\beta)$ (corresponding to the nonzero components $\{\overset{(p)}{\Phi}_\circ(\alpha,\beta)\}_{(\alpha,\beta)\in\mathcal{G}_p}$) by $\mathcal{G}_p^\circ$. In some cases, it is not technically easy to exactly real-

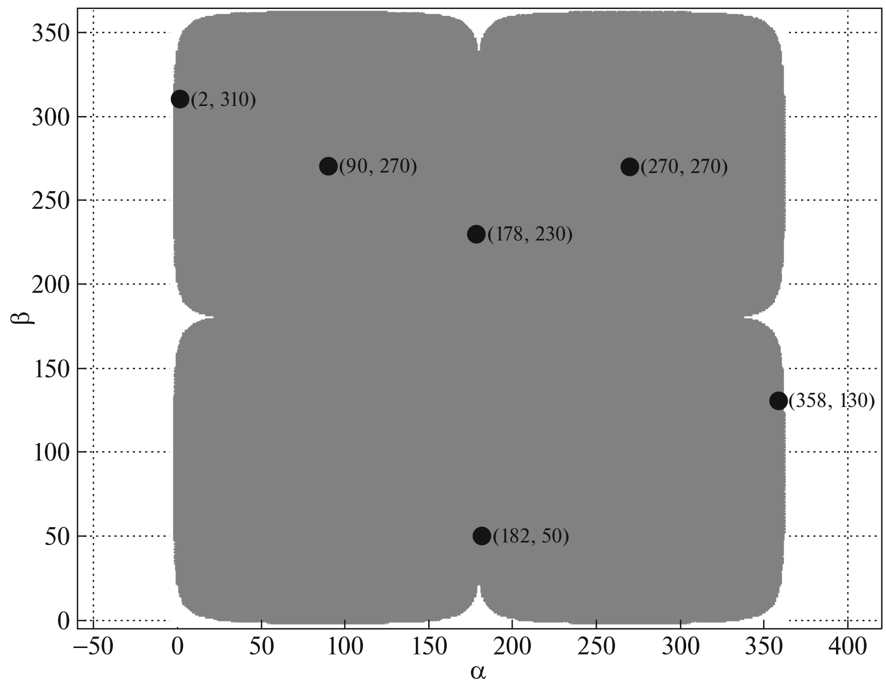


**Fig. 1.** Domain $\mathcal{G}_1$ of admissible angles (in degrees).

ize the required optimal angular positions $\mathcal{G}_p^\circ$, and we can put the test bench in the close angular positions $(\alpha + \overset{(p)}{\delta\alpha}, \beta + \overset{(p)}{\delta\beta})$ for $(\alpha,\beta) \in \mathcal{G}_p^\circ$.

When the angles are set with errors, unbiasedness conditions (2.5), generally speaking, will no longer be satisfied:

$$\sum_{p=1}^{3}\sum_{(\alpha,\beta)\in\mathcal{G}_p^\circ}\overset{(p)}{\mathcal{F}}(\alpha + \overset{(p)}{\delta\alpha}, \beta + \overset{(p)}{\delta\beta})\overset{(p)}{\Phi}_\circ(\alpha,\beta) - a = b \neq 0, \quad b = \mathrm{col}(b_1,\ldots,b_{18}). \tag{4.1}$$

If the set $\mathcal{G}^\circ = \cup_{p=1}^{3}\mathcal{G}_p^\circ$ consists of 18 elements and the corresponding set $\{\overset{(p)}{\mathcal{F}}(\alpha,\beta)\}_{(\alpha,\beta)\in\mathcal{G}_p}^{p=1,2,3}$ is linearly independent, then, due to the smallness of $(\overset{(p)}{\delta\alpha}, \overset{(p)}{\delta\beta})$, the set $\{\overset{(p)}{\mathcal{F}}(\alpha + \overset{(p)}{\delta\alpha}, \beta + \overset{(p)}{\delta\beta})\}_{(\alpha,\beta)\in\mathcal{G}_p}^{p=1,2,3}$ will also be linearly independent. The unbiased suboptimal estimator is found from the unbiasedness conditions. Clearly, the guaranteed estimation error of the suboptimal estimator will be close to the optimal one.

Let us now consider the case where the set $\mathcal{G}^\circ$ contains less than 18 elements. Let it be additionally known that $|\mathcal{X}_\nu| \le \mathcal{X}_\nu^{\max}$, $\nu = \overline{1,18}$ (for brevity, we denote these conditions as $|\mathcal{X}| \le \mathcal{X}^{\max}$). Then, given (2.3), the guaranteed value of the parameter estimation error $l$ takes the form

$$\max_{|\mathcal{X}|\le\mathcal{X}^{\max},|\varrho(\alpha,\beta)|\le\sigma}|\tilde{l} - l| = \sum_{\nu=1}^{18}|b_\nu|\mathcal{X}_\nu^{\max} + \sigma\sum_{p=1}^{3}\sum_{(\alpha,\beta)\in\mathcal{G}_p^\circ}|\overset{(p)}{\Phi}_\circ(\alpha,\beta)|.$$

If

$$\sum_{\nu=1}^{18}|b_\nu|\mathcal{X}_\nu^{\max} \ll \sigma\sum_{p=1}^{3}\sum_{(\alpha,\beta)\in\mathcal{G}_p^\circ}|\overset{(p)}{\Phi}_\circ(\alpha,\beta)|,$$

then the inaccuracy in the setting of the angles of the test bench is acceptable. Otherwise, it is necessary to use the compensation scheme described and substantiated in [9]. Let us assume the quantities to be estimated are $l^{(\nu)} = a^{(\nu)T}\mathcal{X}$, $\nu = \overline{1,18}$, and the vectors $a^{(1)},\ldots,a^{(18)}$ are linearly independent (in our case, this

**Table 1.** Results of calculations of parameter estimates (a) and (b) for case (A1)

| Parameter | Angular positions, degrees | Weight coefficients | Accuracy |
|---|---|---|---|
| $\mathscr{X}_1$ | $p$ = 1: (87.178); (89.2); (120.182); | 0.113004; –0.153208; 0.091198; | 1.00σ |
| | (124.2); (279.2); (288.182); | –0.066789; –0.079964; 0.219288; | |
| | (294.358) | –0.123528 | |
| | $p$ = 2: (56.88); (122.88); (124.268); | 0.000111; 0.044698; –0.038232; | |
| | (260.268); (262.88) | –0.038584; 0.032007 | |
| $\mathscr{X}_2$ | $p$ = 3: (92.45); (268.225) | 0.500305; –0.500305 | 1.00σ |
| $\mathscr{X}_3$ | $p$ = 1: (10.170); (170.190); (190.10) | 0.257773; –0.257773; 0.257773 | 1.00σ |
| | (350.350) | –0.257773; | |
| $\mathscr{X}_4$ | $p$ = 1: (2.310); (90.270); (178.230); | –0.256867; 0.027469; –0.256867; | 2.05σ |
| | (182.50); (270.270); (358.130); | –0.256867; 1.000000; – 0.256867; | |
| $\mathscr{X}_5$ | $p$ = 1: (182.130); (182.310); (270.90); | 0.256867; 0.256867; –1.013734 | 2.05σ |
| | (270.270); (358.50); (358.230); | –0.013734; 0.256867; 0.256867 | |
| $\mathscr{X}_9$ | $p$ = 2: (182.40); (182.320); (270.180); | –0.256867; –0.256867; 1.000000 | 2.05σ |
| | (270.360); (358.40); (358.320); | 0.027469; –0.256867; –0.256867 | |
| $\mathscr{X}_{10}$ | $p$ = 2: (182.140); (182.220); (270.180); | 0.256867; 0.256867; –0.027469; | 2.05σ |
| | (270.360); (358.140); (358.220); | –1.000000; 0.256867; 0.256867 | |
| $\mathscr{X}_{14}$ | $p$ = 3: (92.53); (92.225); (180.277); | –0.447556; –0.070525; 0.036162; | 2.07σ |
| | (268.1); (268.89); (360.2); | –0.214207; –0.303874; 1.000000 | |
| $\mathscr{X}_{15}$ | $p$ = 3: (88.45); (88.200); (180.5); | 0.436524; 0.081556; –1.000000; | 2.07σ |
| | (272.19); (272.114); (360.274) | 0.327419; 0.190662; –0.036162 | |

is obviously the case). Then the suboptimal estimates $\tilde{l}^{(\nu)}$ of parameters $l^{(\nu)}$, $\nu = \overline{1,18}$ should be constructed using a formula similar to the corresponding formula in [9]:

$$\tilde{l}^{(\nu)} = a^{(\nu)T}\mathscr{W}^{-1}\cdot\mathscr{Z}, \tag{4.2}$$

where

$$\mathscr{W} = \begin{pmatrix} \sum\limits_{p=1}^{3}\sum\limits_{(\alpha,\beta)\in\mathscr{G}_p^\circ} \overset{(p)}{\Phi}{}_{\circ}^{(1)}(\alpha,\beta)\overset{(p)}{\mathscr{F}}{}^{\mathrm{T}}(\alpha+\overset{(p)}{\delta\alpha},\beta+\overset{(p)}{\delta\beta}) \\ \vdots \\ \sum\limits_{p=1}^{3}\sum\limits_{(\alpha,\beta)\in\mathscr{G}_p^\circ} \overset{(p)}{\Phi}{}_{\circ}^{(18)}(\alpha,\beta)\overset{(p)}{\mathscr{F}}{}^{\mathrm{T}}(\alpha+\overset{(p)}{\delta\alpha},\beta+\overset{(p)}{\delta\beta}) \end{pmatrix} \in \mathbb{R}^{18\times18},$$

$$\mathscr{Z} = \begin{pmatrix} \sum\limits_{p=1}^{3}\sum\limits_{(\alpha,\beta)\in\mathscr{G}_p^\circ} \overset{(p)}{\Phi}{}_{\circ}^{(1)}(\alpha,\beta)\,\overset{(p)}{z}(\alpha+\overset{(p)}{\delta\alpha},\beta+\overset{(p)}{\delta\beta}) \\ \vdots \\ \sum\limits_{p=1}^{3}\sum\limits_{(\alpha,\beta)\in\mathscr{G}_p^\circ} \overset{(p)}{\Phi}{}_{\circ}^{(18)}(\alpha,\beta)\,\overset{(p)}{z}(\alpha+\overset{(p)}{\delta\alpha},\beta+\overset{(p)}{\delta\beta}) \end{pmatrix} \in \mathbb{R}^{18}.$$

The set $\{\overset{(p)}{z}(\alpha+\overset{(p)}{\delta\alpha},\beta+\overset{(p)}{\delta\beta})\}_{(\alpha,\beta)\in\mathscr{G}_p^\circ}^{p=1,2,3}$ is formed by measurements in positions close to the optimal angular positions and the sets $\{\overset{(p)}{\Phi}{}_{\circ}^{(\nu)}(\alpha,\beta)\}_{(\alpha,\beta)\in\mathscr{G}_p^\circ}^{p=1,2,3}$ consist of nonzero components of the optimal solutions to the

**Table 2.** Results of calculations of parameter estimates (c)–(e) for case (A1)

| Parameter | Angular positions, degrees | Weight coefficients | Accuracy |
|---|---|---|---|
| $\mathscr{X}_6 + \mathscr{X}_8$ | $p = 1$: (90.178); (270.182) | –0.500305; 0.500305 | 2.00σ |
| | $p = 2$: (90.88); (90.272); (270.88); (270.272) | 0.250152; –0.250152; –0.250152; 0.250152 | |
| $\mathscr{X}_7 + \mathscr{X}_{12}$ | $p = 1$: (2.68); (2.251); (182.50); (182.245); (182.270); (358.263); (358.302) | 0.036440; 0.041929; –0.051752; –0.078714; –0.369839; 0.343043; 0.078892 | 2.00σ |
| | $p = 3$: (88.270); (92.90); (92.270); (272.270) | –0.407391; 0.092914; –0.092914; 0.407391 | |
| $\mathscr{X}_{11} + \mathscr{X}_{13}$ | $p = 2$: (2.13); (2.166); (2.179); (178.15); (178.153); (178.332); (182.354); (358.330) | 0.020308; 0.125096; 0.275617; –0.006597; –0.099269; –0.015193; –0.379245; 0.079284 | 2.00σ |
| | $p = 3$: (272.180); (272.360) | 0.500305; –0.500305 | |
| $\mathscr{X}_8$ | $p = 1$: (90.178); (270.182) | –0.500305; 0.500305 | 1.00σ |
| $-\mathscr{X}_{12}$ | $p = 1$: (2.56); (2.233); (178.90); (178.124); (182.55); (182.257); (358.268); (358.292) | –0.011289; –0.097459; 0.302084; 0.093598; 0.095018; 0.009606; –0.230376; –0.161180 | 1.00σ |
| $\mathscr{X}_{13}$ | $p = 2$: (2.194); (2.350); (178.177); (178.193); (182.146); (182.341); (358.14); (358.336) | 0.053382; 0.062675; –0.215213; –0.155860; –0.074451; –0.054781; 0.277628; 0.106621 | 1.00σ |
| $\mathscr{X}_{16}$ | $p = 1$: (2.230); (178.310); (182.230); (270.90); (270.270); (358.310) | 0.256867; 0.256867; 0.256867; –0.013734; –0.013734; 0.256867 | 1.05σ |
| $\mathscr{X}_{17}$ | $p = 2$: (182.140); (182.320); (270.180); (270.360); (358.140); (358.320) | 0.256867; 0.256867; –0.013734; –0.013734; 0.256867; 0.256867 | 1.05σ |
| $\mathscr{X}_{18}$ | $p = 3$: (88.84); (92.161); (180.5); (272.153); (272.317); (360.274) | 0.061464; 0.456617; –0.015936; 0.495707; 0.022374; –0.020226 | 1.07σ |

variational problem (2.4) and (2.5) for $a = a^{(\nu)}$, $\nu = \overline{1,18}$. Similarly to [9], it can be shown that for sufficiently small $(\overset{(p)}{\delta\alpha}, \overset{(p)}{\delta\beta})$, a matrix inverse to $\mathscr{W}$ exists, the obtained suboptimal estimates are unbiased, and their guaranteed estimation errors are close to optimal.

## 5. BRIEF DESCRIPTION OF PROCESSING OF EXPERIMENTS

The developed theory was applied to calibrate a real accelerometer unit on a high-precision test bench. At the start, the correctness of the raw data obtained in the experiments was checked. First, the stability of the angular positions of the accelerometer unit was confirmed. For this, the time-averaged values and spread of readings of the angle sensors in each angular position were calculated. In addition, the angular data were tested for a linear trend. The calculations showed the almost complete absence of a linear trend and high level of stability of the angular positions. Therefore, the positions of the unit for each pair of angles can be considered rigidly fixed relative to the Earth.

Second, for each accelerometer in all positions, the time-averaged values and the spread of their readings were calculated; in addition, the corresponding linear trend was estimated. The experimental data showed the presence of a large initial time interval of temperature instability of the readings of the accelerometers and made it possible to determine the instants from which the average readings of the accelerometers could be considered reliable. The results of the experiments also confirmed the need for averaging the accelerometer readings in each position and indicated the presence of a slight residual linear (apparently temperature) trend. In this case, the variations in the average values of the readings of the accelerometers, treated as measurement errors, obviously belonged to the required accuracy range.

The calibration technique described above used the linearization of the corresponding relations and thus assumed the smallness of the values of the estimated parameters. At the same time, the initial readings of the unit were noticeably affected by the excessively large errors in the scale factors, which cast doubt on the accuracy of the linearization and thus prevented the direct application of the theory developed in this article. Therefore, the accelerometer unit was preliminarily roughly calibrated, for which both the asymmetry of the scale factor errors and accelerometer misalignments were ignored. After that, the initial unit reading model was adjusted as necessary to fit the proposed methodology. Thus, the solution to the estimation problem was preceded by a laborious stage of initial data processing to prepare them for exact calibration. As a result, using the precision calibration procedure, estimates of the errors of the accelerometer unit were obtained, which, in particular, revealed significant asymmetry of the scale factor error in one accelerometer of the unit.

## CONCLUSIONS

The guaranteed approach was used to calibrate an accelerometer unit in the case of asymmetry of scale factor errors. This made it possible to quite simply obtain the optimal design of the angular calibration positions. The effectiveness of the guaranteed approach is confirmed by real experiments on a test bench.

## FUNDING

This study was supported by the Russian Foundation for Basic Research (project no. 18-01-00054-a).

## CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

## REFERENCES


1. A. Yu. Ishlinskii, *Orientation, Gyroscopes and Inertial Navigation* (Nauka, Moscow, 1976) [in Russian].
2. A. A. Golovan and N. A. Parusnikov, *Mathematical Foundations of Navigation Systems. Part I. Mathematical Models of Inertial Navigation* (MAKS Press, Moscow, 2011) [in Russian].
3. P. A. Akimov, A. V. Derevyankin, and A. I. Matasov, *Guaranteed Approach and $l_1$-Approximation in Problems of SDINS Parameter Estimation under Bench Testing* (Mosk. Gos. Univ., Moscow, 2012) [in Russian].
4. G. I. Bobrik and A. I. Matasov, “Optimal guaranteed estimation of accelerometer unit parameters,” Izv. Akad. Nauk, Mekh. Tverd. Tela, No. 5, 8–14 (1993).
5. Yu. V. Bolotin, V. P. Golikov, S. V. Larionov, and A. V. Trebukhov, “Calibration algorithm for platform inertial navigation system,” Girosk. Navig., No. 3, 13–27 (2008).
6. N. B. Vavilova, N. A. Parusnikov, and I. Yu. Sazonov, “Calibration of strapdown inertial systems using rough one-degree test benches,” in *Modern Problems of Mathematics and Mechanics. Applied Research* (Mosk. Gos. Univ., Moscow, 2009), Vol. 1, pp. 212–223 [in Russian].
7. K. K. Veremeenko and I. A. Galai, “Development of an algorithm for the calibration of an inertial navigation system on a biaxial test bench,” Tr. MAI, No. 63 (2013). http://www.trudymai.ru.
8. A. A. Golovan and A. I. Matasov, “Guaranteed approach for determining the optimal design of accelerometer unit calibration,” Fundam. Prikl. Mat. **22** (2), 133–145 (2018).
9. A. A. Golovan and A. I. Matasov, “Application of the guaranteeing approach to the accelerometer unit calibration problem,” Autom. Remote Control **81**, 686 (2020).
10. V. Z. Gusinskii, V. M. Lesyuchevskii, Yu. A. Litmanovich, and A. A. Stolbov, “Calibration algorithm for a triaxial accelerometer unit designed for use in SINS,” Giroskn Navig., No. 4 (31), 86 (2000).
11. A. V. Derevyankin and A. I. Matasov, “Formalizing a sequential calibration scheme for a strapdown inertial navigation system,” Autom. Remote Control **79**, 51 (2018).
12. E. V. Dranitsyna, “Calibration of the measuring unit according to the SINS navigation solution: The choice of the test bench motion design,” in *Proceedings of the 24th SPb. International Conference on Integrated Navigation Systems* (TsNII Elektropribor, St. Petersburg, 2017), pp. 235–240.
13. Yu. G. Egorov and E. A. Popov, “Research on accelerometers unit minimally excess calibration programs,” Aviakosm. Priborostr., No. 6, 3–8 (2016).
14. G. I. Emel’yantsev and A. P. Stepanov, *Integrated Inertial-Satellite Orientation and Navigation Systems* (TsNII Elektropribor, St. Petersburg, 2016) [in Russian].

15. V. S. Ermakov, D. A. Dunaev, A. A. Shirokov, et al., "Calibration of strapdown inertial navigation and attitude control systems," Aerokosm. Tekh., Vestn. PGTU, No. 18, 25–30 (2004).
16. E. A. Izmailov, S. N. Lepe, A. V. Molchanov, and E. F. Polikovskii, "Scalar method of calibration and balancing of strapdown inertial navigation systems," in *Proceedings of the 15th Anniversary SPb International Conference on Integrated Navigation Systems* (TsNII Elektropribor, St. Petersburg, 2008), pp. 145–154.
17. I. Yu. Sazonov and I. Kh. Shaimardanov, "Calibration of a strapdown inertial navigation system on micromechanical sensors of accelerometers and gyroscopes," Vopr. Oboron. Tekh., Ser. 9: Spets. Sist. Upravl., Sledyashch. Privody Elem., No. 3, 73–82 (2010).
18. Q. Cai, G. Yang, N. Song, and Y. Lin, "Systematic calibration for ultra-high accuracy inertial measurement unit," Sensors **16**, 940 (2016).
19. Moon-Sik Kim, Si-Bok Yu, and Kwang-Soo Lee, "Development of high-precision calibration method for inertial measurement unit," Int. J. Precision Eng. Manuf. **15**, 567–575 (2014).
20. G. Panahandeh, I. Skog, and M. Jansson, "Calibration of the accelerometer triad of an inertial measurement unit, maximum likelihood estimation and Cramer-Rao bound," in *Proceedings of the International Conference on Indoor Positioning and Indoor Navigation, Zurich, Switzerland, 2010.*
21. S. Paternain, M. Tailanian, and R. Canetti, "Calibration of an inertial measurement unit," in *Proceedings of the 16th International Conference on Advanced Robotics. Montevideo, Uruguay, 2013.*
22. G. Secer and B. Barshan, "Improvements in deterministic error modeling and calibration of inertial sensors and magnetometers," Sens. Actuators, A **247**, 522–538 (2016).
23. Z. F. Syed, P. Aggarwal, C. Goodall, X. Niu, and N. El-Sheimy, "A new multi-position calibration method for MEMS inertial navigation systems," Meas. Sci. Technol., No. 18, 1897–1907 (2007).
24. Y. Xu, Y. Wang, Y. Su, and X. Zhu, "Research on the calibration method for micro inertial measurement unit for engineering applications," J. Sensors **2016**, 9108197 (2016).
25. M. L. Lidov, "To a priori estimates of the accuracy of determining parameters using the least squares method," Kosm. Issled. **2**, 713–715 (1964).
26. M. L. Lidov, "Minimax methods for estimation," KIAM Preprint No. 71 (Keldysh Inst. Appl. Math. RAS, Moscow, 2010).
27. N. N. Krasovskii, "On the theory of controllability and observability of linear dynamical systems," Prikl. Mat. Mekh. **28** (1), 3–14 (1964).
28. N. N. Krasovskii, *The Theory of Motion Control* (Nauka, Moscow, 1968) [in Russian].
29. A. B. Kurzhanskii, *Control and Observation under Conditions of Uncertainty* (Nauka, Moscow, 1977) [in Russian].
30. F. L. Chernous'ko, *State Estimation for Dynamic Systems* (Nauka, Moscow, 1988; CRC, Boca Raton, FL, 1993).
31. A. I. Matasov, *Guaranteed Estimation Method* (Mosk. Gos. Univ., Moscow, 2009) [in Russian].
32. A. I. Matasov, *Estimators for Uncertain Dynamic Systems* (Springer Science, Dordrecht, 2013).
33. S. A. Ashmanov, *Linear Programming* (Nauka, Moscow, 1981) [in Russian].
34. M. L. Lidov, "On the modification of the simplex method of linear programming in the case of degeneration," Kosm. Issled. **29**, 499–508 (1991).
35. B. Ts. Bakhshiyan, A. I. Matasov, and K. S. Fedyaev, "On the solution of degenerate problems of linear programming," Autom. Remote Control **61**, 98 (2000).
36. S. Boyd and L. Vandenberghe, *Convex Optimization* (Cambridge Univ. Press, Cambridge, 2004).