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ESTIMATING THE INSTABILITY RADII OF POLYNOMIALS OF ARBITRARY DEGREE

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We consider the determination of the instability radius of polynomials. Sufficient conditions are stated for robust instability of a family of polynomials. A lower bound on the instability radius is given in the general case and the exact value of the instability radius is obtained for polynomials of fifth degree. The proof relies on the geometric properties of continuous curves in a plane combined with parametric properties of the roots of a family of polynomials and the apparatus of the Tsypkin – Polyak hodograph. Applications of the results are illustrated.

1. Introduction

One of the important problems of linear control is the problem of feedback stabilization, in particular stabilization by a regulator of a given structure. In this setting, assuming that the characteristic polynomial of the closed-loop system depends on parameters, we need to elucidate how these parameters should be chosen so that the polynomial is stable. The parameter region can be specified in advance. In general, this is a fairly complex problem and so far it has not been solved.

In the present study we consider a particular case of the problem, when the parameters are the coefficients of the characteristic polynomial of a closed-loop system that may vary inside some region in the $(n+1)$-dimensional real space, where n is the degree of the polynomial. The main result provides a bound on the distance from the given unstable polynomial to the nearest stable polynomial in some metric in the parameter space.

2. Robust Instability of Polynomials

Consider the set P_n of polynomials of the form

$$p(s) = a_0 + a_1 s + \ldots + a_n s^n, \qquad a_i \in \mathbb{R}, \quad a_0 \neq 0,$$

which are identified with the vector $a = (a_0, \ldots, a_n)$ in the space $\mathbb{R}^{n+1}$ with the norm $\|a\| = \max\limits_i |a_i|$. To link the polynomial $p(s)$ with a point in the space $\mathbb{R}^{n+1}$ we denote it by $p(s, a)$.

The polynomial $p(s, a) \in P_n$ is called *stable* if all its root in the complex plane lie strictly in the left halfplane; otherwise the polynomial is called *unstable*.

Given is a family of polynomials

$$\mathcal{P}_n(A) = \big\{ p(s, a) = a_0 + a_1 s + \ldots + a_n s^n, \;\; a \in A \big\}, \tag{1}$$

whose coefficients vary in some admissible set $A \subset \mathbb{R}^{n+1}$. This family is called *robustly unstable* if $p(s, a)$ are unstable for all $a \in A$.

Consider the value set of the family of polynomials (1)

$$I(A) = \big\{ p(j\omega, a) \colon \; a \in A, \; 0 \leqslant \omega < \infty \big\}.$$

The following theorem establishes sufficient conditions of robust instability of family (1).

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Theorem 1. *Assume that the set A is connected and for some $a^0 \in A$ the polynomial $p(s, a^0)$ is unstable and does not have purely imaginary roots. Moreover, let $a_n \neq 0$ for all $a \in A$. Then the condition*

$$0 \notin I(A) \tag{2}$$

is sufficient for robust instability of family (1).

Proof. Assume the contrary: condition (2) is satisfied, but there exists a stable polynomial $p(s, a^1),\ a^1 \in A$. Since A is connected, we can construct a continuous curve $a(t) \in A,\ 0 \leqslant t \leqslant 1,\ a(0) = a^0,\ a(1) = a^1$. Let $s_i(t)$ be the roots of the polynomial $p\big(s, a(t)\big)$. Define the function $\varphi(t) = \max \operatorname{Re} s_i(t)$. Since $a_n \neq 0$ for all $a \in A$ and applying the theorem on continuous dependence of the roots of a polynomial of degree n on its coefficients, we see that $\varphi(t)$ is a continuous function. Here $\varphi(1) < 0$ (because all the roots of the polynomial $p(s, a^1)$ are in the left halfplane), and $\varphi(0) > 0$ (by instability of the polynomial $p(s, a^0)$ and nonexistence of purely imaginary roots). Thus, there exist $0 \leqslant t^* \leqslant 1$ and $a^* = a(t^*) \in A$ such that $\varphi(t^*) = 0$. But this implies that $p(s, a^*)$ has a purely imaginary root $j\omega$. We have thus obtained that $p(j\omega^*, a^*) = 0$ for some $a^* \in A$ and $\omega^* \geqslant 0$, which contradicts condition (2). Q.E.D.

3. Robust Instability of Interval Family of Polynomials

Given is an interval family of polynomials

$$\begin{gathered} \mathcal{P}_n\big(A(a^0)\big) = \big\{p(s, a) = a_0 + a_1 s + \ldots + a_n s^n, \quad |a_i - a_i^0| \leqslant \gamma, \quad i = 0, 1, \ldots, n\big\}, \\ a^0 = (a_0^0,\ a_1^0, \ldots, a_n^0). \end{gathered} \tag{3}$$

Assumption. $a^0 \in \mathbb{R}_+^{n+1}$ *(i.e., $a_i^0 > 0$ for all i).*

Consider the Tsypkin – Polyak hodograph [1] for the polynomial $p(s, a^0)$:

$$z(\omega, a^0) = x_0(\omega) + j y_0(\omega), \quad 0 \leqslant \omega \leqslant \infty,$$

where

$$x_0(\omega) = \frac{g_0(\omega)}{r(\omega)}, \qquad y_0(\omega) = \frac{h_0(\omega)}{q(\omega)}, \tag{4}$$

$$g_0(\omega) = a_0^0 - a_2^0\omega^2 + a_4^0\omega^4 - \ldots, \qquad h_0(\omega) = a_1^0 - a_3^0\omega^2 + a_5^0\omega^4 - \ldots,$$

$$r(\omega) = 1 + \omega^2 + \omega^4 + \ldots, \qquad q(\omega) = 1 + \omega^2 + \omega^4 + \ldots,$$

$$\deg r(\omega) = \deg g_0(\omega), \qquad \deg q(\omega) = \deg h_0(\omega).$$

In this notation,

$$p(j\omega, a^0) = g_0(\omega) + j\omega h_0(\omega).$$

Theorem 2. *For robust instability of the family of polynomials (3) it is sufficient to have the following conditions:*

1. *The polynomial $p(s, a^0)$ is without purely imaginary roots;*

2. $a_n^0 > \gamma$;

3. *The hodograph* $z(\omega, a^0)$ *does not cross the square with the vertices* $(\pm\gamma, \pm\gamma)$ *as* ω *varies from* 0 *to* ∞.

Proof. Note that if the value of every polynomial from the family (3) at the point $j\omega$ is representable in the form

$$p(j\omega, a) = g(\omega) + j\omega h(\omega),$$

$$g(\omega) = a_0 - a_2\omega^2 + a_4\omega^4 - \ldots, \qquad h(\omega) = a_1 - a_3\omega^2 + a_5\omega^4 - \ldots,$$

then for $\omega \geqslant 0$ we have the inequalities

$$\underline{g}(\omega) \leqslant g(\omega) \leqslant \bar{g}(\omega), \qquad \underline{h}(\omega) \leqslant h(\omega) \leqslant \bar{h}(\omega),$$

where

$$\underline{g}(\omega) = (a_0^0 - \gamma) - (a_2^0 + \gamma)\omega^2 + (a_4^0 - \gamma)\omega^4 - \ldots,$$

$$\bar{g}(\omega) = (a_0^0 + \gamma) - (a_2^0 - \gamma)\omega^2 + (a_4^0 + \gamma)\omega^4 - \ldots,$$

$$\underline{h}(\omega) = (a_1^0 - \gamma) - (a_3^0 + \gamma)\omega^2 + (a_5^0 - \gamma)\omega^4 - \ldots,$$

$$\bar{h}(\omega) = (a_1^0 + \gamma) - (a_3^0 - \gamma)\omega^2 + (a_5^0 + \gamma)\omega^4 - \ldots.$$

Thus, $p(j\omega, a)$ for every fixed ω may vary in the rectangle with the vertices

$$z_1 = \underline{g}(\omega) + j\omega\underline{h}(\omega), \qquad z_2 = \underline{g}(\omega) + j\omega\bar{h}(\omega),$$

$$z_3 = \bar{g}(\omega) + j\omega\bar{h}(\omega), \qquad z_4 = \bar{g}(\omega) + j\omega\underline{h}(\omega).$$

Therefore, the value set of the family (3) for each ω is the rectangle

$$I\big(\omega; A(a^0)\big) = \big\{z = x + jy\colon |x - g_0(\omega)| \leqslant \gamma r(\omega),\ |y - \omega h_0(\omega)| \leqslant \gamma\omega q(\omega)\big\}.$$

The condition that the hodograph $z(\omega, a^0)$ does not cross the square with the vertices $(\pm\gamma, \pm\gamma)$ is equivalent to the condition

$$\left[\begin{array}{l} \begin{cases} |x_0(\omega)| > \gamma, \\ |y_0(\omega)| > \gamma, \end{cases} \\ \begin{cases} |x_0(\omega)| > \gamma, \\ |y_0(\omega)| \leqslant \gamma, \end{cases} \\ \begin{cases} |x_0(\omega)| \leqslant \gamma, \\ |y_0(\omega)| > \gamma, \end{cases} \end{array}\right. \qquad \text{for all} \quad \omega \geqslant 0. \tag{$*$}$$

At the same time, the condition $0 \in I\big(A(a^0)\big)$ is equivalent to the existence of $\omega^* > 0$ such that $0 \in I\big(\omega^*; A(a^0)\big)$. But then for ω^* we have

$$\begin{cases} |g_0(\omega^*)| \leqslant \gamma r(\omega^*), \\ |h_0(\omega^*)| \leqslant \gamma q(\omega^*), \end{cases}$$

or

$$\begin{cases} |x_0(\omega^*)| \leqslant \gamma, \\ |y_0(\omega^*)| \leqslant \gamma. \end{cases}$$

Hence, condition $(*)$ is equivalent to $0 \notin I\big(A(a^0)\big)$. But then by Theorem 1 the family (3) is robustly unstable. Q.E.D.

4. Lower Bound for Instability Radii

Denote by S and U the sets of points in the space $\mathbb{R}^{n+1}$ corresponding to stable and unstable polynomials, respectively. For an unstable polynomial $p(s, a)$ the *instability radius* is defined as

$$R_U(a) = \inf_{b \in S} \|a - b\|.$$

Theorem 2 not only establishes sufficient conditions for robust instability of family (3), but also leads to a bound for the instability radius $R_U(a^0)$ of the polynomial $p(s, a^0)$ with positive coefficients.

For the polynomial $p(s, a^0)$ consider the function

$$\psi(\omega) = \max\big\{|x_0(\omega)|,\, |y_0(\omega)|\big\}, \tag{5}$$

defined for $\omega \geqslant 0$ (the functions $x_0(\omega)$ and $y_0(\omega)$ are defined by (4)). Denote by $\omega_1, \ldots, \omega_k$ all the positive roots of the following equations:

$$g_0(\omega)q(\omega) - h_0(\omega)r(\omega) = 0, \qquad g_0(\omega)q(\omega) + h_0(\omega)r(\omega) = 0, \tag{6a}$$

$$g_0'(\omega)r(\omega) - g_0(\omega)r'(\omega) = 0, \qquad h_0'(\omega)q(\omega) - h_0(\omega)q'(\omega) = 0. \tag{6b}$$

Let

$$\Phi(a^0) = \min_{1 \leqslant i \leqslant k} \psi(\omega_i).$$

Theorem 3. *Let $p(s, a^0) = a_0^0 + a_1^0 s + \ldots + a_n^0 s^n$ be an unstable polynomial with positive coefficients and without purely imaginary roots. Then we have the bound*

$$R_U(a^0) \geqslant \gamma(a^0), \tag{7}$$

where

$$\gamma(a^0) = \min\big\{a_0^0,\, a_1^0,\, a_{n-1}^0,\, a_n^0,\, \Phi(a^0)\big\}.$$

Proof. Theorem 2 gives the following lower bound for the instability radius of the polynomial $p(s, a^0)$:

$$R_U(a^0) \geqslant \min\{\gamma^*, a_n^0\},$$

where γ^* is the size (side half-length) of the largest square $\{|x| \leqslant \gamma^*, |y| \leqslant \gamma^*\}$ inscribed in the Tsypkin – Polyak hodograph $z(\omega, a^0)$ for the polynomial $p(s, a^0)$.

Define the following metric in the complex plane:

$$\rho(z_1,\, z_2) = \max\big\{|\mathrm{Re}\, z_1 - \mathrm{Re}\, z_2|,\, |\mathrm{Im}\, z_1 - \mathrm{Im}\, z_2|\big\}.$$

Then the number γ^* can be determined as the distance from the origin to the nearest point of the hodograph $z(\omega)$, i.e.,

$$\gamma^* = \min_{0 \leqslant \omega \leqslant \infty} \rho\big(0,\, z(\omega)\big).$$

Since $z(\omega) = x(\omega) + iy(\omega)$, we determine γ^* by finding the minimum of the function $\psi(\omega) = \max\{|x_0(\omega)|, |y_0(\omega)|\}$ on the interval $[0,\, \infty)$. Note that $\psi(\omega)$ is bounded, because it is continuous and

$$\lim_{\omega \to \infty} \psi(\omega) = \max\big\{a_n^0,\, a_{n-1}^0\big\}.$$

The function $\psi(\omega)$ is differentiable everywhere, possibly with the exception of points where $|x_0(\omega)| = |y_0(\omega)|$. Therefore

$$\min_{0 \leqslant \omega \leqslant \infty} \psi(\omega) = \min\big\{\psi(0),\, \psi(\infty),\, \psi(\omega_1), \ldots, \psi(\omega_k)\big\},$$

where $\psi(\infty) = \lim\limits_{\omega \to \infty} \psi(\omega)$, $\omega_1,\, \ldots,\, \omega_k$ are the roots of Eqs. (6). Note that one of the following two conditions holds at the points ω_i:

1. $x_0(\omega) = y_0(\omega)$ or $x_0(\omega) = -y_0(\omega)$ (i.e., $|x_0(\omega)| = |y_0(\omega)|$);
2. $x_0'(\omega) = 0$ or $y_0'(\omega) = 0$.

Case 1 corresponds to Eqs. (6a), case 2 corresponds to Eqs. (6b). Noting that $\psi(0) = \max\{a_0^0,\, a_1^0\}$, we obtain (7). Q.E.D.

5. Instability Radii for Polynomials of Degree Not Higher Than 5

Using the example of polynomials of degree not higher than 5 we will show that bounds for instability radii can be found for some sets of polynomials. Note that for polynomials of first and second degree a necessary and sufficient condition of stability is positivity of all their coefficients. We should therefore consider polynomials of degree higher than 2.

We consider unstable polynomials with positive coefficients

$$p(s, a) = a_0 + a_1 s + \ldots + a_n s^n, \quad 3 \leqslant n \leqslant 5. \tag{8}$$

For these polynomials, using the function (5), we introduce the notation

$$\Psi(a) = \min\big\{\psi(\omega_1),\, \ldots,\, \psi(\omega_l)\big\},$$

where $\omega_1, \ldots, \omega_l$ are the roots of Eqs. (6a),

$$\Lambda(a) = \min\left\{\psi(\omega_{l+1}), \ldots, \psi(\omega_k), a_0, a_1, a_{n-1}, a_n\right\},$$

where $\omega_{l+1}, \ldots, \omega_k$ are the roots of Eqs. (6b).

Theorem 4. *Assume that the following conditions hold for the unstable polynomial (8) with positive coefficients:*

1. $a_1 > a_0$;
2. $a_{n-1} > a_n$;
3. $\gamma(a) = \Psi(a) < \Lambda(a)$;
4. $\operatorname{Re} p(i\omega, a) \neq 0,\ \operatorname{Im} p(i\omega, a) \neq 0,\ \omega > 0.$

Then

$$R_U(a) = \gamma(a).$$

Proof. For polynomials of third and fourth degree, the assertion of the theorem follows from [2]. We will prove the theorem for polynomials of degree 5.

Consider the Tsypkin – Polyak hodograph $z(\omega, a) = x(\omega) + jy(\omega)$ for the unstable polynomial

$$p(s, a) = a_0 + a_1 s + a_2 s^2 + a_3 s^3 + a_4 s^4 + a_5 s^5, \quad a_i > 0.$$

Note that by condition 4 the hodograph lies in the first quadrant of the complex plane (i.e., $\operatorname{Re} z(\omega, a) = x(\omega) > 0$, $\operatorname{Im} z(\omega, a) = y(\omega) > 0$). Now, by conditions 1, 2, the hodograph crosses the bisector of the first quadrant, because

$$z(0, a) = a_0 + ja_1, \qquad z(\infty, a) = a_4 + ja_5, \quad a_1 > a_0, \quad a_4 > a_5. \tag{9}$$

This means that the hodograph starts above the first-quadrant bisector and ends below the bisector. The hodograph crosses the bisector at a unique value ω^* $(x(\omega^*) = y(\omega^*))$. Indeed, the intersection with the bisector may occur only for positive ω which are the roots of the equation

$$(a_1 - a_0) - (a_3 - a_2)\omega^2 + (a_5 - a_4)\omega^4 = 0.$$

But given conditions 1, 2 this equation has only one positive root ω^*:

$$\omega^* = \sqrt{\frac{(a_3 - a_2) - \sqrt{(a_3 - a_2)^2 - 4(a_1 - a_0)(a_5 - a_4)}}{2(a_5 - a_4)}}.$$

Thus,

$$\gamma(a) = \operatorname{Re} z(\omega^*, a) = \operatorname{Im} z(\omega^*, a).$$

Now consider the polynomial

$$p(s,\tilde{a}) = \big(a_0 - \gamma(a)\big) + \big(a_1 - \gamma(a)\big)s + \big(a_2 + \gamma(a)\big)s^2$$

$$+ \big(a_3 + \gamma(a)\big)s^3 + \big(a_4 - \gamma(a)\big)s^4 + \big(a_5 - \gamma(a)\big)s^5.$$

By condition 3 all its coefficients are positive. Note that the hodograph $z(\omega,\tilde{a})$ may be obtained from the hodograph $z(\omega,a)$ by parallel translation along the first-quadrant bisector. We will show that $\tilde{a} \in \partial S$ (i.e., the polynomial $p(s,\tilde{a})$ is contained in the boundary of the region of stable polynomials).

Indeed, the Tsypkin – Polyak hodograph $z(\omega,\tilde{a})$ passes through the point $z = 0$ of the complex plane by condition 3 and the equalities

$$\operatorname{Re} z(\omega,\tilde{a}) = \operatorname{Re} z(\omega,a) - \gamma(a),$$

$$\operatorname{Im} z(\omega,\tilde{a}) = \operatorname{Im} z(\omega,a) - \gamma(a),$$

which imply that

$$\operatorname{Re} z(\omega^*,\tilde{a}) = 0, \qquad \operatorname{Im} z(\omega^*,\tilde{a}) = 0.$$

Let $\tilde{\gamma} = \gamma(a) + \varepsilon$ ($\varepsilon > 0$ is a sufficiently small number) and consider the polynomial

$$p(s,a^*) = (a_0 - \tilde{\gamma}) + (a_1 - \tilde{\gamma})s + (a_2 + \tilde{\gamma})s^2$$

$$+ (a_3 + \tilde{\gamma})s^3 + (a_4 - \tilde{\gamma})s^4 + (a_5 - \tilde{\gamma})s^5.$$

For the hodograph $z(\omega,a^*)$ of this polynomial we have the equalities

$$\operatorname{Re} z(\omega,a^*) = \operatorname{Re} z(\omega,a) - \tilde{\gamma},$$

$$\operatorname{Im} z(\omega,a^*) = \operatorname{Im} z(\omega,a) - \tilde{\gamma}.$$

By condition 3 ε may be chosen sufficiently small so that

$$\operatorname{Re} z(0,a^*) = a_0 - \gamma(a) - \varepsilon > 0,$$

$$\operatorname{Im} z(0,a^*) = a_1 - \gamma(a) - \varepsilon > 0,$$

$$\operatorname{Re} z(\infty,a^*) = a_4 - \gamma(a) - \varepsilon > 0,$$

$$\operatorname{Im} z(\infty,a^*) = a_5 - \gamma(a) - \varepsilon > 0.$$

But

$$\operatorname{Re} z(\omega^*,a^*) = -\varepsilon < 0, \qquad \operatorname{Im} z(\omega^*,a^*) = -\varepsilon < 0,$$

because the hodograph $z(\omega^*,a^*)$ may be obtained from the hodograph $z(\omega,\tilde{a})$ by parallel translation along the first-quadrant bisector. Therefore, using condition (9) and noting the uniqueness of the intersection point of the hodograph $z(\omega, a)$ with the first-quadrant bisector, we obtain that the Tsypkin – Polyak hodograph of the polynomial $p(s, a^*)$ crosses five quadrants as ω varies from 0 to ∞. Moreover, all the coefficients of the polynomial $p(s, a^*)$ are positive. Therefore, $\tilde{a} \in \partial S$ and $R_U(a) = \gamma(a)$. Q.E.D.

Example 1. Given is an unstable polynomial of degree 7,

$$p(s) = s^7 + 2s^6 + 6s^5 + s^4 + 6s^3 + s^2 + 3s + 1.$$

Denoting $a = (1, 2, 6, 1, 6, 1, 3, 1),$ we obtain by Theorem 3 the following bound for the instability radius:

$$R_U(a) \geqslant 0.5.$$

Example 2. Given is an unstable polynomial of degree 5:

$$p(s) = s^5 + 3s^4 + s^3 + s^2 + 2s + 1.$$

Denoting $b = (1, 3, 1, 1, 2, 1),$ we obtain by Theorem 4

$$\Psi(b) = 0.8123 < 1 = \Lambda(b)$$

and thus

$$R_U(a) = 0.8123.$$

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