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On linear extension operators

We construct a countably dimensional Hausdorff locally convex topological vector space E and a stratifiable closed linear subspace $F \subset E$ such that any linear extension operator from $C_b(F)$ to $C_b(E)$ is unbounded ($C_b(X)$ is the Banach space of bounded real-valued functions on X).

I Introduction

By a space we always mean a Tychonoff topological space. All linear spaces are assumed to be real. All topological vector spaces are assumed to be Hausdorff.

Let X be a space, $Y \subset X$, E be a locally convex topological vector space (LCS). We say that *the Dugundji theorem is valid for the triple* (X, Y, E) if there exists a linear operator $\Phi : C(Y, E) \to C(X, E)$ such that $\Phi(f)\big|_Y = f$ and $\Phi(f)(X)$ is contained in the convex hull of $f(F)$ for any $f \in C(Y, E)$ (here and below $C(X, E)$ is the space of all continuous maps from X to E). We say that *Dugundji theorem is valid for the space* X if for any closed set $Y \subset X$ and any LCS E, Dugundji theorem is valid for the triple (X, Y, E).

Dugundji theorem is valid for any metrizable space [1], [2]. Borges [3] introduced the class of stratifiable spaces, which contains metrizable spaces and proved that Dugundji theorem is valid for any stratifiable space. Different equivalent definitions of stratifiability can be also found in [4, 5, 11]. In [11] it is shown that the majority of non-metrizable LCS's, naturally appearing in analysis (such as the space $\mathcal{D}(\mathbb{R}^n)$ of infinitely differentiable functions with compact support, the space $\mathcal{D}'(\mathbb{R}^n)$ of generalized functions on $\mathbb{R}^n$ the space $\mathcal{S}(\mathbb{R}^n)$) of Schwarz distributions etc.), are stratifiable.

Let $C_b(X)$ be the Banach space of bounded real-valued continuous functions on a space X with the norm $\|f\| = \sup |f|$. If $Y \subseteq X$ then a map $T : C_b(Y) \to C_b(X)$ is called an *extension operator* if $Tf\big|_Y = f$ for all $f \in C_b(Y)$.

Definition 1. Let X be a space, $Y \subset X$, $c \in [1, +\infty)$. We say that Y is $\mathcal{D}_c$-*embedded* into X and write $Y \underset{\mathcal{D}_c}{\subset} X$ if there exists a linear extension operator $\Phi : C_b(F) \to C_b(X)$ such that $\|\Phi\| \leqslant c$. Let also

$$\lambda(Y, X) = \inf\{c \geqslant 1 : Y \underset{\mathcal{D}_c}{\subset} X\}, \tag{1}$$

$$\lambda(X) = \sup\{\lambda(Y, X) : Y \text{ is a closed subset of } X\}. \tag{2}$$

Evidently $Y \underset{\mathcal{D}_1}{\subset} X$ and $\lambda(Y, X) = 1$ if Dugundji theorem is valid for the triple $(X, Y, \mathbb{R})$. In particular $\lambda(X) = 1$ for any stratifiable space X.

Remind that the *network* [6] in a space X is a family $\mathcal{F}$ of subsets of X such that any open set $U \subseteq X$ is a union of a subfamily of $\mathcal{F}$. Any space with countable network is perfectly normal and Lindelöf [6]. *Dimension* of a linear space is the cardinality of a Hamel basis [7] of this space. Evidently, any countably dimensional topological vector space has a countable network.

Tietze–Urysohn theorem [6] implies that any bounded continuous function on a closed subspace Y of a normal space X admits a bounded continuous extension to whole X. Moreover, as it was observed by Heath and Lutzer [8], there exists a linear extension operator $T : C_b(Y) \to C_b(X)$ (apply Tietze–Urysohn theorem to elements of the Hamel basis of $C_b(Y)$).

Van Dowen [9] constructed a space X with countable network and a closed subset Y of X such that $\lambda(Y, X) = \infty$, i.e. any linear extension operator $T : C_b(Y) \to C_b(X)$ is unbounded.

We prove a stornger statement:

Theorem 1. *There exists a countable space X and a closed stratifiable subspace Y of X such that $\lambda(Y, X) = \infty$.*

Using this example we prove

Theorem 2. *There exists a countably dimensional LCS E and a stratifiable closed linear subspace $F \subset E$ such that $\lambda(F, E) = \infty$.*

Remark 1. For E and F of Theorem 2 we have that Dugundji theorem is invalid for the triple $(E, F, \mathbb{R})$. This gives a negative solution of one problem, posed by A. V. Arkhangel'skii, who asked whether Dugundji theorem is valid for any triple $(E, F, \mathbb{R})$, where E is a LCS with countable network and F is its closed linear subspace.

Remind that a subspace Y of a space X is said to be P-embedded if any continuous pseudometric on Y admits an extension to a continuous pseudometric on X. The following fact is proved in [10].

Lemma 1. *Let a space X be collectionwise normal* [6] *and Y be its closed subspace. Then Y is P-embedded into X.*

Remark 2. Let E be a LCS, X be a collectionwise normal space and Y be its closed metrizable G_δ subspace. Then Dugundji theorem is valid for the triple (X, Y, E) and hence $Y \underset{\mathcal{D}_1}{\subset} X$. Indeed, Lemma 1 implies the existence of a continuous metric ρ on X such that its restriction to Y defines the initial topology of Y. Since Y is a G_δ set, ρ can be chosen in such a way that Y is closed with respect to ρ. Then the usual Dugundji theorem and the fact that the initial topology is stronger then the ρ-topology imply the validity of Dugundji theorem for the triple (X, Y, E).

Remark 3. Theorems 1 and 2 show that metrizability condition is essential in the last statement (any space with countable network is collectionwise normal and perfect [6]).

II (k,n)-embeddings

Let (X, τ_X) be a space, $Y \subseteq X$ and $y \in Y$. By $i(y, Y, X)$ we denote the least $n \in \mathbb{N}$ for which there exists $\varkappa : \tau_Y \to \tau_X$ such that

$$\varkappa(U) \cap Y = U \quad \text{for all} \quad U \in \tau_Y; \tag{3}$$

$$\begin{gathered}\bigcap_{j=0}^{n} \varkappa(U_j) = \varnothing \quad \text{for any} \quad U_0, \ldots, U_n \in \tau_Y,\ y \in U_0 \\ \text{such that} \quad U_i \cap U_j = \varnothing \quad \text{for} \quad 0 \leqslant i, j \leqslant n,\ i \neq j.\end{gathered} \tag{4}$$

If such an n does not exist, we put $i(y, Y, X) = \infty$.

Definition 2. Let (X, τ_X) be a space, $Y \subseteq X$, $k, n \in \mathbb{N}$, $k \leqslant n$. We say that Y is (k,n)-*embedded* into X ($Y \underset{k,n}{\subset} X$), if there exists $\varkappa : \tau_Y \to \tau_X$ satisfying (3) and the following condition

$$\begin{gathered}\bigcap_{j=0}^{n} \varkappa(U_j) = \varnothing \quad \text{for any} \quad U_0, \ldots, U_n \in \tau_Y \\ \text{such that} \bigcap_{j=0}^{k} U_{i_j} = \varnothing \text{ for } 0 \leqslant i_0 < i_1 < \cdots < i_k \leqslant n.\end{gathered} \tag{5}$$

Proposition 1. 1) *The number $i(y, Y, X)$ is a local invariant, i.e. if $y_1 \in Y_1 \subset X_1$, $y_2 \in Y_2 \subset X_2$, and there exist neighborhoods U_1 and U_2 of y_1 and y_2 in X_1 and X_2 respectively and a homeomorphism $h : U_1 \to U_2$ such that $h(Y_1 \cap U_1) = Y_2 \cap U_2$ and $h(y_1) = y_2$ then $i(y_1, Y_1, X_1) = i(y_2, Y_2, X_2)$.*

2) *If Y is dense in X then $Y \underset{k,n}{\subset} X$ for all $k, n \in \mathbb{N}$, $k \leqslant n$.*

3) *If Y is a retract of X then $Y \underset{\mathcal{D}_1}{\subset} X$.*

4) *If $k, n \in \mathbb{N}$, $k \leqslant n$, $c \in \left[1, \frac{2n+2}{k} - 1\right)$ and $Y \underset{\mathcal{D}_c}{\subset} X$, then $Y \underset{k,n}{\subset} X$.*

5) *If* $n \geqslant m \geqslant k$, *then* $Y \underset{k,m}{\subset} X \Longrightarrow Y \underset{k,n}{\subset} X$ *and* $Y \underset{m,n}{\subset} X \Longrightarrow Y \underset{k,n}{\subset} X$.

6) $Z \underset{k,m}{\subset} Y \underset{m,n}{\subset} X \Longrightarrow Z \underset{k,n}{\subset} X$.

7) *If* $Y \underset{1,n}{\subset} X$, *then* $i(y, Y, X) \leqslant n$ *for all* $y \in Y$.

8) $Y \underset{\mathcal{D}_a}{\subset} Z \underset{\mathcal{D}_b}{\subset} X \Longrightarrow Y \underset{\mathcal{D}_{ab}}{\subset} X$.

9) *If* $Y \underset{\mathcal{D}_a}{\subset} X$ *and* $Y \subset Z \subset X$ *then* $Y \underset{\mathcal{D}_a}{\subset} Z$.

10) *If* $Y \underset{k,n}{\subset} X$ *and* $Y \subset Z \subset X$ *then* $Y \underset{k,n}{\subset} Z$.

Proof. Statements 1), 5), 7), 9) and 10) are obvious.

2) If Y is dense in X then any map $\varkappa : \tau_Y \to \tau_X$, satisfying (3), satisfies also (5) for any $k, n \in \mathbb{N}$, $k \leqslant n$.

3) Let $r : X \to Y$ be a retraction. Then the operator $\Phi : C_b(Y) \to C_b(X)$, $\Phi(f)(x) = f(r(x))$ is a linear extension operator with norm 1.

4) Since $c \in \left[1, \frac{2n+2}{k} - 1\right)$, we can pick $q, \delta > 0$ such that

$$\frac{k(c+1)}{2(n+1)} < q < 1, \quad \delta > 0 \quad \text{and} \quad 2q(n+1) > k(c+1+\delta). \tag{6}$$

Let $\Phi : C_b(Y) \to C_b(X)$ be a linear extension operator and $\|\Phi\| \leqslant c$. Let also $e \in C_b(Y)$, $e(y) = 1$ for all $y \in Y$ and

$$W = \{x \in X : |\Phi e(x) - 1| < \delta\}.$$

Clearly W is open in X and $Y \subseteq W$. For any open set $U \subseteq Y$ let

$$\begin{aligned} \mathcal{F}_U &= \{f \in C_b(Y) : f(Y) \subseteq [0,1],\ f(Y \setminus U) \subseteq \{0\}\}, \\ \forall\, x \in X \quad g_U(x) &= \sup\{\Phi f(x) : f \in \mathcal{F}_U\}, \\ \varkappa(U) &= \{x \in X : g_U(x) > q\} \cap W. \end{aligned}$$

First, we shall prove that $\varkappa(U)$ is open in X and $\varkappa(U) \cap Y = U$ for any $U \in \tau_Y$. Let $x \in \varkappa(U)$. Then $g_U(x) > q$. Hence there exists $f \in \mathcal{F}_U$ such that $\Phi f(x) > q$. Since Φf is continuous the set $V = \{y \in X : \Phi f(y) > q\}$ is open. By definition of $\varkappa(U)$ we have $x \in V \cap W \subseteq \varkappa(U)$. Thus, $\varkappa(U)$ is open. Let $x \in U$. Since Y is a Tychonoff space there exists $f \in \mathcal{F}_U$ such that $f(x) = 1$. Hence, $g_U(x) \geqslant 1 > q$. Therefore $x \in \varkappa(U)$. Let $x \in Y \setminus U$. Then $\Phi f(x) = f(x) = 0$ for any $f \in \mathcal{F}_U$. Hence $g_U(x) = 0 < q$, i.e., $x \notin \varkappa(U)$. The equality $\varkappa(U) \cap Y = U$ is proved.

Let now $U_0, \ldots, U_n \in \tau_Y$ and $\bigcap\limits_{j=0}^{n} U_j = \varnothing$. It remains to verify that $\bigcap\limits_{j=0}^{n} \varkappa(U_j) = \varnothing$. Suppose that there exists $x \in \bigcap\limits_{j=0}^{n} \varkappa(U_j)$. Then for any j, $0 \leqslant j \leqslant n$ there exists $f_j \in \mathcal{F}_{U_j}$ such that $\Phi f_j(x) > q$. Let $f = \sum\limits_{j=0}^{n} f_j$. Then

$$\Phi f(x) = \sum_{j=0}^{n} \Phi f_j(x) > q(n+1). \tag{7}$$

Formula (5) implies that $0 \leqslant f(y) \leqslant k$ for all $y \in Y$ (all $f_j(y)$ belong to [0,1] and at most k of these numbers differ from 0). Hence, $\|2f - ke\| \leqslant k$. Using (7) and the inclusion $x \in W$ we have

$$2q(n+1) < \Phi f(x) = \Phi(2f - ke)(x) + k\Phi e(x) \leqslant \|\Phi\|\|2f - ke\| + k(1+\delta) \leqslant ck + k + k\delta.$$

This inequality contradicts (6).

6) Let $\varkappa_1 : \tau_Z \to \tau_Y$ be a map satisfying conditions (3) and (5) with m instead of n, $\varkappa_2 : \tau_Y \to \tau_X$ be a map satisfying the same conditions with m instead of k and $\varkappa : \tau_Z \to \tau_X$, $\varkappa(U) = \varkappa_2(\varkappa_1(U))$. It is straightforward to verify that $\varkappa$ satisfies (3) and (5).

8) Let $\Phi_1 : C_b(Y) \to C_b(Z)$, $\Phi_2 : C_b(Z) \to C_b(X)$ be extension operators such that $\|\Phi_1\| \leqslant a$, $\|\Phi_2\| \leqslant b$. Then $\Phi = \Phi_2 \circ \Phi_1 : C_b(Y) \to C_b(X)$ is an extension operator and $\|\Phi\| \leqslant ab$. ■

Recall that a space (Y, τ) is called *monotonically regular in* $y_0 \in Y$ (see e.g. [11]), if there exists a mapping Ψ, which maps any (open) neighborhood U of y_0 to another neighborhood $\Psi(U)$ of y_0 such that

$$\forall U \quad \overline{\Psi(U)} \subseteq U;$$
$$U \subseteq V \Longrightarrow \Psi(U) \subseteq \Psi(V).$$

The space Y is called *monotonically regular* if it is monotonically regular in all points.

Lemma 2 (see [11]). *A space (Y, τ) is monotonically regular at $y_0 \in Y$ if and only if there exists a map $\Lambda : Y \setminus \{y_0\} \to \tau$ such that*

$$\forall\, y \in Y \setminus \{y_0\} \quad y \in \Lambda(y), \tag{8}$$
$$y_0 \in U \in \tau \implies y_0 \notin \overline{\bigcup_{y \in Y \setminus U} \Lambda(y)}. \tag{9}$$

Let $Z = \prod_{\alpha \in A} Z_\alpha$ be a Tychonoff product of spaces Z_α. A set $W \subset Z$ is said to be *factorizable through* $B \subseteq A$ if $W = \pi_B^{-1}(\pi_B(W))$, where $\pi_B : Z \to \prod_{\alpha \in B} Z_\alpha$ is the natural projection.

Proposition 2. *Let $Z = \prod_{\alpha \in A} Z_\alpha$, where Z_α are separable and metrizable, $n \in \mathbb{N}$, $X \underset{n,n}{\subset} Z$, Y be a countable subspace of X monotonically regular in $y_0 \in Y$, $B = \{y \in Y : i(y, Y, X) \geqslant n\}$. Let also the space $B \cup \{y_0\}$ be non-first-countable in y_0. Then $i(y_0, Y, X) \geqslant n + 1$.*

Proof. Suppose that $i(y_0, Y, X) \leqslant n$. Let $\varkappa_1 : \tau_X \to \tau_Z$ be a map satisfying (3) and (5) with $k = n$ and $\varkappa_2 : \tau_Y \to \tau_X$ be a map satisfying (3) and (4). According to Lemma 2 there exists a map $\Lambda : Y \setminus \{y_0\} \to \tau_Y$ satisfying (8) and (9). Since $i(y, Y, X) \geqslant n$ for any $y \in B$, we have that for any $y \in B \setminus \{y_0\}$, the set

$$\mathcal{U}_y = \left\{ (U_1^y, \ldots, U_n^y) \in \tau_Y^n : y \in U_1^y,\ U_i^y \cap U_j^y = \varnothing \ \text{ for } \ i \neq j, \quad U_j^y \subseteq \Lambda(y),\ \bigcap_{j=1}^{n} \varkappa_2(U_j^y) \neq \varnothing \right\}$$

is nonempty. Moreover, y belongs to the closure in X of the set $\bigcup_{(U_1^y, \ldots, U_n^y) \in \mathcal{U}_y} \bigcap_{j=1}^{n} \varkappa_2(U_j^y)$. Let

$$W_y = \bigcup_{(U_1^y, \ldots, U_n^y) \in \mathcal{U}_y} \bigcap_{j=1}^{n} \varkappa_1(\varkappa_2(U_j^y)).$$

Clearly W_y is open in Z. Since closure of any open subset of a product of a family of separable metrizable spaces is factorizable through a countable set [6], the set $\overline{W_y}$ is factorizable through a countable set $A_y \subset A$. Let $A' = \bigcup_{y \in Y \setminus \{y_0\}} A_y$. Since Y is countable, A' is also countable. Let $y_0 \in U \in \tau_Y$, $V = Y \setminus \overline{\bigcup_{y \in Y \setminus U} \Lambda(y)}$. According to Lemma 2 $y_0 \in V \in \tau_Y$.

Consider
$$W = Z \setminus \overline{\bigcup_{y\in B\setminus U} W_y}.$$
Evidently, W is an open subset of Z. Let us show that

(Q) W is factorizable through A', $V \subset W$ and $W \cap B \subset U \cap B$.

Since $\overline{W_y}$ are factorizable through $A' \supset A_y$ we have that the sets $\bigcup_{y\in B\setminus U} \overline{W_y}$ are also factorizable through A'. Hence, the set $\overline{\bigcup_{y\in B\setminus U} W_y} = \overline{\bigcup_{y\in B\setminus U} \overline{W_y}}$, is factorizable through A'. Therefore W is also factorizable through A'. Since $y \in \overline{W_y}$ for any $y \in B \setminus \{y_0\}$, we have that $B \setminus U \subset \overline{\bigcup_{y\in B\setminus U} W_y}$. Hence, $W \cap B \subset U \cap B$. It remains to show that $V \subset W$. Suppose that $V \not\subset W$. Then $V \cap \overline{\bigcup_{y\in B\setminus U} W_y} \neq \varnothing$. Since V is open, we have $V \cap \bigcup_{y\in B\setminus U} W_y \neq \varnothing$. Hence, there exists $y \in B \setminus U$ such that
$$V \cap W_y = V \cap \left(\bigcup_{(U_1^y,\dots,U_n^y)\in\mathcal{U}_y} \bigcap_{j=1}^{n} \varkappa_1(\varkappa_2(U_j^y)) \right) \neq \varnothing.$$
Thus, there exists $(U_1^y,\dots,U_n^y) \in \mathcal{U}_y$ such that
$$V \cap \bigcap_{j=1}^{n} \varkappa_1(\varkappa_2(U_j^y)) \neq \varnothing.$$
Therefore
$$\varkappa_1(\varkappa_2(V)) \cap \bigcap_{j=1}^{n} \varkappa_1(\varkappa_2(U_j^y)) \neq \varnothing.$$
Since $\varkappa_1$ satisfies (3) and (5) with $k = n$, we obtain
$$\varkappa_2(V) \cap \bigcap_{j=1}^{n} \varkappa_2(U_j^y) \neq \varnothing.$$
By definition of $\varkappa_2$, the family $\{V\} \cup \{U_j^y : 1 \leqslant j \leqslant n\}$ of open sets is not (pairwise) disjoint. According to the definition of $\mathcal{U}_y$ the family $\{U_j^y : 1 \leqslant j \leqslant n\}$ is disjoint. Hence, there exists $j \in \mathbb{N}$, $1 \leqslant j \leqslant n$ such that $U_j^y \cap V \neq \varnothing$. This contradicts the relations $U_j^y \subset \Lambda(y)$, $\Lambda(y) \cap V = \varnothing$. Thus, (Q) is proved.

Since the set of open subsets of Z factorizable through A' is a base of a (non-Hausdorff) topology defined by one pseudometric, the condition (Q) implies that there exists a countable base of neighborhoods of y_0 in $B \cup \{y_0\}$, which is a contradiction. ■

Corollary 1. *Let $n \in \mathbb{N}$, $Z = \prod_{\alpha\in A} Z_\alpha$, where all Z_α are separable and metrizable, $X \underset{m,m}{\subset} Z$ for any $m \in \mathbb{N}$, $1 \leqslant m \leqslant n$ and $Y \subset X$ be countable, monotonically regular and non-first-countable in any point $y \in Y$. Then Y is not $(1,n)$-embedded into X and is not $\mathcal{D}_c$-embedded into X for any $c < 2n+1$.*

P r o o f. According to Proposition 1 (statements 4 and 7) it suffices to show that there exists $y_0 \in Y$ for which $i(y_0, Y, X) \geqslant n+1$.

Using induction with respect to n, we shall prove that $i(y, Y, X) > n$ for any $y \in Y$. First, let $n = 1$. The set B from Proposition 2 in this case coincides with Y. According to Proposition 2,

$i(y,Y,X) \geqslant 2$ for all $y \in Y$. Let now $n \geqslant 2$ and for $n-1$ our statement is already proved. This means that $i(y,Y,X) \geqslant n$ for all $y \in Y$. Hence, the set B from Proposition 2 coincides with Y. According to Proposition 2, $i(y,Y,X) \geqslant n+1$ for all $y \in Y$. ■

Corollary 2. *Let $n \in \mathbb{N}$, $Z = \prod\limits_{\alpha \in A} Z_\alpha$, where all Z_α are separable and metrizable, $X \underset{m,m}{\subset} Z$ for any $m \in \mathbb{N}$, $1 \leqslant m \leqslant n$ and $Y \subset X$ be countable, monotonically regular homogeneous and non-metrizable. Then Y is not $(1,n)$-embedded into X and is not $\mathcal{D}_c$-embedded into X for any $c < 2n+1$.*

P r o o f. Since Y is countable and non-metrizable, there exists a point $y_0 \in Y$ such that Y is non-first-countable in y_0 (otherwise Y has a countable base and therefore is metrizable according to the Alexandrov theorem [6]). Since Y is homogeneous, Y is non-first-countable in any $y \in Y$. It remains to apply Corollary 1. ■

III Auxiliary Lemmas

Lemma 3. *Let $Z = [-1,1]^{[0,1]}$ (with Tychonoff product topology), Y is a countable space. Then there exists a dense countable set $X \subset Z$ and a closed (in X) set $Y_1 \subset X$ such that Y is homeomorphic to Y_1.*

P r o o f. Let $Z_1 = \{x \in Z : x(0) = 1\}$. Evidently Z_1 is also homeomorphic to a product of continuum (2^ω) of closed intervals. Hence, there exists a countable set $Y_1 \subset Z_1$ such that Y_1 is homeomorphic to Y (the weight of Y does not exceed 2^ω and hence Y admits a homeomorphic embedding into a product of 2^ω closed segments [6]). From the other hand $Z \setminus Z_1$ is homeomorphic to $[0,1) \times Z$ and therefore is separable as a product of 2^ω of separable spaces [6]. Thus, there exists a countable dense set $X_1 \subset Z \setminus Z_1$. Let $X = X_1 \cup Y_1$. Since Z_1 is closed in Z and $Y_1 = X \cap Z_1$, we have that Y_1 is closed in X. Since X_1 is dense in $Z \setminus Z_1$ and $Z \setminus Z_1$ is dense in Z we have that X is dense in Z. ■

Definition 3. Let X be a space $X_n \subset X$ be closed, $X_n \subseteq X_{n+1}$ for all $n \in \mathbb{N}$ and $X = \bigcup\limits_{n=1}^{\infty} X_n$. The space X is said to be *the inductive limit of X_n* if the topology of X is the strongest topology inducing the initial topology on any X_n (equivalently: a set $U \subseteq X$ is open if and only if $U \cap X_n$ is open in X_n for any $n \in \mathbb{N}$).

Lemma 4 (see e.g. [12])**.** *Inductive limits of sequences of stratifiable spaces are stratifiable.*

Definition 4. *Free abelian topological group* $A(X)$ over a space X is the free abelian group over the set X endowed with the strongest group topology inducing the initial topology on X (see [13,14]).

Definition 5. *Free locally convex space* $L(X)$ over a space X is the free linear space over the set X (i.e. the space of formal linear combinations of elements of X) endowed with the strongest locally convex topology inducing the initial topology on X (see [13,14]).

The following lemma is proved by Uspenskii [13,14].

Lemma 5. *Let X be a space, $Y \subset X$. Then*

(1) The topology of $A(X)$ coincides with the topology induced from $L(X)$.

(2) The topology of

$$L_0(X) = \Big\{ \sum_j \lambda_j x_j : \sum_j \lambda_j = 0 \Big\}$$

is defined by the family of seminorms

$$q_d(\mu) = \inf\Big\{ \sum_j |\lambda_j| d(u_j - v_j) : \mu = \sum_j \lambda_j (u_j - v_j) \Big\}, \quad d \in \mathcal{D}, \quad d \in \mathcal{D}, \tag{10}$$

where $\mathcal{D}$ is the set of all continuous pseudometrics on X.

(3) *The topology of* $L(Y)$ *coincides with the topology induced from* $L(X)$ *if and only if* Y *is* P*-embedded into* X.

Remind that $L(X)$ can be naturally interpreted as the space of all measures on X with finite support ($x \in X$ in this interpretation is identified with the probability measure with the support $\{x\}$). We'll use this interpretation without additional comments.

Lemma 6. *Let* X *be a space with countable network,* $x_0 \in X$. *Then topological group* $A(X)$ *is (Weyl) complete,* $\overline{L}(X)$ *is naturally isomorphic to the space of all* σ*-additive Borel real-valued (not necessarily positive) measures on* X *supported on metrizable compact subsets of* X, *endowed with the topology defined by the family of seminorms*

$$p_d(\mu) = \sup_{f \in \mathcal{F}_d} \int\limits_X f\,d\mu, \quad d \in \mathcal{D},$$

where $\mathcal{D}$ *is the set of all continuous pseudometrics on* X *and* $\mathcal{F}_d$ *is the set of all* $f \in C_b(X)$ *such that* $|f(x_0)| \leqslant 1$ *and* $|f(x) - f(y)| \leqslant d(x, y)$ *for all* $x, y \in X$.

Proof. Completeness of $A(X)$ for Dieudonne complete spaces [6] X had been proved by V. V. Uspenskii [13,14]. Since any space with countable network is Dieudonne complete [6], we have that $A(X)$ is complete. The description of the completion of free LCS's obtained by V. V. Uspenskii [13,14] also implies that the space $\overline{L}(X)$ for Dieudonne complete X is naturally isomorphic to the space of functionals from $C_b^*(X)$, which admit a representation as inegration with respect to a σ-additive Borel measure on X supported on compact subsets of X. Since any compact space with countable network is metrizable, this implies the required description of $\overline{L}(X)$.

Let $L_0(X)$ be the space of Lemma 5, i.e., the set of elements of $L(X)$ with zero integral. Evidently $L_0(X)$ is a closed hyperplane in $L(X)$. Let q_d be seminorms defined by (10).

For any $\mu = \sum\limits_j \lambda_j(u_j - v_j) \in L_0(X)$, $d \in \mathcal{D}$ and $f \in \mathcal{F}_d$ we have

$$\left| \int\limits_X f\,d\mu \right| = \left| \sum_j \lambda_j (f(u_j) - f(v_j)) \right| \leqslant \sum_j |\lambda_j| d(u_j, v_j) \Longrightarrow \left| \int\limits_X f\,d\mu \right| \leqslant q_d(\mu).$$

Hence, $p_d(\mu) \leqslant q_d(\mu)$.

According to the definition of topology of $L(X)$ a linear functional on $L(X)$ is continuous if and only if its restriction to X is continuous. Therefore the dual of $L(X)$ is $C(X)$ and the action of $f \in C(X)$ on $L(X)$ is given by

$$\langle \mu, f \rangle = \int\limits_X f\,d\mu.$$

The dual of $L_0(X)$ then can be naturally identified with $C_0(X) = \{f \in C(X) : f(x_0) = 0\}$. Let U_d be the set of $f \in C_0(X)$ such that $|\langle \mu, f \rangle| \leqslant 1$ for all $\mu \in L_0(X)$ such that $q_d(\mu) \leqslant 1$. For any $x, y \in X$ consider $\mu = (x - y)/d(x, y) \in L_0(X)$. Then $q_d(\mu) = 1$. Therefore $|\langle \mu, f \rangle| = |f(x) - f(y)|/d(x, y) \leqslant 1$ for all $f \in U_d$. Hence $|f(x) - f(y)| \leqslant d(x, y)$ for all $f \in U_d$. On the other hand if $f \in C_0(X)$ and $|f(x) - f(y)| \leqslant d(x, y)$ for all $x, y \in X$ then (as we have already verified) $|\langle \mu, f \rangle| \leqslant q_d(\mu)$ for any $\mu \in L_0(X)$. Thus, $f \in U_d$ if and only if $f \in C_0(X)$ and $|f(x) - f(y)| \leqslant d(x, y)$ for all $x, y \in X$. Therefore $U_d = \mathcal{F}_d$. Hence, polars in $C_0(X)$ of closed unit balls with respect to q_d and the restriction of p_d to $L_0(X)$ are equal. Both seminorms are continuous (q_d due to Lemma 5 and the restriction of p_d to $L_0(X)$ because of the inequality $p_d(\mu) \leqslant q_d(\mu)$). Therefore q_d coincides with the restriction of p_d to $L_0(X)$. It remains to use the facts that $L_0(X)$ is a closed hyperplane in $L(X)$ and that seminorms q_d define the topology of $L_0(X)$. ■

Lemma 7. *Let X be an inductive limit of a sequence of metrizable compact spaces $K_n \subseteq X$. Then $A(X)$ and the closure $\overline{L}(X)$ of the free LCS over X are also inductive limits of sequences of metrizable compact spaces.*

P r o o f. In [13,14] it is proved that the topology $A(X)$ coincides with the topology induced from $\overline{L}(X)$. According to Lemmas 5 and 6 $A(X)$ is complete and therefore is closed in $\overline{L}(X)$. Since a closed subspace of an inductive limit of a sequence of metrizable compact spaces is itself an inductive limit of a sequence of metrizable compact spaces, it suffices to show that $\overline{L}(X)$ is an inductive limit of metrizable compact spaces.

Let K_n be metrizable compact subspaces of X such that X is the inductive limit of the sequence K_n. Let

$$M_n = \{\mu \in \overline{L}(X) \,:\, \operatorname{supp}\mu \subseteq K_n \;\text{ and }\; \|\mu\| \leqslant n\},$$

where $\|\mu\|$ is the complete variation of the measure μ and the inclusion $\operatorname{supp}\mu \subseteq K_n$ means that $\mu(A) = 0$ if $A \cap K_n = \varnothing$.

According to Lemma 6 $\bigcup\limits_{n=1}^{\infty} M_n = \overline{L}(X)$. It remains to verify that M_n are compact and metrizable and that $\overline{L}(X)$ is the inductive limit of M_n. According to Alaoglu theorem [7] and Riesz theorem on the general form of continuous linear functionals on $C(K)$ for compact metrizable K we have that M_n are metrizable and compact with respect to the weak topology $\sigma(\overline{L}(K_n), C(K_n))$. According to Arceli–Ascoli theorem [7] the set $\mathcal{F}_d^n = \{f\big|_{K_n} : f \in \mathcal{F}_d\}$ is compact in $C(K_n)$ for any continuous pseudometric d on X ($\mathcal{F}_d$ are sets of functions from Lemma 6). According to Lemmas 1, 5 and 7 the topology of $\overline{L}(K_n)$ coincides with the topology induced from $\overline{L}(X)$ and coincides with the topology $t(\overline{L}(K_n), C(K_n))$ of uniform convergence [7] over compact subsets of the Banach space $C(K_n)$. According to Banach–Dieudonné theorem [17] the compact subsets of $\overline{L}(K_n)$ with respect to topologies $t(\overline{L}(K_n), C(K_n))$ and $\sigma(\overline{L}(K_n), C(K_n))$ are the same and the restrictions of these topologies to compact subsets coincide. Hence, M_n are compact and metrizable in $\overline{L}(X)$. Let τ_1 be the inductive limit topology of the sequence M_n and τ_2 be the natural topology of $\overline{L}(X)$. We have to show that $\tau_1 = \tau_2$. By definition of the inductive limit topology $\tau_2 \subseteq \tau_1$. On the other hand, since $X \cap M_n = K_n$, the restrictions of τ_1 and τ_2 on X coincide. Since τ_2 is the strongest locally convex topology, inducing the initial topology on X, it suffices to verify that the topology τ_1 is locally convex. Let ξ be the topology on $C(X)$ with the basis $\{U_n = M_n^\circ : n \in \mathbb{N}\}$ of neighborhoods of zero. Then $(C(X), \xi)$ is a metrizable LCS. Using Banach–Dieudonné theorem [17] again, we obtain that the topology τ_3 on $\overline{L}(X)$ of uniform convergence over precompact subsets of $(C(X), \xi)$ coincides with the inductive limit topology of the sequence of sets $U_n^\circ = M_n$ endowed with the topology $\sigma(\overline{L}(X), C(X))$. Since (as we have already shown) the restrictions of τ_2 and $\sigma(\overline{L}(X), C(X))$ to M_n coincide, we obtain that $\tau_1 = \tau_3$ and τ_1 is locally convex, since τ_3 is obviously locally convex. ■

IV Proof of Theorems 1 and 2

Let S be an infinite countable compact space (such a space is automatically metrizable [6]), $G = A(S)$. Then G is a countable abelian topological group. Lemma 3 implies the existence of $Y \subset X \subset Z = [-1,1]^{[0,1]}$ such that X is countable and dense in Z, Y is closed in X and homeomorphic to G.

A free abelian topological group over an infinite space is always non-metrizable [15,16]. According to Lemma 7 G is an inductive limit of a sequence of metric compact spaces and therefore is stratifiable due to Lemma 4. Any stratifiable space is monotonically regular [11]. Hence, Y (which is homeomorphic to G) is countable, homogeneous, monotonically regular and non-metrizable. Statement 2 of Proposition 1 implies that $X \underset{nn}{\subset} Z$ for any $n \in \mathbb{N}$. According to Corollary 2 Y is not $(1,n)$-embedded into X for any $n \in \mathbb{N}$ and is not $\mathcal{D}_c$-embedded into X for any $c \geqslant 1$. This proves Theorem 1 (the pair (X, Y) is the desired example).

Let now $E = L(X)$ and F be the linear hull of Y in E. Clearly E is a countably dimensional LCS and F is its closed linear subspace. For the proof of Theorem 2 it suffices to verify that F is not $\mathcal{D}_c$-embedded into E for any $c \geqslant 1$ (then the pair (E, F) is the desired example).

According to Lemmas 1 and 5, F is naturally isomorphic to $L(Y)$, which is stratifiable (see Lemmas 7 and 4). Borges theorem [3] then implies that $Y \underset{\mathcal{D}_1}{\subset} F$. Suppose that $F \underset{\mathcal{D}_c}{\subset} E$ for some $c \geqslant 1$. Then according to Proposition 1 (Statements 8 and 9) $Y \underset{\mathcal{D}_c}{\subset} E$ and hence $Y \underset{\mathcal{D}_c}{\subset} X$, which is a contradiction. Theorem 2 is proved.

V. Open problems

Problem 1. For which $c \in [1, +\infty)$ there exists a countable space (or at least a space with a countable network) with $\lambda(X) = c$?

Problem 2. For which $c \in [1, +\infty)$ there exist a countable Hausdorff topological group (or at least a Hausdorff topological group with a countable network) X and a closed subgroup $Y \subset X$ with $\lambda(Y, X) = c$?

Problem 3. For which $c \in [1, +\infty)$ there exist a countably dimensional LCS (or at least a LCS with a countable network) X and a closed linear subspace $Y \subset X$ with $\lambda(Y, X) = c$?

Remark 4. For $c = 1$ and for $c = \infty$ the desired spaces do exist for all three problems. Indeed, for $c = 1$ one has to apply the Dugundji–Borges theorem. For $c = \infty$ it follows from Theorems 1 and 2 (for Problem 2 one have to consider free abelian topological groups over Y and X from Theorem 1).

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REFERENCES

[1] J. Dugundji, *An extension of Tietze's theorem,* Pacific J. Math., 1951, vol. 1, p.353–367

[2] C. Bessaga, A. Pelczinsci, *Selected topics in infinite dimensional topology,* Warszawa, Polish scientific publishers, 1975

[3] C. J. R. Borges, *On stratifiable spaces,* Pacific J. Math., 1966, vol. 17, no. 1, pp. 1–16

[4] C. J. R. Borges, *Continuous extensions,* Proc. Amer. Math. Soc., 1967, vol. 18, no. 5, pp. 874–878

[5] R. W. Heath and R. H. Hodel, *Caracterization of σ-spaces,* Fund. Math., 1973, vol. 77, pp. 271–275

[6] R. Engelking, *General Topology,* Heldermann Verlag, Berlin, 1989

[7] A. P. Robertson, W. J. Robertson, *Topological Vector Spaces,* Cambridge University Press, 1964

[8] R. W. Heath and D. J. Lutzer, *Dugunji extension theorems for linearly ordered spaces,* Pacific Journ. Math., 1974, vol. 55, pp. 419–425

[9] E. van Dowen, *Simultaneous linear extension of continuous functions,* General Topology and its Applications, 1975, vol. 5, pp. 297–319

[10] T. Przymusinski, *Collectionwise normality and extension of continuous functions,* Fundam. Math., 1978, vol. 98, no. 1, pp. 75–81

[11] S. A. Shkarin, *On stratifiable locally convex spaces,* Russian Journ. of Math. Phys, 1999, vol. 6, no. 4, pp. 435–460

[12] *Handbook of set-theoretic topology,* ed. K. Kunen, J. E. Vaughan, Ch. 10: *Generalized metric spaces* by Gary Gruenhage, North Holland, Amsterdam, New-York, Oxford, Tokio, 1984

[13] V. V. Uspenskii, *Free topological groups of metrizable spaces,* Math. USSR Izv., 1991, vol. 37, no. 3, pp. 657–680

[14] V. V. Uspenskii, *On the topology of a free locally convex space,* Soviet Math. Dokl., 1983, vol. 27, no. 3 pp. 771–775

[15] O. V. Sipacheva, *Stratifiability of free abelian topological groups,* Topology Proc., 1993, vol. 18, pp. 271–311

[16] O. V. Sipacheva, *Free topological groups and their subspaces,* Topology Appl., 2000, vol. 101, no. 3, pp. 181–212

[17] G. Köthe, *Topological vector spaces I,* Springer, New York, 1969; *Topological vector spaces II,* Springer, New York, 1979