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On Descriptive Characterizations of an Integral Recovering a Function from Its L^r-Derivative

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Abstract—The notion of L^r-variational measure generated by a function $F \in L^r[a,b]$ is introduced and, in terms of absolute continuity of this measure, a descriptive characterization of the HK_r-integral recovering a function from its L^r-derivative is given. It is shown that the class of functions generating absolutely continuous L^r-variational measure coincides with the class of ACG_r-functions which was introduced earlier, and that both classes coincide with the class of the indefinite HK_r-integrals under the assumption of L^r-differentiability almost everywhere of the functions consisting these classes.

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1. INTRODUCTION

There are many areas in analysis which require nonabsolutely convergent integration processes more powerful than Lebesgue integration. In most cases, such integrals are introduced to solve the problem of recovering primitives defined in terms of respective generalized derivatives (note that the Lebesgue integral does not solve this problem even in the case of the ordinary derivative). For example, in harmonic analysis, the problem of recovering coefficients of series with respect to an orthogonal system from their sums can often be reduced to the integration of an appropriate generalized derivative chosen in accordance with the considered system. In classical harmonic analysis, integration of the approximate symmetric derivative solves the problem of recovering coefficients of trigonometric series (see [1]), while in the case of series with respect to the system of characters of the dyadic Cantor group or of its generalizations, the dyadic and p-adic derivatives and derivatives with respect to various derivate bases do the job (see [2]–[8]).

Various versions of nonabsolute integrals were introduced to integrate each of those generalized derivatives. Constructive definitions of Denjoy– or Henstock–Kurzweil-type integrals usually turn out to be equivalent to the Perron-type approach (see [4], [6], [9] and [10]), as well as to a Lusin-type descriptive definition. For some useful properties of those integrals, see [11]–[13]. These definitions were also extended to functions defined or ranging in some kind of abstract spaces (see [14], [15]).

All descriptive definitions of nonabsolute integrals are in fact generalizations of the known descriptive characterization of the indefinite Lebesgue integral in terms of absolutely continuous functions, which can be formulated in the following equivalent form: a function f is L-integrable on $[a,b]$ if and only if there exists a function F of bounded variation on $[a,b]$ that generates an absolutely continuous

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Lebesgue–Stieltjes measure and $F'(x) = f(x)$ a.e.; the function $F(x) - F(a)$ being the indefinite L-integral of f (this is in fact the one-dimensional case of the Radon–Nikodym theorem).

In the case of nonabsolute generalizations of the Lebesgue integrals (of Denjoyp–Perron- or Henstock–Kurzweil-type) indefinite integrals fail to be of bounded variation and so cannot generate a finite Stieltjes measure. In this case, a descriptive characterization can be obtained in terms of some generalized σ-finite outer measure (so-called variational measure) generated by the indefinite integral.

Variational measures can be defined with respect to various derivate bases. In the simplest case of the full interval basis and of the original Henstock–Kurzweil integral, it was introduced by B. Thomson in [16] and was used to give a full descriptive characterization of this integral in [17] (also see [18]–[21]).

In this paper, we introduce and investigate a descriptive characterization of Henstock–Kurzweil-type integrals, the HK_r-integral, which serves to integrate a derivative defined in the space L^r, $1 \le r < \infty$. This L^r-derivative was introduced in [22] by Calderon and Zygmund to be used in obtaining some estimates for solutions of elliptic partial differential equations. Later L. Gordon [23] described a Perron-type integral, the P_r-integral, that recovers a function from its L^r-derivative. In 2004, Musial and Sagher [24] extended the P_r-integral by the L^r-Henstock–Kurzweil integral, the HK_r-integral, which turned out to be strictly wider than the P_r-integral [25]. They also defined the Lusin-type class of ACG_r-functions and showed that indefinite HK_r-integrals belong to this class. Some other properties of this integral were investigated in [26]–[28].

In Sec. 3 of the present paper, we define the L^r-variational measure generated by a function belonging to $L^r[a,b]$ and obtain a characterization of the indefinite HK_r-integral in terms of absolute continuity of this measure. Namely, we show that the class of the indefinite HK_r-integrals coincides with the class of functions in $L^r[a,b]$ that generate an absolutely continuous L^r-variational measure, under the additional assumption that the functions are L^r-differentiable almost everywhere. In Sec. 4, we prove that any absolutely continuous L^r-variational measure is σ-finite. In the case of the variational measure defined by the usual increment of a function, an analogous statement was established in [29], and we use the same tetechniq here (also see [18, Theorem 7.37]). This result is crucial for establishing in the next Sec. 5 that the class of functions generating absolutely continuous L^r-variational measure coincides with the class of ACG_r-functions. Note that both classes coincide with the class of the indefinite HK_r-integrals under assumption of L^r-differentiability almost everywhere of functions of the classes. But the proof in Sec. 5 does not depend on this assumption. The problem on L^r-differentiability almost everywhere of functions of those classes is left open.

2. PRELIMINARIES

We work in a fixed closed interval $[a,b]$. The symbol I denotes a nondegenerate closed subinterval of $[a,b]$. A *tagged interval* is a pair (I,x), where $x \in I$, is a *tag*, and a *gauge* is a strictly positive function δ on $[a,b]$ (or on a subset of $[a,b]$). We say that a tagged interval (I,x) is *δ-fine* if $I \subset (x-\delta(x), x+\delta(x))$.

A *δ-fine partition* is a finite collection π of δ-fine tagged intervals, where distinct elements $(I', \mathbf{x}')$ and $(I'', \mathbf{x}'')$ in π have nonoverlapping I' and I'', i.e., they have no interior points in common. A partition π is *tagged in a set* $E \subset [a,b]$ if $x \in E$ for each element (I,x) of π.

Throughout this paper, we assume that $r \ge 1$ and consider the respective Lebesgue spaces $L^r[a,b]$. By μ we denote the Lebesgue measure on $[a,b]$. We recall the definition of the L^r-Henstock–Kurzweil integral given in [24] and some related notions from [23].

Definition 1. A function $f\colon [a,b] \to \mathbb{R}$ is *L^r-Henstock–Kurzweil integrable* (*HK_r-integrable*) on $[a,b]$ if there exists a function $F \in L^r[a,b]$ such that, for any $\varepsilon > 0$, there exists a gauge δ such that, for any δ-fine partition $\pi = \{([c_i,d_i],x_i), 1 \le i \le q\}$, we have

$$\sum_{i=1}^{q}\biggl(\frac{1}{d_i-c_i}\int_{c_i}^{d_i}|F(y)-F(x_i)-f(x_i)(y-x_i)|^r\,dy\biggr)^{1/r} < \varepsilon. \tag{2.1}$$

By Theorem 5 in [24], the function F is unique up to an additive constant, so, putting $F(a)=0$, we can consider the *indefinite HK_r-integral*

$$F(x) = (HK_r)\int_a^x f \qquad \forall\, x \in (a,b].$$

As it was done in the case of the classical Henstock–Kurzweil integral (see [18, Corollary 2.80]), it can be easily checked that the value of HK_r-integral does not depend on the values of the function f on a set of measure zero (under the condition that only finite values of f are considered).

Definition 2. A function $F \in L^r[a,b]$ is said to be L^r-*continuous at* $x \in [a,b]$ if

$$\lim_{h\to 0} \frac{1}{2h} \int_{x-h}^{x+h} |F(y) - F(x)|^r \, dy = 0.$$

If F is L^r-continuous for all $x \in E$, we say that F is L^r-*continuous on* E.

Definition 3. A function $F \in L^r[a,b]$ is said to be L^r-*differentiable* at x if there exists a real number α such that

$$\left(\frac{1}{h} \int_{-h}^{h} |F(x+t) - F(x) - \alpha t|^r \, dt\right)^{1/r} = o(h).$$

In this case we say that α is the L^r-*derivative* of F at x and denote $F'_r(x) = \alpha$.

It was proved in [24] that if F is the indefinite HK_r-integral of f, then F is L^r-continuous on $[a,b]$ and $F'_r(x)$ exists and is equal to $f(x)$ a.e. on $[a,b]$.

For $F \in L^r[a,b]$ and a tagged interval (I,x), we denote

$$\Delta_r F(I,x) = \left(\frac{1}{\mu(I)} \int_I |F(y) - F(x)|^r \, dy\right)^{1/r}.$$

The following generalization of the notion of absolute continuity of a function was considered in [24] in order to give a Lusin-type description of the class of the indefinite HK_r-integrals.

Definition 4. Let $E \subset [a,b]$. We say that $F \in AC_r(E)$, i.e., is an AC_r-*function on the set* E, if for all $\varepsilon > 0$, there exist $\eta > 0$ and a gauge δ defined on E such that for any δ-fine partition $\{I_i, x_i),\ 1 \le i \le q\}$, tagged in E and such that $\sum_{i=1}^{q} \mu(I_i) < \eta$, we have

$$\sum_{i=1}^{q} \Delta_r F(I_i, x_i) < \varepsilon.$$

Definition 5. We say that $F \in ACG_r(E)$ if E can be written as $E = \bigcup_{n=1}^{\infty} E_n$ and $F \in AC_r(E_n)$ for all n.

3. DESCRIPTIVE CHARACTERIZATION OF HK_r-INTEGRAL IN TERMS OF THE L^r-VARIATIONAL MEASURE

Here we introduce the notion of the L^r-variational measure generated by a function $F \in L^r[a,b]$ and apply it to give a descriptive characterization of the indefinite HK_r-integral.

Definition 6. For $F \in L^r[a,b]$, a set $E \subset [a,b]$ and a fixed gauge δ on E, we define the δ-*variation* of F on E as

$$\mathrm{Var}(E, F, \delta, r) = \sup \sum_{i=1}^{q} \Delta_r F(I_i, x_i), \tag{3.1}$$

where the sup is taken over all δ-fine partitions $\{(I_i, x_i)\}$ in $[a,b]$ tagged in E.

Definition 7. The L^r-*variational measure* generated by $F \in L^r[a,b]$ of a set $E \subset [a,b]$ is defined as

$$\mathrm{V}_F(E) = \inf_{\delta} \mathrm{Var}(E, F, \delta, r),$$

where the inf is taken over all gauges tagged in E.

Definition 8. An L^r-variational measure V_F is said to be *absolutely continuous* on a set $E \subset [a,b]$ with respect to the Lebesgue measure μ if $\mathrm{V}_F(N) = 0$ for any $N \subset E$ such that $\mu(N) = 0$.

By techniques similar to those used in [20], we can show that V_F is a metric outer measure, and if it is absolutely continuous with respect to the Lebesgue measure, it is a σ-additive measure when restricted to Lebesgue measurable subsets of $[a,b]$.

We first give a simple proof establishing that an L^r-differentiable almost everywhere function F is (up to an additive constant) an indefinite HK_r-integral of its L^r-derivative if and only if the L^r-variational measure generated by F is absolutely continuous.

Theorem 1. *A function f is HK_r-integrable on $[a,b]$ if and only if there exists a function F on $[a,b]$ which generates an absolutely continuous L^r-variational measure and which is L^r-differentiable almost everywhere with $F'_r(x) = f(x)$ a.e.,; the function $F(x) - F(a)$ being the indefinite HK_r-integral of f.*

Proof. Let f be an HK_r-integrable function on $[a,b]$ with $F(x)$ being its indefinite HK_r-integral. Fix an arbitrary set $A \subset [a,b]$, $\mu(A) = 0$. We can assume that $f(x) = 0$ if $x \in A$. Then, having found a gauge δ for arbitrary ε and applying the inequality (2.1) from Definition 1 for a δ-fine partition $\{(I_i, x_i)\}_{i=1}^q$ tagged in A, we obtain

$$\sum_{i=1}^{q} \left(\frac{1}{\mu(I_i)} \int_{I_i} |F(y) - F(x_i)|^r \, dy \right)^{1/r} < \varepsilon; \tag{3.2}$$

This implies that $\mathrm{V}_F(A) = 0$ and proves the absolute continuity of V_F.

Conversely, let F be a function generating an absolutely continuous L^r-variational measure and let E be the set on which its L^r-derivative exists. Then $\mu(E^c) = \mu([a,b] \setminus E) = 0$. We will show that $F(x) - F(a)$ is the indefinite HK_r-integral of the function

$$f(x) = \begin{cases} F'_r(x) & \text{at } x \in E, \\ 0 & \text{at } x \in E^c. \end{cases}$$

Having fixed an arbitrary ε and using (3.1), we first define a gauge δ on E^c such that inequality (3.2) holds for any δ-fine partition $\{(I_i, x_i)\}_{i=1}^q$ tagged in E^c. To define the gauge on E, we note that, according to Definition 3, for each $x \in E$ there exists $\delta(x)$ such that for any interval I, $x \in I \subset (x - \delta(x), x + \delta(x))$, we have

$$\left(\frac{1}{\mu(I)} \int_I |F(y) - F(x) - f(x)(y - x)|^r \, dy \right)^{1/r} < \varepsilon |I|.$$

Now the gauge δ is defined for all $x \in [a,b]$ and for any δ-fine partition $\{(I_i, x_i)\}_{i=1}^n$ in $[a,b]$ we finally obtain

$$\begin{aligned} &\sum_{i=1}^{n} \left(\frac{1}{\mu(I_i)} \int_{I_i} |F(y) - F(x_i) - f(x)(y - x)|^r \, dy \right)^{1/r} \\ &\quad = \sum_{i\colon x_i \in E^c} \left(\frac{1}{\mu(I_i)} \int_{I_i} |F(y) - F(x_i)|^r \, dy \right)^{1/r} \\ &\qquad + \sum_{i\colon x_i \in E} \left(\frac{1}{\mu(I_i)} \int_{I_i} |F(y) - F(x_i) - f(x_i)(y - x_i)|^r \, dy \right)^{1/r} \\ &\quad < \varepsilon + \varepsilon \sum_{i\colon x_i \in E} \mu(I_i) = \varepsilon(1 + b - a). \end{aligned}$$

According to Definition 1, this means that F is the indefinite HK_r-integral of f.. $\square$

4. σ-FINITENESS OF THE L^r-VARIATIONAL MEASURE

Here we prove that any absolutely continuous L^r-variational measure is σ-finite.

Theorem 2. *If the L^r-variational measure generated by a function $F \in L^r[a,b]$ is absolutely continuous on a closed set $E \subset [a,b]$, then V_F is σ-finite on E.*

Proof. Suppose that V_F is not σ-finite on E, and let us show that this leads us to a contradiction with the absolute continuity of V_F.

Let T be a set of all points $x \in E$ such that V_F is σ-finite on no nonempty portion $E \cap I$ defined by a closed interval I, with $x \in I$, and let P be a set of all two-sided limit points of T. It is clear that T is a perfect set. Then $T \setminus P$ is a countable set and so V_F is σ-finite on this set. This implies that V_F s finite on no nonempty portion $P \cap I$ of the set P defined by a closed interval I. Assuming that T is not empty, we construct a set $N \subset P$ of measure zero with $\mathrm{V}_F(N) \geq 1$, and obtain the desired contradiction.

Our assumptions imply that V_F is not finite on P. Then it follows from Definition 6 and 7 that we can find a family of nonoverlapping closed intervals $\{I_j^{(1)}\}_{j=1}^{m_1}$, which are subintervals of $[a,b]$ so that, for each $j = 1, 2, \dots, m_1$, we have

$$P \cap I_j^{(1)} \neq \varnothing \qquad \text{and} \qquad \sum_{j=1}^{m_1} |\Delta_r F(I_j^{(1)})| > 1.$$

We can suppose that $m_1 > 1$ and

$$\sum_{j=1}^{m_1} \mu(I_j^{(1)}) < \frac{1}{2}\,.$$

The last inequality is possible since, in case of need, we can replace the set P by its portion defined by a small enough interval.

Now we proceed by induction. We suppose that we have already constructed in $[a,b]$ a family of nonoverlapping closed intervals $\{I_i^{(k-1)}\}_{i=1}^{m_{k-1}}$, $k > 1$, such that $P \cap I_i^{(k-1)} \neq \varnothing$ for each i. Repeating the construction used in the first step, we obtain a family of nonoverlapping closed intervals $\{I_j^{(k)}\}_{j=1}^{m_k}$ such that

(a) $P \cap I_j^{(k)} \neq \varnothing$ for each $j = 1, 2, \dots, m_k$;

(b) each $I_j^{(k)}$ is contained in some $I_i^{(k-1)}$;

(c) each $I_i^{(k-1)}$ contains at least two intervals from the family $\{I_j^{(k)}\}_{j=1}^{m_k}$;

(d) $\sum_{j=1}^{m_k} \mu(I_j^{(k)}) < 1/2^k$;

(e) $\sum_{j\colon I_j^{(k)} \subset I_i^{(k-1)}} |\Delta_r F(I_j^{(k)})| > 1$ for each $i = 1, 2, \dots, m_{k-1}$.

We put

$$N = \bigcap_{k=1}^{\infty} \bigcup_{j=1}^{m_k} I_j^{(k)}.$$

Condition (d) implies that N is a set of measure zero. Using (a)–(d) it is not difficult to check that N is a perfect subset of P. For an arbitrary gauge δ on N and for any $n \in \mathbb{N}$, we define

$$N_n = \left\{x \in N\colon \delta(x) > \frac{1}{n}\right\}.$$

Applying the Baire category theorem, we find an n such that the set N_n is dense in some portion $J \cap N$ of the set N. There exist l and i such that $I_i^{(l)} \subset J$ and $|I_i^{(l)}| < 1/n$. The density of N_n in $J \cap N$ implies that for each $I_j^{(l+1)} \subset I_i^{(l)}$ there exists a $x_j \in I_j^{(l+1)} \cap N_n$. Since $\delta(x_j) > 1/n$, the family of pairs

$$\{(I_j^{(l+1)}, x_j) \colon I_j^{(l+1)} \subset I_i^{(l)}\}$$

forms δ-fine partition tagged in N. Using (e) with k replaced by $l+1$, we obtain

$$\operatorname{Var}(F, N, \delta, r) \geq \sum_{j \colon I_j^{(l+1)} \subset I_i^{(l)}} |\Delta_r F(I_j^{(l+1)})| > 1.$$

The gauge δ was arbitrary. So $\mathrm{V}_F(N) \geq 1$, which yields the desired contradiction with the absolute continuity of V_F on E. □

5. COMPARISON OF DESCRIPTIVE CHARACTERIZATIONS OF THE HK_r-INTEGRAL

The next two theorems show that the class $ACG_r([a,b])$ coincides with the class of functions which generate absolutely continuous L^r-variational measures.

Theorem 3. *Suppose that the L^r-variational measure V_F generated by a function F of class $L^r[a,b]$ is finite on a set $E \subset [a,b]$. Then it is absolutely continuous on E if and only if $F \in AC_r(E)$.*

Proof. To prove the *necessity*, note that if $\mu(E) = 0$, then $\mathrm{V}_F(E) = 0$ and $F \in AC_r(E)$ by the definitions. So we assume that. $\mu(E) > 0$ and fix an arbitrary $\varepsilon > 0$. Since $\mathrm{V}_F(E) < +\infty$, we can choose a gauge δ on E such that

$$\operatorname{Var}(E, F, \delta, r) < \mathrm{V}_F(E) + \frac{\varepsilon}{3}\,. \tag{5.1}$$

Since V_F is a metric outer measure, it can be treated as a σ-additive measure on σ-algebra $\mathfrak{B}_E$ of Borel subsets of the set E, regarded as a metric space. Then absolute continuity of V_F implies that there exists a $\eta \in (0, \mu(E)/2)$ such that

$$\mathrm{V}_F(Y) < \frac{\varepsilon}{3} \qquad \text{if} \quad Y \in \mathfrak{B}_E, \quad \mu(Y) < \eta. \tag{5.2}$$

Let $\boldsymbol{\pi}_1 = \{(M_i, \xi_i)\}_{i=1}^p$ be a δ-partition tagged in E with $\sum_{i=1}^p \mu(M_i) < \eta$. Then $E \cap (\bigcup_{i=1}^p M_i) \in \mathfrak{B}_E$ and

$$\mu\biggl(E \cap \biggl(\bigcup_{i=1}^p M_i\biggr)\biggr) \leq \sum_{i=1}^p \mu(M_i) < \eta.$$

Then (5.2) implies

$$\mathrm{V}_F\biggl(E \cap \biggl(\bigcup_{i=1}^p M_i\biggr)\biggr) < \frac{\varepsilon}{3}\,. \tag{5.3}$$

Consider the set $Z = E \setminus (\bigcup_{i=1}^p M_i)$. Since $\eta \in (0, \mu(E)/2)$, we have $\mu(Z) > 0$. Define a gauge $\delta_0 \colon Z \to (0, +\infty)$ having the following properties:

(i) $\delta_0(x) \leq \delta(x)$ for each $x \in Z$;

(ii) $(x - \delta_0(x), x + \delta_0(x)) \cap (\bigcup_{i=1}^p M_i) = \varnothing$ for each $x \in Z$;

(iii) $\operatorname{Var}(Z, F, \delta_0, r) < +\infty$ (since $\mathrm{V}_F(E) < +\infty$).

By property (iii), there exists δ_0-partition $\boldsymbol{\pi}_2 = \{(L_i, y_i)\}_{i=1}^q$, on Z such that

$$\sum_{i=1}^{q} |\Delta_r F(L_i)| > \mathrm{Var}(Z, F, \delta_0, r) - \frac{\varepsilon}{3}\,. \tag{5.4}$$

By (ii), none of intervals L_i intersect $\bigcup_{i=1}^p M_i$. Then $\boldsymbol{\pi}_1 \cup \boldsymbol{\pi}_2$ is a δ-partition tagged in E (see (i)). Hence by (3.1) and (5.1), we have

$$\sum_{i=1}^{p} |\Delta_r F(M_i)| + \sum_{i=1}^{q} |\Delta_r F(L_i)| \le \mathrm{Var}(E, F, \delta, r) < \mathrm{V}_F(E) + \frac{\varepsilon}{3}\,.$$

Combining this inequality with (5.3) and (5.4), we obtain

$$\sum_{i=1}^{p} |\Delta_r F(M_i)| < \mathrm{V}_F(E) - \sum_{i=1}^{q} |\Delta_r F(L_i)| + \frac{\varepsilon}{3} < \mathrm{V}_F(E) - \mathrm{Var}(Z, F, \delta_0, r) + \frac{2\varepsilon}{3}$$
$$\le \mathrm{V}_F(E) - \mathrm{V}_F(Z) + \frac{2\varepsilon}{3} \le \mathrm{V}_F\left(E \cap \left(\bigcup_{i=1}^{p} M_i\right)\right) + \frac{2\varepsilon}{3} < \varepsilon.$$

This means that $F \in AC_r(E)$.

To prove the sufficiency, consider a set $A \subset E$ of measure zero. Then for each $\varepsilon > 0$, we can find a gauge δ' on A, a number $\eta > 0$, and an open set G covering A such that

$$\mu(G) < \eta, \tag{5.5}$$

$$\sum_{i=1}^{p} |\Delta_r F(M_i)| < \varepsilon \tag{5.6}$$

for any δ'-partition $\{(M_i, \xi_i)\}_{i=1}^p$, tagged in A provided

$$\sum_{i=1}^{p_n} \mu(M_i) < \eta. \tag{5.7}$$

Put $\delta(x) = \min(\delta'(x), \rho(x, \mathbb{R} \setminus G))$ on A. Then, by (5.5), any δ-partition $\pi = \{(M_i, \zeta_i)\}_{i=1}^p$ tagged in A satisfies inequality (5.7). Therefore, (5.6) holds for π, and $\mathrm{Var}(A, F, \delta, r) \le \varepsilon$. This proves that $\mathrm{V}_F(A) = 0$. □

Remark 1. Note that the proof of sufficiency in the above theorem does not require the finiteness of V_F on E. Hence the condition $F \in AC_r(E)$ always implies absolute continuity of L^r-variational measure V_F on E.

Theorem 4. *A function $F \in L^r[a, b]$ generates an absolutely continuous L^r-variational measure on $[a, b]$ if and only if $F \in ACG_r([a, b])$.*

Proof. If F generates an absolutely continuous L^r-variational measure on $[a, b]$, then by Theorem 2, $[a, b] = \bigcup_{n=1}^\infty E_n$ and $\mathrm{V}_F(E_n) < +\infty$. By Theorem 3, $F \in AC_r(E_n)$ for each n. Therefore $F \in ACG_r([a, b])$.

Conversely, let $F \in ACG_r([a, b])$. Then, by definition, there exists a sequence $\{E_n\}_{n=1}^\infty$ such that $[a, b] = \bigcup_{n=1}^\infty E_n$ and $F \in AC_r(E_n)$ for each n. Taking into consideration Remark 1, we see that the variational measure V_F is absolutely continuous on each E_n and so, by its subadditivity, on $[a, b]$. □

The class $ACG_r([a, b])$ was used in [24] to give the following descriptive characterization of the HK_r-integral.

Theorem 5. *A function f is HK_r-integrable on $[a, b]$ if and only if there exists $F \in ACG_r[a, b]$ such that $F'_r = f$ a.e.; the function $F(x) - F(a)$ being the indefinite HK_r-integral of f.*

Combining Theorem 4 with Theorem 5, we obtain the following new characteristic property of the HK_r-integral.

Theorem 6. *If a function $F \in L^r[a,b]$ is L^r-differentiable almost everywhere on $[a,b]$, then it generates an absolutely continuous variational measure on $[a,b]$ if and only if it is the indefinite HK_r-integral of its L^r-derivative F'_r.*

This theorem is clearly equivalent to Theorem 1 which was proved in Sec. 3 in a simpler way, independently of the more technical and deeper Theorems 2–4.

Remark 2. It is an open problem whether the a priori assumption on L^r-differentiability of F can be dropped in Theorems 1, 5 and 6. More precisely: is the class of functions generating absolutely continuous L^r-variational measure (or, equivalently, the class $ACG_r[a,b]$) coincide with the class of indefinite HK_r-integrals? In other words: is each function in those classes L^r-differentiable almost everywhere? Note that in the case of the classical Kurzweil–Henstock integral, the answer to these questions is positive (see [17], [18]).

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