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SIMPLE GERMS OF SKEW-SYMMETRIC MATRIX FAMILIES WITH ODDNESS OR EVENNESS PROPERTIES

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We consider families of skew-symmetric matrices depending analytically on parameters. We obtain necessary existence conditions and find normal forms for such families. The results obtained are generalized to the case of families possessing the evenness or oddness property in the totality of variables. Bibliography: 13 *titles.*

1 Introduction

Families of matrices depending analytically on parameters naturally arise in the study of problems of differential geometry. In [1], Arnold solved the problem of finding a simple normal form to which some given matrix (and any family of close matrices) can be reduced by using a transformation depending smoothly on matrix entries. The reduction of matrices to such a normal form was applied to the study of singularities of bifurcation diagram of matrix families of general position.

The results obtained in the theory of matrix singularities are closely connected with the results of the classical singularity theory. For example, the listed below publications use the results of [2] where the classification of simple germs of analytic functions of many variables with a critical point at the origin was obtained.

It is natural to consider matrix families up to a $\widetilde{\mathscr{G}}$- or $\mathscr{G}$-equivalence, i.e., up to a linear parameter-dependent changes of the basis and analytic (in both directions) changes of parameters (see Definitions 2.5 and 2.6).

This paper is devoted to the study of $\mathscr{G}$-simple germs, i.e., germs with finitely many adjacent orbits (see Definitions 2.7 and 2.8) of skew-symmetric matrix families. In particular, we obtain necessary conditions for the existence of $\mathscr{G}$-simple germs, necessary conditions for the

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$\mathscr{G}$-simplicity of a germ with a given rank of the 1-jet, and normal forms of $\mathscr{G}$-simple germs with 1-jet of corank 0. We also generalize these problems to the case of germs of skew-symmetric matrix-valued functions that are equivariant or invariant under the action of the group $\mathbb{Z}_2$.

Interest in this problem was caused by the following two results. The result due to Bruce and Tari [3] who obtained a complete classification of $\mathscr{G}$-simple germs of square matrices. The classification of $\mathscr{G}$-simple germs of symmetric matrices was obtained by Bruce in [4]. It turned out that the discriminants of some $\mathscr{G}$-simple germs of both families correspond to simple germs A_k, D_k and E_6, E_7, E_8 obtained by Arnold in [2].

Furthermore, G. J. Haslinger established (Ph.D. Thesis, 2001) a connection between the classification of $\mathscr{G}$-simple germs of skew-symmetric matrices of order 2, 3 and the classification of simple germs of maps, as well as a complete classification of $\mathscr{G}$-simple germs of one-parameter skew-symmetric matrix families of any order, two-parameter skew-symmetric matrix families of order 4, and a partial classification of $\mathscr{G}$-simple germs of three-parameter skew-symmetric matrix families of order 4.

We emphasize that the results of this paper does not repeat the results of G. J. Haslinger.

The paper is organized as follows. Section 2 contains the main definition and notation. In Section 3, we formulate the problem and known results for the problem under consideration and some other related problems. The main results of this paper are formulated in Section 4 and are proved in Section 5.

2 Definitions and Notation

We introduce the notation: $\mathbb{K}$ is a field of real $\mathbb{R}$ or complex $\mathbb{C}$ numbers, $n \geqslant 2$ is the matrix order, and $r \geqslant 1$ is the number of parameters on which the entries of matrices depend.

2.1. Germs of analytic functions. We denote by $\mathscr{O}_r$ the ring of germs of analytic functions $f\colon (\mathbb{K}^r, 0) \to (\mathbb{K}, 0)$ and by $\mathrm{j}_k\mathscr{O}_r$ the space of k-jets of germs of functions $f \in \mathscr{O}_r$. Let G be a group with action on $(\mathbb{K}^r, 0)$ and $(\mathbb{K}, 0)$.

Definition 2.1. A germ of a function $f \in \mathscr{O}_r$ is *equivariant* under the actions of the group G if for all $\mathbf{x} \in \mathbb{K}^r$ and $g \in G$

$$f(g \cdot \mathbf{x}) = g \cdot f(\mathbf{x}).$$

On the set of germs of analytic functions that are equivariant under the actions of the group G on the preimage and image, a natural equivalence relation arises.

Definition 2.2. Two equivariant germs $f, g \in \mathscr{O}_r$ are called *equivariantly right-equivalent* (or $\mathscr{R}^G$*-equivalent*) under the actions of G if there exists an equivariant (under the same action) germ of a diffeomorphism $\Phi\colon (\mathbb{K}^r, 0) \to (\mathbb{K}^r, 0)$ such that $g = f \circ \Phi$.

Germs of analytic functions with a critical point at the origin can degenerate in a complicated way, which means that the classification of such germs contains moduli (continuous parameters). However, there are specific degenerations in a neighborhood of which there are no moduli.

Definition 2.3. A germ $f \in \mathscr{O}_r$ is *simple* under given actions of G (G*-simple*) if for sufficiently large $k \in \mathbb{N}$

(1) a sufficiently small neighborhood of some point in its orbit (under the action of the group G) in $\mathrm{j}_k\mathscr{O}_r$ intersects only a finite number of other orbits, called *adjacent* to the orbit of the germ f,

(2) the number of orbits remains bounded as $k \to \infty$.

2.2. Germs of matrix-valued functions. We introduce similar notions for germs of matrix-valued functions.

We consider germs of matrix-valued analytic functions $A\colon \mathbb{K}^r \to \mathrm{Mat}_{\,m\times l}(\mathbb{K}) \cong \mathbb{K}^N$, where $N = m \cdot l$. They are matrices such that each its entry is a germ of some analytic function of r variables (parameters).

Let $V \cong W \cong \mathbb{K}^n$ be n-dimensional vector spaces over the field $\mathbb{K}$. Introduce the following notation:

— $\mathrm{GL}\,(V) \cong \mathrm{GL}\,(W) \cong \mathrm{GL}_{\,n}(\mathbb{K})$ is the group of invertible $n \times n$-matrices over the field $\mathbb{K}$,

— $\mathrm{Hom}\,(V, W)$ is the space of linear mappings $V \to W$, equipped with the direct product of groups $\mathrm{GL}\,(V) \times \mathrm{GL}\,(W)$ (namely, the action of $(X, Y) \in \mathrm{GL}\,(V) \times \mathrm{GL}\,(W)$ on a mapping $A \in \mathrm{Hom}\,(V, W)$ is given by $(X, Y) \cdot A = X^{-1}AY$). Each element of $\mathrm{Hom}\,(V, W)$ is given by a square $n \times n$-matrix.

The following spaces can be identified with spaces of matrices of the corresponding form:

— $\mathrm{Hom}\,(V, W)$ is the space of square $n \times n$-matrices over $\mathbb{K}$,

— $(S^2V)^*$ is the space of symmetric $n \times n$-matrices over $\mathbb{K}$,

— $(\bigwedge^2 V)^*$ is the space of skew-symmetric $n \times n$-matrices over $\mathbb{K}$.

Remark 2.1. Each of the above spaces can be identified with the space $\mathbb{K}^N$, where N is the dimension of the space of matrices of the corresponding form.

Definition 2.4. Let A be a germ of an analytic matrix-valued function of r parameters. Its 1-jet $\mathrm{j}_1 A$ can be regarded as a linear mapping from $\mathbb{K}^r$ to $\mathbb{K}^N$, where N is the dimension of the space of matrices of the corresponding form. By the *rank of 1-jet* we mean the rank of this mapping.

We introduce the notation for groups whose actions define equivalence relations on the set of matrix-valued germs:

— $\mathscr{R}$ is the group of germs of zero-preserving analytic (in both sides) automorphisms of the pregame $(\mathbb{K}^r, 0) \to (\mathbb{K}^r, 0)$;

— $\mathscr{L}$ is the group of germs of zero-preserving analytic (in both sides) autormorphisms of the image $(\mathbb{K}^N, 0) \to (\mathbb{K}^N, 0)$;

— $\mathscr{H}$ is the group of germs of analytic mappings $(\mathbb{K}^r, 0) \to \mathrm{GL}\,(V)$, and $\widetilde{\mathscr{H}}$ is the group of germs of analytic mappings $(\mathbb{K}^r, 0) \to \mathrm{GL}\,(V) \times \mathrm{GL}\,(W)$,

— $\mathscr{G} = \mathscr{R} \ltimes \mathscr{H}$, $\widetilde{\mathscr{G}} = \mathscr{R} \ltimes \widetilde{\mathscr{H}}$ is the semidirect product of groups,

— $\mathscr{C}$ is the group of germs of automorphisms $\{\varphi_x \in \mathscr{L} \mid x \in \mathbb{K}^r\} = \{\varphi\colon (\mathbb{K}^r \times \mathbb{K}^N, 0) \to (\mathbb{K}^r \times \mathbb{K}^N, 0)\}$, where φ preserves the subspace $\mathbb{K}^r \times \{0\}$ and the projection on $\mathbb{K}^r$,

— $\mathscr{K} = \mathscr{R} \ltimes \mathscr{C}$ is the *contact group.*

Remark 2.2. The group $\mathscr{G}$ can be regarded as a subgroup of the corresponding contact group $\mathscr{K}$: $\mathscr{G} = \mathscr{R} \ltimes \mathscr{H} \subset \mathscr{R} \ltimes \mathscr{C} = \mathscr{K}$ (see [5]).

We introduce the notions of $\widetilde{\mathscr{G}}$- and $\mathscr{G}$-equivalence for square and symmetric (skew-symmetric) matrices.

Definition 2.5. Germs of analytic functions $A, B\colon (\mathbb{K}^r, 0) \to \mathrm{Hom}\,(V, W)$ are said to be $\widetilde{\mathscr{G}}$-*equivalent* if there exists an element $\big(\varphi, (X, Y)\big) \in \widetilde{\mathscr{G}}$ such that $B = X^{-1}(A \circ \varphi^{-1})Y$.

Definition 2.6. Germs of analytic functions A, $B\colon (\mathbb{K}^r, 0) \to (S^2V)^*$ (or $(\mathbb{K}^r, 0) \to (\bigwedge^2 V)^*$ are said to be $\mathscr{G}$-*equivalent* if there exists an element $(\varphi, X) \in \mathscr{G}$ such that $B = X^T(A \circ \varphi^{-1})X$.

We denote by E_s the block-diagonal matrix with standard skew-symmetric 2×2-blocks on the main diagonal

$$E_s = \underbrace{\begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & \text{\Large 0} \\ & \ddots & \\ \text{\Large 0} & & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{pmatrix}}_{2\times 2-blocks}. \tag{2.1}$$

By the *direct sum* of square matrices A and B of order m and n respectively we mean the block matrix $C = A \oplus B$ of order $m + n$ defined by

$$C = A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}. \tag{2.2}$$

Remark 2.3. For square, symmetric, and skew-symmetric matrices with parameters the splitting lemma holds (see [3, Proposition 4.1], [4, Proposition 4.1], and Proposition 4.3.5 in Ph.D. Thesis, 2001 by G. J. Haslinger respectively), which allows us to reduce the general case to consideration only those germs of matrix-valued analytic functions of the corresponding form which vanish for the zero parameters. We formulate this lemma only in the skew-symmetric case discussed in this paper.

Lemma 2.1. *Any germ of an analytic function* $A\colon (\mathbb{K}^r, 0) \to (\bigwedge^2 \mathbb{K}^n)^*$ *is* $\mathscr{G}$*-equivalent to a germ of the form* $E_s \oplus B$*, where the matrix* E_s *has the form* (2.1)*, the direct sum of matrices is defined by formula* (2.2)*, and* $B\colon (\mathbb{K}^r, 0) \to (\bigwedge^2 \mathbb{K}^{n-2s})^*$ *is a germ of an analytic function such that* $B(0) = 0$.

In the case of germs of matrix-valued functions, the notions of an adjacent orbit and a simple germ take the following form.

Definition 2.7. An orbit $\mathrm{j}_k(\widetilde{\mathscr{G}}\cdot B)$ (respectively, $\mathrm{j}_k(\mathscr{G}\cdot B)$) of a germ $B\colon (\mathbb{K}^r, 0) \to \mathrm{Hom}\,(V, W)$ (respectively, $B\colon (\mathbb{K}^r, 0) \to (S^2V)^*$ or $B\colon (\mathbb{K}^r, 0) \to (\bigwedge^2 V)^*$) in the space of k-jets of germs of square (respectively, symmetric or skew-symmetric) matrix-valued analytic functions is said to be *adjacent* to an orbit $\mathrm{j}_k(\widetilde{\mathscr{G}} \cdot A)$ (respectively, $\mathrm{j}_k(\mathscr{G} \cdot A)$) of a germ $A\colon (\mathbb{K}^r, 0) \to \mathrm{Hom}\,(V, W)$ (respectively, $A\colon (\mathbb{K}^r, 0) \to (S^2V)^*$ or $(\mathbb{K}^r, 0) \to (\bigwedge^2 V)^*$) if any neighborhood (in the jet space) some (and, consequently, any) point of the orbit $\mathrm{j}_k(\widetilde{\mathscr{G}} \cdot A)$ (respectively, $\mathrm{j}_k(\mathscr{G} \cdot A)$) intersects the orbit $\mathrm{j}_k(\widetilde{\mathscr{G}} \cdot B)$ (respectively, $\mathrm{j}_k(\mathscr{G} \cdot B)$).

Definition 2.8. A germ of an analytic functions $A\colon (\mathbb{K}^r, 0) \to \mathrm{Hom}\,(V, W)$ (respectively, $A\colon (\mathbb{K}^r, 0) \to (S^2V)^*$ or $(\mathbb{K}^r, 0) \to (\bigwedge^2 V)^*$) is said to be $\widetilde{\mathscr{G}}$-*simple* (respectively, $\mathscr{G}$-*simple*) if the number of orbits adjacent to the orbit $\mathrm{j}_k(\widetilde{\mathscr{G}} \cdot A)$ (respectively, $\mathrm{j}_k(\mathscr{G} \cdot A)$) in the space of k-jets of germs of square (respectively, symmetric or skew-symmetric) matrix-valued analytic functions is finite for all sufficiently large $k \in \mathbb{N}$ and is bounded as $k \to \infty$.

The notion of finite definiteness can be also extended to germs of matrix-valued functions. Since we mainly deal with skew-symmetric matrices, we adapt the definition to this case.

Definition 2.9. A germ A of an analytic function $A\colon (\mathbb{K}^r,0)\to(\bigwedge^2 V)^*$ is *k-$\mathscr{G}$-definite* if any germ $B\colon (\mathbb{K}^r,0)\to(\bigwedge^2 V)^*$ with the same k-jet is $\mathscr{G}$-equivalent to the germ A. If there exists $k\in\mathbb{N}$ such that the germ A is k-$\mathscr{G}$-definite, then A is said to be *$\mathscr{G}$-finitely definite*.

2.3. Germs of matrix-valued functions, even or odd in the totality of parameters. Let $A\colon (\mathbb{K}^r,0)\to(\bigwedge^2 V)^*$ be a germ of a skew-symmetric matrix-valued analytic function. Assume that a finite Abelian group G linearly acts on the space of parameters $\mathbb{K}^r$ and the image $\mathbb{K}^N$, $N=\frac{n(n-1)}{2}$.

Definition 2.10. A germ of a function $A\colon \mathbb{K}^r\to\mathbb{K}^N$ is *equivariant* under the actions of the group G if for all $\mathbf{x}\in\mathbb{K}^r$ and $g\in G$

$$A(g\cdot\mathbf{x})=g\cdot A(\mathbf{x}).$$

If the action of G on $\mathbb{K}^N$ is trivial, we say that A is *invariant* under the action of G on $\mathbb{K}^r$.

If A is a germ of a skew-symmetric matrix-valued analytic function that is even in the totality of variables i.e., for any $\mathbf{x}=(x_1,\dots,x_r)\in\mathbb{K}^r$ the equality $A(\mathbf{x})=A(-\mathbf{x})$ holds, then it is invariant under the action of the group $\mathbb{Z}_2$ on the space of parameters $\mathbb{K}^r_{(x_1,\dots,x_r)}$, given by

$$\sigma\cdot(x_1,\dots,x_r)=(-x_1,\dots,-x_r)$$

(here and below, σ denotes the generator of the group $\mathbb{Z}_2$). We denote by $\mathscr{O}^{\text{even}}_{r,N}$ the ring of such germs with respect to the standard operation of addition and multiplication of matrices and by $\mathscr{O}^{\text{even}}_{r}$ the ring of germs of analytic functions $f\colon(\mathbb{K}^r,0)\to(\mathbb{K},0)$, even in the totality of variables.

If A is a germ of a skew-symmetric matrix-values analytic function, odd in the totality of variables, i.e., for any $\mathbf{x}=(x_1,\dots,x_r)\in\mathbb{K}^r$ the equality $A(\mathbf{x})=-A(-\mathbf{x})$ holds, then it is equivariant under the nontrivial scalar actions of the group $\mathbb{Z}_2$ on the space of parameters $\mathbb{K}^r_{(x_1,\dots,x_r)}$ and the image $\mathbb{K}^N_{(y_1,\dots,y_N)}$ given by the formula

$$\sigma\cdot(x_1,\dots,x_r;y_1,\dots,y_N)=(-x_1,\dots,-x_r;-y_1,\dots,-y_N).$$

The space of such germs is denoted by $\mathscr{O}^{\text{odd}}_{r,N}$. This space is not a ring, but it has structure of an $\mathscr{O}^{\text{even}}_{r,N}$-module.

We denote by $\mathscr{O}^{\text{odd}}_{r}$ the space of germs of analytic functions $f\colon(\mathbb{K}^r,0)\to(\mathbb{K},0)$, odd in the totality of variables. We denote by $\mathscr{H}^{\text{even}}$, $\mathscr{R}^{\text{odd}}$, $\mathscr{G}^{\text{odd}}$, $\mathscr{K}^{\text{odd}}$ the subgroup of the corresponding group $\mathscr{R}$, $\mathscr{H}$, $\mathscr{G}$, $\mathscr{K}$ generated by germs of the corresponding mappings, even or odd in the totality of variables.

The symbol j_k indicates that we deal with quotient spaces (group) of k-jets of elements of this space (group).

Definition 2.11. A germ $A\in\mathscr{O}^{\text{odd}}_{r,N}$ (respectively, $A\in\mathscr{O}^{\text{even}}_{r,N}$) is said to be *$\mathscr{G}^{\text{odd}}$-simple* if the number of orbits adjacent to the orbit $\mathrm{j}_k(\mathscr{G}^{\text{odd}}\cdot A)$ in the space $\mathrm{j}_k\mathscr{O}^{\text{odd}}_{r,N}$ (respectively, $\mathrm{j}_k\mathscr{O}^{\text{even}}_{r,N}$) is finite for all sufficiently large $k\in\mathbb{N}$ and is bounded as $k\to\infty$.

As in the case of $\mathscr{G}$-equivalence, we can introduce the equivalence relation in the space of germs of skew-symmetric matrix-valued analytic functions with preserving evenness or oddness of a germ of a matrix-valued function in $\mathscr{O}^{\text{even}}_{r,N}$ and $\mathscr{O}^{\text{odd}}_{r,N}$ respectively.

Definition 2.12. Germs of analytic functions $A, B\colon (\mathbb{K}^r, 0) \to (\bigwedge^2 V)^*$ are $\mathscr{G}^{\,\text{odd}}$- *equivalent* if there exists $(\varphi, X) \in \mathscr{G}^{\,\text{odd}} = \mathscr{R}^{\,\text{odd}} \ltimes \mathscr{H}^{\,\text{even}}$ such that $B = X^T(A \circ \varphi^{-1})X$.

We also introduce the $V^{\,\text{odd}}$-equivalence relation on the set of germs of functions $\mathscr{O}_r$.

Definition 2.13. Germs $f, g \in \mathscr{O}_r$ are $V^{\,\text{odd}}$-*equivalent* if there exists a germ $\varphi \in \mathscr{R}^{\,\text{odd}}$ and a germ of the invertible function $h + c$, where $h \in \mathscr{O}_r^{\text{even}}$, $c \in \mathbb{K} \setminus \{0\}$, such that $f \circ \varphi = g \cdot h$.

Remark 2.4. Germs f and g are $V^{\,\text{odd}}$-equivalent if and only if germs of the hypersurfaces $\{f = 0\}$ and $\{g = 0\}$ are $\mathscr{R}^{\,\text{odd}}$-equivalent.

The $V^{\,\text{odd}}$-equivalence relation is associated with the notion of the $V^{\,\text{odd}}$-simplicity of a germ.

Definition 2.14. A germ $f \in \mathscr{O}_r^{\text{odd}}$ (respectively, $f \in \mathscr{O}_r^{\text{even}}$) is $V^{\,\text{odd}}$-*simple* if the number of the $V^{\,\text{odd}}$-equivalence classes intersecting a sufficiently small neighborhood of some element of the $V^{\,\text{odd}}$-equivalence class of the jet $\mathrm{j}_k f$ in the space $\mathrm{j}_k\mathscr{O}_r^{\text{odd}}$ (respectively, $\mathrm{j}_k\mathscr{O}_r^{\text{even}}$), is finite for all sufficiently large $k \in \mathbb{N}$ and bounded as $k \to \infty$.

3 Statement of the Problem and Survey of Results

It is a general problem to classify up to a $\widetilde{\mathscr{G}}$- or $\mathscr{G}$-equivalence relation germs of matrix-valued functions $A\colon (\mathbb{K}^r, 0) \to (\mathbb{K}^N, 0)$ with a critical point at the origin, where r is the number of parameters and N is the dimension of the space of matrices of the corresponding form.

The complete classification was obtained in [3, Theorem 1.1] for square $\mathscr{G}$-simple matrix families and in [4, Theorem 1.1] for symmetric $\mathscr{G}$-simple matrix families. The study of simple singularities of symmetric matrices was continued in [6]. It turned out that these singularities are classified by subgroups generated by mappings in the Weyl groups A_ν, D_ν, E_ν (see [6, Table 1]). Furthermore, the corresponding groups of even monodromy of a determinant curve (see [6, § 5]) were described for each singularity.

G. J. Haslinger considered (Ph.D. Thesis, 2001) skew-symmetric $\mathscr{G}$-simple matrix families in the following cases:

1) $r = 1$,
2) $n = 2$ and $r > 1$,
3) $n = 3$ and $r > 1$,
4) $n = 4$ and $r = 2$ (the complete classification of simple families,
5) $n = 4$ and $r = 3$ (the partial classification of simple families).

The classification of $\mathscr{R}$-simple germs $(\mathbb{C}^r, 0) \to (\mathbb{C}, 0)$ was obtained by Arnold [2, Theorem 2.10]. Now, we list some classifications with references to the corresponding works.

The classification of simple projections of surfaces on manifolds of not greater dimension and, respectively, the classification of $\mathscr{K}$-simple germs $(\mathbb{C}^r, 0) \to (\mathbb{C}^3, 0)$ [7].

The classification of simple singularities of functions on a manifold with boundary (simple boundary singularities), i.e., invariant under the action of the group $\mathbb{Z}_2$ on $\mathbb{C}^n$ with respect to the first variable [8].

The classification of simple odd singularities i.e., those that are equivariantly simple with respect to nontrivial scalar actions of the group $\mathbb{Z}_2$ on $\mathbb{K}^r$ and $\mathbb{K}$:

$$\sigma \cdot (z_1, \dots, z_r; w) = (-z_1, \dots, -z_r; -w).$$

In particular, it was proved that there are no such singularities if $r \geqslant 3$ [9].

As shown in [10], any germ of an analytic function $(\mathbb{C}^{r_1+r_2}, 0) \to (\mathbb{C}, 0)$ with critical point $0 \in \mathbb{C}^{r_1+r_2}$ is invariantly simple under the action of the group $G = \mathbb{Z}_2^{r_1}$ with generators $\sigma_1, \ldots, \sigma_{r_1}$ on $\mathbb{C}^{r_1+r_2}$:

$$\sigma_j \cdot (x_1, \ldots, x_{r_1}, y_1, \ldots, y_{r_2}) = (x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_{r_1}, y_1, \ldots, y_{r_2}),$$

equivalent to one of the simple boundary singularities after a suitable permutation of variables.

The classification of germs of analytic functions $(\mathbb{K}^r, 0) \to (\mathbb{K}, 0)$ that are equivariantly simple under the actions of the group $G = (\mathbb{Z}_2)^r$ with generators $\sigma_1, \ldots, \sigma_r$ on the preimage and image:

$$\sigma_j \cdot (x_1, \ldots, x_r; y) = (x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_r, \varepsilon_j y), \quad j \in \{1, \ldots, r\},$$

where $\varepsilon_j = -1$ if $1 \leqslant j \leqslant p$ and $\varepsilon_j = 1$ if $p+1 \leqslant j \leqslant r$ [11].

Within the framework of the general problem, the main goal of this paper is to obtain new results on classification of simple germs of skew-symmetric matrices, up to a given equivalence relation. In particular, it is required to find

1) necessary conditions for the existence of $\mathscr{G}$-simple germs of skew-symmetric matrix-valued functions in terms of the matrix sizes and the number of parameters,
2) normal forms of $\mathscr{G}$-simple germs of skew-symmetric matrix-valued functions with 1-jet of corank 0,
3) necessary conditions for the $\mathscr{G}$-simplicity of a germ of a skew-symmetric matrix-valued function in terms of the matrix sizes and the rank of 1-jet,
4) classification of $\mathscr{G}^{\mathrm{odd}}$-simple germs of skew-symmetric matrix-valued functions with one parameter, even or odd with respect to the parameter,
5) connection between the classification of $\mathscr{G}^{\mathrm{odd}}$-simple germs of families of skew-symmetric matrices of sizes 2×2 and 3×3, even or odd in the totality of variables and the classification of simple germs of the mappings of corresponding vector space.

4 The Main Results

Denote by $n \geqslant 2$ the matrix order and by $r \geqslant 1$ the number of parameters. We recall that we consider only germs A such that $A(0) = 0$ (see Remark 2.3). In this section, $N = n(n-1)/2$.

4.1. Necessary condition for the existence of simple germs.

Theorem 4.1. 1. *If one of the following conditions is satisfied*:

1) $n = r = 5$,
2) $n \geqslant 6$ *and* $3 \leqslant r \leqslant \frac{n(n-1)}{2} - 3$,

then there are no $\mathscr{G}$*-simple germs of skew-symmetric matrix-valued analytic functions of order* n *that depend on* r *parameters and vanish for the zero value of the parameter.*

2. *If one of the following conditions is satisfied*:

1) $n \geqslant 3$ *and* $r \geqslant 3$,

2) $n \geqslant 5$ *and* $r = 2$.

then there are no $\mathscr{G}^{\,\mathrm{odd}}$*-simple germs of skew-symmetric matrix-valued analytic functions of order* n *that depend on* r *parameters, are even in the totality of variables, and vanish for the zero parameters.*

3. *If one of the following conditions is satisfied*:

1) $n = r = 5$,
2) $n \geqslant 6$ *and* $3 \leqslant r \leqslant \frac{n(n-1)}{2} - 3$,

then there are no $\mathscr{G}^{\,\mathrm{odd}}$*-simple germs of skew-symmetric matrix-valued analytic functions that have order* n*, depend on* r *parameters, and are odd in the totality of parameters.*

4.2. Normal form of germs with 1-jet of corank 0.

Theorem 4.2. *Let* A *be a germ of a skew-symmetric matrix-valued analytic function of* n *order that depends on* r *parameters and vanishes for the zero parameters. Let a germ* A *have the rank of* 1*-jet* (*see Definition* 2.4) *equal to* $s = n(n-1)/2$*, i.e., maximal among all possible ones.*

1. *A germ* A *is* $\mathscr{G}$*-simple and* $\mathscr{G}$*-equivalent to the germ*

$$\begin{pmatrix} 0 & x_{1,2} & \dots & \dots & x_{1,n} \\ -x_{1,2} & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & x_{n-1,n} \\ -x_{1,n} & \dots & \dots & -x_{n-1,n} & 0 \end{pmatrix}, \tag{4.1}$$

where $x_{i,j} (1 \leqslant i < j \leqslant n)$ *are pairwise distinct parameters.*

2. *If, under the above assumptions, the germ* A *is also a germ of a function that is odd in the totality of parameters, then* A *is* $\mathscr{G}^{\,\mathrm{odd}}$*-simple and* $\mathscr{G}^{\,\mathrm{odd}}$*-equivalent to a germ of the form* (4.1).

Remark 4.1. Since for germs of functions that are even in the totality of variables the rank of 1-jet is always equal to zero, for such germs there are no counterparts of Theorem 4.2.

4.3. Necessary condition for the simplicity of a germ in terms of matrix sizes and rank of 1-jet.

Theorem 4.3. *Let* A *be a germ of a skew-symmetric matrix-valued analytic function of order* n *that depends on* r *parameters and vanishes for the zero parameters. Let the germ* A *have the rank of* 1*-jet* s (*see Definition* 2.4).

1. *If one of the following conditions is satisfied*:

1) $n = s = 5$,
2) $n \geqslant 6$ *and* $3 \leqslant s \leqslant N - 3$,
3) $n \geqslant 3, r \geqslant 3$ *and* $s = 0$,

4) $n \geqslant 5, r = 2$ *and* $s = 0$,

then the germ A *is not* $\mathscr{G}$*-simple.*

2. *If the germ* A *is also a germ of a function that is odd in the totality of variables and one of the following conditions is satisfied*:

1) $n = s = 5$,
2) $n \geqslant 6$ *and* $3 \leqslant s \leqslant N - 3$,
3) $n \geqslant 2$, $r \geqslant 4$, *and* $s = 0$,
4) $n \geqslant 4$, $r = 2$, *and* $s = 0$,
5) $n \geqslant 3, r = 3$, *and* $s = 0$,

then the germ A *is not* $\mathscr{G}^{\,\mathrm{odd}}$*-simple.*

Remark 4.2. Since for germs of functions that are even in the totality of variables the rank of 1-jet is always zero, for such germs there are no counterparts of Theorem 4.3.

4.4. Classification of simple germs with one parameter.

Theorem 4.4. 1. *A germ of a skew-symmetric matrix-valued analytic function of order* n *that depends on one parameter, is even with respect to this parameter, and vanishes for the zero parameter is* $\mathscr{G}^{\,\mathrm{odd}}$*-equivalent to the germ*

$$x^{2k_1}E_{s_1} \oplus x^{2k_2}E_{s_2} \oplus \ldots \oplus x^{2k_t}E_{s_t} \oplus 0,$$

where the matrices E_{s_i} *have the form* (2.1), *the direct sum of matrices is defined by formula* (2.2), $s_i \in \mathbb{N}$, $k_i \in \mathbb{N}$ *are such that* $k_1 < k_2 < \ldots < k_t$, *and the last zero block has sizes* $(n - 2\sum i = 1^t s_i)$ *and can be absent. Moreover, if the last zero block has sizes* 1×1 *or is absent, then the germ is* $\mathscr{G}^{\,\mathrm{odd}}$*-simple*; *otherwise, it is not* $\mathscr{G}^{\,\mathrm{odd}}$*-simple.*

2. *A germ of a skew-symmetric matrix-valued analytic function of order* n *that depends on one parameter and is odd with respect to this parameter is* $\mathscr{G}^{\,\mathrm{odd}}$*-equivalent to the germ*

$$x^{2k_1+1}E_{s_1} \oplus x^{2k_2+1}E_{s_2} \oplus \ldots \oplus x^{2k_t+1}E_{s_t} \oplus 0,$$

where E_{s_i} *have the form* (2.1), *the direct sum of matrices is defined by* (2.2), $s_i \in \mathbb{N}$, *the numbers* $k_i \in \mathbb{Z}_{\geqslant 0}$ *are such that* $k_1 < k_2 < \ldots < k_t$, *and the last zero block has sizes* $(n - 2\sum i = 1^t s_i)$ *or can be absent. Moreover, if the last zero block has sizes* 1×1 *or is absent, then the germ is* $\mathscr{G}^{\,\mathrm{odd}}$*-simple*; *otherwise, it is not* $\mathscr{G}^{\,\mathrm{odd}}$*-simple.*

4.5. Connection between the classification of simple germs of families of skew-symmetric 2×2-matrices that are even or odd in the totality of variables and the classification of simple germs of mappings.

Theorem 4.5. *Germs* A *and* B *of skew-symmetric matrix-valued analytic functions of order* $n = 2$ *that depend on* r *parameters, are even* (*odd*) *in the totality of variables, vanish for the zero parameters, and have the form*

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix},$$

where the germs a *and* b *of analytic functions* $(\mathbb{K}^r, 0) \to (\mathbb{K}, 0)$ *that are even* (*odd*) *in the totality of variables are* $\mathscr{G}^{\,\mathrm{odd}}$*-equivalent if and only if* a *and* b *are* $V^{\,\mathrm{odd}}$*-equivalent.*

Corollary 4.1. *The classification of $\mathscr{G}^{\,\mathrm{odd}}$-simple even (odd) germs is determined by the classification of $V^{\,\mathrm{odd}}$-simple even (odd) germs $(\mathbb{K}^r,0) \to (\mathbb{K},0)$.*

4.6. Connection between the classification of simple germs of families of skew-symmetric 3×3-matrices that are even or odd in the totality of variables and the classification of simple germs of mappings.

Theorem 4.6. 1. *A germ A of a skew-symmetric matrix-valued analytic function of order $n = 3$ that depends on r parameters, is even (odd) in the totality of variables, vanishes for the zero parameters, and has the form*

$$A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

is $\mathscr{G}^{\,\mathrm{odd}}$-finitely definite if and only if the corresponding even (odd) germ $(a,b,c)\colon (\mathbb{C}^r,0) \to (\mathbb{C}^3,0)$ is $\mathscr{K}^{\,\mathrm{odd}}$-finitely definite.

2. *Two $\mathscr{G}^{\,\mathrm{odd}}$-finitely definite germs A and B of skew-symmetric matrix-valued analytic functions of order $n = 3$ that depend on r parameters, are even (odd) in the totality of variables, and vanish for the zero parameters are $\mathscr{G}^{\,\mathrm{odd}}$-equivalent if and only if the corresponding even (odd) germs $(\mathbb{C}^r,0) \to (\mathbb{C}^3,0)$ are $\mathscr{K}^{\,\mathrm{odd}}$-equivalent. In particular, the classification of $\mathscr{G}^{\,\mathrm{odd}}$-simple germs is determined by the classification of $\mathscr{K}^{\,\mathrm{odd}}$-simple germs $(\mathbb{C}^r,0) \to (\mathbb{C}^3,0)$.*

5 Proof of the Main Results

Before proving Theorems 4.1–4.6, we prove two auxiliary results.

Let **Sk** be the space of germs of analytic mappings $A\colon (\mathbb{K}^r,0) \to (\bigwedge^2 V)^* \cong \mathbb{K}^N$, where $N = n(n-1)/2$.

The following lemma is a criterion for the existence of moduli (see [3, Proposition 4.4]).

Lemma 5.1. *Let G be an algebraic group acting on the affine space L over the field $\mathbb{K}$, and let S be a smooth irreducible algebraic submanifold in L such that the set $\{s \in S\colon T_sS \subset T_s(G\cdot s)\}$ is a proper algebraic submanifold of S. Then any neighborhood U of any point $s \in S$ contains infinitely many orbits of the action of the group G.*

The following lemma asserts the finite definiteness for matrix-valued functions (the general case can be found in [12, Section 10]).

Lemma 5.2 (on finite definiteness)**.** *A germ of the a function $A\colon (\mathbb{K}^r,0) \to (\mathbb{K}^N,0)$ is $\mathscr{G}$-finitely definite if and only if $\mathscr{M}_r^{k+1}\mathscr{O}_r^N \subset T_A(\mathscr{G}\cdot A)$ for some k.*

Proof of Theorem 4.1. 1. The action of the group $\mathrm{j}_1\mathscr{G}$ on the space of 1-jets in **Sk** can be regarded as the action of the direct product of complete linear groups $\mathrm{GL}\,(\mathbb{K}^r) \times \mathrm{GL}\,(V)$ of dimension $r^2 + n^2$ on the space of linear mappings $\mathrm{Hom}\,(\mathbb{K}^r, (\bigwedge^2 V)^*)$ of dimension $rN = rn(n-1)/2$.

If the dimension of each orbit of a given group is less than the dimension of the space where the group acts, then there exist no simple germs (see Lemma 5.1).

The dimension of each orbit of the group $\mathrm{GL}(\mathbb{K}^r) \times \mathrm{GL}(V)$ does not exceed r^2+n^2-1 since the scalar matrices in the spaces $\mathrm{GL}(\mathbb{K}^r)$ and $\mathrm{GL}(V)$ act in the same way.

Consequently, if

$$r\frac{n(n-1)}{2} > r^2+n^2-1,$$

then there are no simple matrices in **Sk**. We consider this inequality as a square inequality for r with coefficients depending on n. Solving this equation, we find that

1) for $n = 1, 2, 3, 4$ the inequality fails for any r,
2) for $n = 5$ the inequality is satisfied only for $r = 5$,
3) for $n \geqslant 6$ the inequality is satisfied if and only if

$$3 \leqslant r \leqslant \frac{n(n-1)}{2} - 3 = N - 3.$$

2. We consider the action of the group $\mathrm{j}_1\mathscr{G}^{\,\mathrm{odd}} = \mathrm{j}_1(\mathscr{R}^{\,\mathrm{odd}} \ltimes \mathscr{H}^{\,\mathrm{even}})$ on the space of 1-jets in $\mathscr{O}_{r,N}^{\mathrm{even}}$. In this case, the 1-jet $\mathrm{j}_1\mathscr{O}_{r,N}^{\mathrm{even}}$ is zero since it consists of constant matrices such that $A(0) = 0$ (see Remark 2.2).

Since the 1-jet is zero, we consider the space $\mathrm{j}_2\mathscr{O}_{r,N}^{\mathrm{even}}$ of 2-jets consisting of matrices whose entries are homogeneous polynomials of degree 2. The action of the group $\mathrm{j}_1\mathscr{G}^{\,\mathrm{odd}}$ on the space $\mathrm{j}_2\mathscr{O}_{r,N}^{\mathrm{even}}$ of 2-jets can be regarded as the action of the direct product of full linear groups $\mathrm{GL}(\mathbb{K}^r)\times \mathrm{GL}(V)$ of dimension r^2+n^2 on the space of linear mappings $\mathrm{Hom}(\mathbb{K}^r, (\bigwedge^2 V)^*)$ of dimension

$$\frac{r(r+1)}{2}N = \frac{r(r+1)n(n-1)}{4}.$$

If the dimension of each orbit of a given group is less than the dimension of the space where it acts, then there are no simple germs (see Lemma 5.1).

The dimension of each orbit of the group $\mathrm{GL}(\mathbb{K}^r) \times \mathrm{GL}(V)$ does not exceed r^2+n^2-1 since the scalar matrices in $\mathrm{GL}(\mathbb{K}^r)$ and $\mathrm{GL}(V)$ act in the same way.

Consequently, if

$$\frac{n(n-1)r(r+1)}{4} > r^2+n^2-1,$$

then there are no simple matrices in **Sk**.

The obtained inequality is satisfied in the following cases:

1) $n \geqslant 3$ and $r \geqslant 3$,
2) $n \geqslant 5$ and $r = 2$.

3. We consider the action of the group $\mathrm{j}_1\mathscr{G}^{\,\mathrm{odd}} = \mathrm{j}_1(\mathscr{R}^{\,\mathrm{odd}} \ltimes \mathscr{H}^{\,\mathrm{even}})$ on the space $\mathscr{O}_{r,N}^{\mathrm{odd}}$ of 1-jets. We note that the group $\mathrm{j}_1\mathscr{R}^{\,\mathrm{odd}}$ is the group of odd (i.e., all) linear transformations of parameters, $\mathrm{j}_1\mathscr{H}^{\,\mathrm{even}}$ are constant nonsingular matrices of order n and $\mathrm{j}_1\mathscr{O}_{r,N}^{\mathrm{odd}}$ are odd linear mappings $(\mathbb{K}^r, 0) \to (\bigwedge^2 V)^*$. Consequently, this action can be regarded as the action of the direct product of full linear groups $\mathrm{GL}(\mathbb{K}^r)\times\mathrm{GL}(V)$ of dimension r^2+n^2 on the space of linear mappings $\mathrm{Hom}^{\,\mathrm{odd}}(\mathbb{K}^r, (\bigwedge^2 V)^*)$ of dimension $rN = r\frac{n(n-1)}{2}$. The further arguments completely coincide with the first part of the proof. $\square$

Proof of Theorem 4.2. 1. We first consider the 1-jet of the germ A. In the case $s = N$, it has the maximal rank as a mapping $\mathrm{j}_1 A\colon \mathbb{K}^r \to \mathbb{K}^N$. In particular, $r \geqslant s = N$. Making elementary transformations of rows and columns, we can reduce $\mathrm{j}_1 A$ to the required form (4.1).

If $A = \mathrm{j}_1 A + B$, where $B = (b_{i,j}) \in \mathbf{Sk}$ is a matrix with zero 1-jet $\mathrm{j}_1 B = 0$, then, making the change of parameters $\widetilde{x}_{i,j} = x_{i,j} + b_{i,j}$, we can reduce it to the form (4.1).

Let an orbit $\mathrm{j}_k(\mathscr{G} \cdot C)$ in the space $\mathrm{j}_k\mathbf{Sk}$ be adjacent to the orbit $\mathrm{j}_k(\mathscr{G} \cdot A)$ for some $k \geqslant 1$. We also assume that the 1-jet of the germ C is close to the 1-jet of the germ A and, consequently, also has the maximal rank $s = N$. As above, we can reduce $\mathrm{j}_1 C$ and $\mathrm{j}_k C$ to the form (4.1). Thus, any germ $C \in \mathbf{Sk}$ whose orbit is adjacent to the orbit of the germ A in $\mathrm{j}_k\mathbf{Sk}$ is $\mathscr{G}$-equivalent to A. Consequently, A is a $\mathscr{G}$-simple germ.

2. The proof is similar to that of assertion 1. □

Proof of Theorem 4.3. 1. Let $\mathrm{rk}\,(\mathrm{j}_1 A) = s \leqslant N$. We can regard the 1-jet $\mathrm{j}_1 A$ as an element of the space $\mathrm{Hom}\left(\mathbb{K}^r,\ (\bigwedge^2 V)^*\right)$. In this case, the dual mapping $\bigwedge^2 V \to (\mathbb{K}^r)^*$ has kernel of corank s which will be denoted by L_A. The set $\{L \subset \bigwedge^2 V\colon\ \dim L = N - s\}$ of all such subspaces is called the *Grassmanian* and is denoted by $\mathrm{Gr}\,(N-s, N)$ and $\dim \mathrm{Gr}\,(s, N) = s(N-s)$.

On the *Grassmanian* $\mathrm{Gr}\,(N-s, N)$, only the group $\mathrm{GL}\,(V)$ acts since it consists of kernels of dual mappings (the image is the zero matrix). Under the action of a scalar operator, every linear subspace is transformed to itself. Therefore, if $L_A \in \mathrm{Gr}\,(N-s, s)$, then $\dim \mathrm{GL}\,(V) \cdot \mathrm{j}_1 A$ is at most $n^2 - 1$.

Consequently, there are no simple matrices in $\mathbf{Sk}$ if

$$s\Big(\frac{n(n-1)}{2} - s\Big) > n^2 - 1.$$

We note that the obtained inequality is similar up to a replacement of r by s to the inequality obtained in Theorem 4.1. Consequently, we have

1) for $n = 1, 2, 3, 4$ the inequality fails for all s,
2) for $n = 5$ the inequality is satisfied only for $s = 5$,
3) for $n \geqslant 6$ the inequality is satisfied if and only if

$$3 \leqslant s \leqslant \frac{n(n-1)}{2} - 3 = N - 3.$$

Consider the case $s = 0$ where the 1-jet of A is zero. In this case, we pass to the jet space $\mathrm{j}_2\mathbf{Sk}$ consisting of skew-symmetric matrices whose entries are homogeneous polynomials of degree 2. They form a space of dimension $\frac{n(n-1)r(r+1)}{4}$ with the action $\mathrm{j}_2\mathscr{G}$. Similarly, we can reduce the consideration to the action of the group $\mathrm{GL}\,(\mathbb{K}^r) \times \mathrm{GL}\,(V)$ on the space $\mathrm{Hom}\,(\mathbb{K}^r, (\bigwedge^2 V)^*)$, where the orbit dimension does not exceed $r^2 + n^2 - 1$. Consequently, if

$$\frac{n(n-1)r(r+1)}{4} > r^2 + n^2 - 1,$$

then there are no simple matrices in $\mathbf{Sk}$. This inequality is satisfied in the following cases:

1) $n \geqslant 3$ and $r \geqslant 3$,

2) $n \geqslant 5$ and $r = 2$.

2. The case $s \neq 0$ is handled in the same way as in the first part.

We consider the case $s = 0$, i.e., the zero 1-jet of the germ A. Since, in the case of germs of odd functions, the 2-jet is also zero, we immediately pass to consideration of the space of 3-jets $\mathrm{j}_3\mathscr{O}^{\mathrm{odd}}_{r,N}$ whose independent elements are homogeneous polynomials of degree 3. The form a space of dimension $\frac{r(r+1)(r+2)}{6}N$. As above, the action of the group $\mathrm{j}_1\mathscr{G}^{\mathrm{odd}}$ on the space of 3-jets $\mathrm{j}_3\mathscr{O}^{\mathrm{odd}}_{r,N}$ can be regarded as the action of the group $\mathrm{GL}\,(\mathbb{K}^r)\times \mathrm{GL}\,(V)$ on the space $\mathrm{Hom}^{\mathrm{odd}}(\mathbb{K}^r, (\bigwedge^2 V)^*)$ where the orbit dimension does not exceed $r^2 + n^2 - 1$.

Consequently, if

$$\frac{n(n-1)r(r+1)(r+2)}{12} > r^2 + n^2 - 1,$$

then there are no simple matrices in **Sk**. This inequality holds in the following cases:

1) $n \geqslant 2$ and $r \geqslant 4$,

2) $n \geqslant 4$ and $r = 2$,

3) $n \geqslant 3$ and $r = 3$.

The theorem is proved. □

Proof of Theorem 4.4. 1. By Lemma 2.1, it suffices to consider only matrices such that $A(0) = 0$. If A is the zero matrix, then we have the required form (the zero block) and there is nothing to prove.

We prove the theorem by induction on n, where n is the order of the matrix A.

Let the minimal degree of x in the entries of A be equal to $2k_1$, $k_1 \geqslant 1$. This degree is always even since the original matrix A is even in the totality of variables. Then $A = x^{2k_1}A_0$, where $A_0(0) \neq 0$.

Let $\mathrm{rk}\, A_0(0) = 2s_1$, $s_1 \geqslant 1$. Using Lemma 2.2 and replacing the basis by making elementary transformations of rows and columns, we can reduce the matrix A to the form

$$\begin{pmatrix} E_{s_1} & 0 \\ 0 & A_1 \end{pmatrix},$$

where $A_1(0) = 0$ and E_{s_1} is defined by (2.1). If $n = 2s_1$, then we obtain the required form, and the theorem is proved. Otherwise, we assume that the required assertion of the theorem holds for all matrices of order less than n and show that the matrix A_1 of order $n - 2s_1$ can be reduced to the form

$$x^{2l_2}E_{s_2} \oplus \ldots \oplus x^{2l_t}E_{s_t} \oplus 0,$$

where $l_2 < \ldots < l_t$. Then we obtain the required assertion for the matrix A of order n by setting $k_i = l_i + k_1$, $i \geqslant 2$.

If the order of the last zero block is at least 2, then the germ is not $\mathscr{G}^{\mathrm{odd}}$-simple since adjacencies of the form x^k appear in the components of the matrix block and the second condition in Definition 2.8 fails.

2. We argue as above, but, in this case, the minimal degree of the parameter x in the matrix entries is equal to $2k_1 + 1$, $k_1 \geqslant 0$. Consequently, $A = x^{2k_1+1}A_0$, $A_0(0) \neq 0$ and $x^{2l_2+1}E_{s_2} \oplus \ldots \oplus x^{2l_t+1}E_{s_t} \oplus$, where $l_2 < \ldots < l_t$. □

Proof of Theorem 4.5. Let

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix},$$

where $a, b \in \mathscr{O}_r^{\text{even}}$ (respectively, $\mathscr{O}_r^{\text{odd}}$). If the germs of the matrix-valued functions A and B are $\mathscr{G}^{\text{odd}}$-equivalent, then $B = X^T(A \circ \varphi)X$, where $\varphi \in \mathscr{R}^{\text{odd}}$, $X \in \mathscr{H}^{\text{even}}$; namely,

$$\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} 0 & a \circ \varphi \\ -a \circ \varphi & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = (\alpha\delta - \beta\gamma) \begin{pmatrix} 0 & a \circ \varphi \\ -a \circ \varphi & 0 \end{pmatrix}.$$

Consequently, $b = (\alpha\delta - \beta\gamma)(a \circ \varphi)$. Moreover, $h = \alpha\delta - \beta\gamma = \det X \in \mathscr{O}_r^{\text{even}}$, i.e., the function is even in the totality of variables (each entry of the matrix X is an even function) and $h(0) \neq 0$ since $\det X(0) \neq 0$. Therefore, a and b are V^{odd}-equivalent.

Conversely, if a and b are V^{odd}-equivalent, i.e., there exists a germ $\varphi \in \mathscr{R}^{\text{odd}}$ and an invertible function $h + c$, where $h \in \mathscr{O}_r^{\text{even}}$, $c \in \mathbb{K} \setminus \{0\}$, such that $a \circ \varphi = b \cdot h$, then we set

$$X = \begin{pmatrix} h^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $B = X^T(A \circ \varphi)X$, i.e., A and B are $\mathscr{G}^{\text{odd}}$-equivalent. □

Proof of Theorem 4.6. With a germ A of the form (4.6) we can associate the germ of the map $(a, b, c) \colon (\mathbb{K}^r, 0) \to (\mathbb{K}^3, 0)$ which is also even or odd in the totality of variables. Then we argue as above, but with parity correction. Moreover, for germs of functions that are even in the totality of variables, to apply Lemma 5.2, we need consider the jet spaces of only even dimension, which leads to the condition

$$\mathscr{M}_r^{2k} \mathscr{O}_r^N \subset T_A(\mathscr{G} \cdot A) + \mathscr{M}_r^{2k+2} \mathscr{O}_r^N$$

and, for germs of functions that are odd in the totality of variables,

$$\mathscr{M}_r^{2k+1} \mathscr{O}_r^N \subset T_A(\mathscr{G} \cdot A) + \mathscr{M}_r^{2k+3} \mathscr{O}_r^N.$$

The theorem is proved □

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