Family of Smooth Solutions of Hyperbolic Differential-Difference Equation

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Abstract Three-parameter familie of solutions is constructed for hyperbolic differential-difference equation with shift operators of the general-type acting with respect to all spatial variables. We prove theorem showing that the solutions obtained are classical provided that the real part of the symbol of the corresponding differential-difference operator is positive. Classes of equations for which these conditions are satisfied is given.

Keywords Hyperbolic equation · Differential-difference equation · Classical solution · Shift operator · Operational scheme · Fourier transform

1 Introduction

Problems for elliptic differential-difference equations in bounded domains have been studied quite comprehensively by now; the theory for such equations was created and developed by Skubachevskii [1, 2]. Problems for elliptic differential-difference equations in unbounded domains have been studied to a much lesser extent. An extensive study of such problems is presented in Muravnik's papers [3–5]. In particular, boundary value problems for multidimensional elliptic differential-difference equations are considered in [3–5].

Problems for parabolic differential-difference equations were studied in Muravnik's monograph [6]. Vlasov and Medvedev [7] studied hyperbolic differential-difference equations for the case where the shift operators act on the time variable.

As far as the present author is aware, at present, there are few papers dealing with hyperbolic differential-difference equations containing shifts with respect to the spatial variable. In [8–10], families of classical solutions are constructed for hyperbolic equations with shifts in the space variable x; the shifts occur in the potentials.

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In the present paper, we study the existence of smooth solutions of hyperbolic differential-difference equation in the half-space $\{(x,t) | \, x \in \mathrm{R}^n, \; t > 0\}$. The equation contains a sum of differential operators and shift operators with respect to each of the spatial variables,

$$u_{tt}(x,t) = a^2 \sum_{j=1}^{n} u_{x_j x_j}(x,t) - \sum_{j=1}^{n} b_j u(x_1, \dots, x_{j-1}, x_j - h_j, x_{j+1}, \dots, x_n, t), \tag{1}$$

where $a, b_1, \dots, b_n$ and $h_1, \dots, h_n$ are given real numbers.

Definition 1 A function $u(x,t)$ is called a classical solution of Eq. (1) if the derivatives u_{tt} and $u_{x_j x_j}$ $(j = 1, \dots, n)$ exist in the classical sense (i.e., as limits of finited-ifference ratios) at each point of the half-space $\{(x,t) | \, x \in \mathrm{R}^n, \; t > 0\}$ and if Eq. (1) holds at each point of the half-space.

2 Construction of Solutions of Equation (1)

To find solutions of the equation, we use the classical operational scheme [11, Sect.10], whereby one formally applies the Fourier transform with respect to the n-dimensional variable x to Eq. (1),

$$\widehat{f}(\xi) = \int_{\mathrm{R^n}} f(x) e^{i\xi \cdot x} dx,$$

and passes to the dual variable ξ.

In view of the formulas [12, Sect. 9]

$$F_x[\partial_x^\alpha \partial_t^\beta f] = (-i\xi)^\alpha \partial_t^\beta F_x[f], \quad F_x[f(x - x_0)] = e^{ix_0 \cdot \xi} F_x[f],$$

for the function $\widehat{u}(\xi,t) := F_x[u](\xi,t)$ we obtain the initial value problem

$$\frac{d^2 \widehat{u}}{dt^2} = -\left(a^2 |\xi|^2 + \sum_{j=1}^{n} b_j \cos(h_j \xi_j) + i \sum_{j=1}^{n} b_j \sin(h_j \xi_j) \right) \widehat{u}, \quad \xi \in \mathrm{R}^n, \tag{2}$$

$$\widehat{u}(0) = 0, \quad \widehat{u}_t(0) = 1. \tag{3}$$

For convenience, in the subsequent calculations we use the notation

$$\alpha(\xi) := \sum_{j=1}^{n} b_j \cos(h_j \xi_j), \quad \beta(\xi) := \sum_{j=1}^{n} b_j \sin(h_j \xi_j).$$

Then Eq. (2) becomes

$$\frac{d^2\widehat{u}}{dt^2} = -\left(a^2|\xi|^2 + \alpha(\xi) + i\,\beta(\xi)\right)\widehat{u}, \quad \xi \in \mathrm{R}^n,$$

and the roots of the corresponding characteristic equation are determined by the formula

$$k_{1,2} = \pm\, i\sqrt{a^2|\xi|^2 + \alpha(\xi) + i\,\beta(\xi)} = \pm\, i\rho(\xi)e^{i\,\varphi(\xi)},$$

where

$$\rho(\xi) := \left[\left(a^2|\xi|^2 + \alpha(\xi)\right)^2 + \beta^2(\xi)\right]^{1/4}, \tag{4}$$

$$\varphi(\xi) := \frac{1}{2}\mathrm{arctg}\frac{\beta(\xi)}{a^2|\xi|^2 + \alpha(\xi)}. \tag{5}$$

Thus, the general solution of Eq. (2) has the form

$$\widehat{u}(\xi,t) = C_1(\xi)e^{i\,t\,\rho(\xi)[\cos\varphi(\xi)+i\,\sin\varphi(\xi)]} + C_2(\xi)e^{-i\,t\,\rho(\xi)[\cos\varphi(\xi)+i\,\sin\varphi(\xi)]},$$

where $C_1(\xi)$ and $C_2(\xi)$ are arbitrary constants depending on the parameter ξ; to determine these constants, we substitute the function $\widehat{u}(\xi,t)$ into the initial conditions (3). From the system

$$\begin{cases} C_1(\xi) + C_2(\xi) = 0, \\ C_1(\xi) - C_2(\xi) = (i\,\rho(\xi)[\cos\varphi(\xi) + i\,\sin\varphi(\xi)])^{-1}, \end{cases}$$

we find the values of these constants,

$$C_1(\xi) = \frac{e^{-i\,\varphi(\xi)}}{2i\,\rho(\xi)}, \quad C_2(\xi) = -\frac{e^{-i\,\varphi(\xi)}}{2i\,\rho(\xi)}.$$

As a result, the solution of problem (2), (3) is given by the formula

$$\begin{aligned}
\widehat{u}(\xi,t) &= \frac{e^{-i\,\varphi(\xi)}}{2i\,\rho(\xi)}\left[e^{i\,t\,\rho(\xi)[\cos\varphi(\xi)+i\,\sin\varphi(\xi)]} - e^{-i\,t\,\rho(\xi)[\cos\varphi(\xi)+i\,\sin\varphi(\xi)]}\right] = \\
&= \frac{e^{-i\,\varphi(\xi)}}{2i\,\rho(\xi)}\left[e^{-t\,\rho(\xi)\sin\varphi(\xi)}e^{i\,t\,\rho(\xi)\cos\varphi(\xi)} - e^{t\,\rho(\xi)\sin\varphi(\xi)}e^{-i\,t\,\rho(\xi)\cos\varphi(\xi)}\right] = \\
&= \frac{1}{2i\,\rho(\xi)}\left[e^{-t\,\rho(\xi)\sin\varphi(\xi)}e^{i(t\,\rho(\xi)\cos\varphi(\xi)-\varphi(\xi))} - e^{t\,\rho(\xi)\sin\varphi(\xi)}e^{-i(t\,\rho(\xi)\cos\varphi(\xi)+\varphi(\xi))}\right] = \\
&= \frac{1}{2i\,\rho(\xi)}\left[e^{-t\,G_1(\xi)}e^{i(t\,G_2(\xi)-\varphi(\xi))} - e^{t\,G_1(\xi)}e^{-i(t\,G_2(\xi)+\varphi(\xi))}\right],
\end{aligned} \tag{6}$$

where we use the notation

$$G_1(\xi) := \rho(\xi)\sin\varphi(\xi), \quad G_2(\xi) := \rho(\xi)\cos\varphi(\xi). \tag{7}$$

Now we formally apply the inverse Fourier transform F_ξ^{-1} to relation (6) and obtain

$$u(x,t) = \frac{1}{(2\pi)^n}\int\limits_{\mathrm{R}^n} \frac{1}{2i\,\rho(\xi)}\left[e^{-t\,G_1(\xi)}e^{i(t\,G_2(\xi)-\varphi(\xi))} - e^{t\,G_1(\xi)}e^{-i(t\,G_2(\xi)+\varphi(\xi))}\right]e^{-ix\cdot\xi}d\xi =$$
$$= \frac{1}{2i(2\pi)^n}\int\limits_{\mathrm{R}^n} \frac{1}{\rho(\xi)}\left[e^{-t\,G_1(\xi)}e^{i(t\,G_2(\xi)-\varphi(\xi)-x\cdot\xi)} - e^{t\,G_1(\xi)}e^{-i(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi)}\right]d\xi.$$

Since the functions $\alpha(\xi)$, $\rho(\xi)$, and $G_2(\xi)$ are even and the functions $\beta(\xi)$, $\varphi(\xi)$, and $G_1(\xi)$ are odd in each of the variables ξ_j, we transform the last expression as follows:

$$\frac{1}{2i(2\pi)^n}\int\limits_{\mathrm{R}^n} \frac{1}{\rho(\xi)}\left[e^{-t\,G_1(\xi)}e^{i(t\,G_2(\xi)-\varphi(\xi)-x\cdot\xi)} - e^{t\,G_1(\xi)}e^{-i(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi)}\right]d\xi =$$
$$= \frac{1}{2i(2\pi)^n}\left[\int\limits_{\mathrm{R}^n_-} \frac{1}{\rho(\xi)}\left[e^{-t\,G_1(\xi)}e^{i(t\,G_2(\xi)-\varphi(\xi)-x\cdot\xi)} - e^{t\,G_1(\xi)}e^{-i(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi)}\right]d\xi + \right.$$
$$\left.\int\limits_{\mathrm{R}^n_+} \frac{1}{\rho(\xi)}\left[e^{-t\,G_1(\xi)}e^{i(t\,G_2(\xi)-\varphi(\xi)-x\cdot\xi)} - e^{t\,G_1(\xi)}e^{-i(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi)}\right]d\xi\right] =$$
$$= \frac{1}{2i(2\pi)^n}\left[\int\limits_{\mathrm{R}^n_+} \frac{1}{\rho(\xi)}\left[e^{t\,G_1(\xi)}e^{i(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi)} - e^{-t\,G_1(\xi)}e^{-i(t\,G_2(\xi)-\varphi(\xi)-x\cdot\xi)}\right]d\xi + \right.$$
$$\left.\int\limits_{\mathrm{R}^n_+} \frac{1}{\rho(\xi)}\left[e^{-t\,G_1(\xi)}e^{i(t\,G_2(\xi)-\varphi(\xi)-x\cdot\xi)} - e^{t\,G_1(\xi)}e^{-i(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi)}\right]d\xi\right] =$$
$$= \frac{1}{2i(2\pi)^n}\int\limits_{\mathrm{R}^n_+} \frac{1}{\rho(\xi)}\left[2i\,e^{t\,G_1(\xi)}\sin\left(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi\right) + \right.$$
$$\left. 2i\,e^{-t\,G_1(\xi)}\sin\left(t\,G_2(\xi)-\varphi(\xi)-x\cdot\xi\right)\right]d\xi =$$
$$= \frac{1}{(2\pi)^n}\int\limits_{\mathrm{R}^n_+} \frac{1}{\rho(\xi)}\left[e^{t\,G_1(\xi)}\sin\left(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi\right) + e^{-t\,G_1(\xi)}\sin\left(t\,G_2(\xi)-\varphi(\xi)-x\cdot\xi\right)\right]d\xi.$$

We use the resulting representation to prove the following assertion.

3 Existence of Smooth Solutions of the Equation (1)

Theorem 1 *Under condition*

$$a^2|\xi|^2 + \sum_{j=1}^{n} b_j \cos(h_j \xi_j) > 0, \tag{8}$$

for all $\xi \in \mathrm{R}^n$*, the functions*

$$F(x,t;\xi) := e^{t\,G_1(\xi)} \sin(t\,G_2(\xi) + \varphi(\xi) + x\cdot\xi), \tag{9}$$

$$H(x,t;\xi) := e^{-t\,G_1(\xi)} \sin(t\,G_2(\xi) - \varphi(\xi) - x\cdot\xi), \tag{10}$$

where $\varphi(\xi)$ *is determined by formula (5) and* $G_1(\xi)$ *and* $G_2(\xi)$ *are determined by relations (7), satisfy Eq. (1) in the classical sense.*

Proof First, let us substitute the function (9) directly into Eq. (1). To this end, we find the derivatives

$$F_{x_j}(x,t;\xi) = \xi_j e^{t\,G_1(\xi)} \cos(t\,G_2(\xi) + \varphi(\xi) + x\cdot\xi),$$

$$F_{x_jx_j}(x,t;\xi) = -\xi_j^2 e^{t\,G_1(\xi)} \sin(t\,G_2(\xi) + \varphi(\xi) + x\cdot\xi),$$

$$F_t(x,t;\xi) = G_1(\xi)e^{t\,G_1(\xi)} \sin(t\,G_2(\xi) + \varphi(\xi) + x\cdot\xi) + \\ +G_2(\xi)e^{t\,G_1(\xi)} \cos(t\,G_2(\xi) + \varphi(\xi) + x\cdot\xi),$$

$$F_{tt}(x,t;\xi) = \left[G_1^2(\xi) - G_2^2(\xi)\right] e^{t\,G_1(\xi)} \sin(t\,G_2(\xi) + \varphi(\xi) + x\cdot\xi) + \\ +2G_1(\xi)G_2(\xi)e^{t\,G_1(\xi)} \cos(t\,G_2(\xi) + \varphi(\xi) + x\cdot\xi).$$

Now let us evaluate the expressions $2G_1(\xi)G_2(\xi)$ and $G_1^2(\xi) - G_2^2(\xi)$. Since $G_1(\xi)$ and $G_2(\xi)$ are defined in (7), we conclude that

$$2G_1(\xi)G_2(\xi) = \rho^2(\xi)\sin 2\varphi(\xi).$$

It follows from formula (5) that $|2\varphi(\xi)| < \pi/2$ and hence $\cos 2\varphi(\xi) > 0$. Then we have

$$\sin 2\varphi(\xi) = \frac{\operatorname{tg} 2\varphi(\xi)}{\sqrt{1+\operatorname{tg}^2 2\varphi(\xi)}} =$$

$$= \operatorname{tg}\left(\operatorname{arctg}\frac{\beta(\xi)}{a^2|\xi|^2+\alpha(\xi)}\right)\left[1+\operatorname{tg}^2\left(\operatorname{arctg}\frac{\beta(\xi)}{a^2|\xi|^2+\alpha(\xi)}\right)\right]^{-1/2} =$$

$$= \frac{\beta(\xi)}{a^2|\xi|^2+\alpha(\xi)}\left[1+\frac{\beta^2(\xi)}{\left(a^2|\xi|^2+\alpha(\xi)\right)^2}\right]^{-1/2} =$$

$$= \frac{\beta(\xi)}{a^2|\xi|^2+\alpha(\xi)}\left[\frac{\left(a^2|\xi|^2+\alpha(\xi)\right)^2}{\left(a^2|\xi|^2+\alpha(\xi)\right)^2+\beta^2(\xi)}\right]^{1/2} =$$

$$= \frac{\beta(\xi)}{a^2|\xi|^2+\alpha(\xi)}\frac{|a^2|\xi|^2+\alpha(\xi)|}{\rho^2(\xi)}.$$

By virtue of condition (8), from the last relation we obtain

$$\sin 2\varphi(\xi) = \frac{\beta(\xi)}{a^2|\xi|^2+\alpha(\xi)}\frac{a^2|\xi|^2+\alpha(\xi)}{\rho^2(\xi)} = \frac{\beta(\xi)}{\rho^2(\xi)},$$

and hence

$$2G_1(\xi)G_2(\xi) = \beta(\xi). \tag{11}$$

With the inequality $\cos 2\varphi(\xi) > 0$ established above and under condition (8), now we find

$$G_1^2(\xi) - G_2^2(\xi) = \rho^2(\xi)\left[\sin^2\varphi(\xi) - \cos^2\varphi(\xi)\right] =$$

$$= -\rho^2(\xi)\cos 2\varphi(\xi) = -\frac{\rho^2(\xi)}{\sqrt{1+\operatorname{tg}^2 2\varphi(\xi)}} = \tag{12}$$

$$= -\rho^2(\xi)\left[\frac{\left(a^2|\xi|^2+\alpha(\xi)\right)^2}{\left(a^2|\xi|^2+\alpha(\xi)\right)^2+\beta^2(\xi)}\right]^{1/2} = -a^2|\xi|^2-\alpha(\xi).$$

In view of the expressions (11) and (12), the function F_{tt} becomes

$$F_{tt}(x,t;\xi) = \left[-(a^2|\xi|^2+\alpha(\xi))\sin\left(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi\right)+\right.$$
$$\left.+\beta(\xi)\cos\left(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi\right)\right]e^{t\,G_1(\xi)}.$$

Now let us substitute the derivatives $\widetilde{F}_{tt}$ and $\widetilde{F}_{x_jx_j}$ into Eq. (1),

$$
\begin{aligned}
F_{tt}(x,t;\xi) - a^2 \sum_{j=1}^{n} F_{x_j x_j}(x,t;\xi) = \\
= [-(a^2|\xi|^2 + \alpha(\xi)) \sin(t\,G_2(\xi) + \varphi(\xi) + x\cdot\xi) + \\
+\beta \cos(t\,G_2(\xi) + \varphi(\xi) + x\cdot\xi) + \\
+a^2 \sum_{j=1}^{n} \xi_j^2 \sin(t\,G_2(\xi) + \varphi(\xi) + x\cdot\xi)]e^{t\,G_1(\xi)} = \\
= -[\alpha(\xi)\sin(t\,G_2(\xi) + \varphi(\xi) + x\cdot\xi) - \\
-\beta(\xi)\cos(t\,G_2(\xi) + \varphi(\xi) + x\cdot\xi)]e^{t\,G_1(\xi)} = \\
= -\left[\sum_{j=1}^{n} b_j \cos(h_j\xi_j)\sin(t\,G_2(\xi) + \varphi(\xi) + x\cdot\xi) - \right. \\
\left. - \sum_{j=1}^{n} b_j \sin(h_j\xi_j)\cos(t\,G_2(\xi) + \varphi(\xi) + x\cdot\xi)\right] e^{t\,G_1(\xi)} = \\
= -\sum_{j=1}^{n} b_j \sin(t\,G_2(\xi) + \varphi(\xi) + x\cdot\xi - h_j\xi_j)e^{t\,G_1(\xi)} = \\
= -\sum_{j=1}^{n} b_j \sin(t\,G_2(\xi) + \varphi(\xi) + x_1\xi_1 + \cdots + x_n\xi_n - h_j\xi_j)e^{t\,G_1(\xi)} = \\
= -\sum_{j=1}^{n} b_j \sin(t\,G_2(\xi) + \varphi(\xi) + x_1\xi_1 + \cdots + x_{j-1}\xi_{j-1} + \\
+(x_j - h_j)\xi_j + x_{j+1}\xi_{j+1} + \cdots + x_n\xi_n)e^{t\,G_1(\xi)} = \\
= -\sum_{j=1}^{n} b_j \sin(t\,G_2(\xi) + \varphi(\xi) + \\
+(x_1,\dots,x_{j-1},x_j - h_j, x_{j+1},\dots,x_n)\cdot\xi)e^{t\,G_1(\xi)} = \\
= -\sum_{j=1}^{n} b_j F(x_1,\dots,x_{j-1},x_j - h_j, x_{j+1},\dots,x_n,t;\xi).
\end{aligned}
$$

Next, let us substitute the function (10) into Eq. (1). To this end, we find the derivatives

$$
H_{x_j}(x,t;\xi) = -\xi_j e^{-t\,G_1(\xi)} \cos(t\,G_2(\xi) - \varphi(\xi) - x\cdot\xi),
$$

$$
H_{x_j x_j}(x,t;\xi) = -\xi_j^2 e^{-t\,G_1(\xi)} \sin(t\,G_2(\xi) - \varphi(\xi) - x\cdot\xi),
$$

$$
\begin{aligned}
H_t(x,t;\xi) = -G_1(\xi)e^{-t\,G_1(\xi)} \sin(t\,G_2(\xi) - \varphi(\xi) - x\cdot\xi) + \\
+G_2(\xi)e^{-t\,G_1(\xi)} \cos(t\,G_2(\xi) - \varphi(\xi) - x\cdot\xi),
\end{aligned}
$$

$$H_{tt}(x,t;\xi)=\left[G_1^2(\xi)-G_2^2(\xi)\right]e^{-t\,G_1(\xi)}\sin\left(t\,G_2(\xi)-\varphi(\xi)-x\cdot\xi\right)-$$
$$-2G_1(\xi)G_2(\xi)e^{-t\,G_1(\xi)}\cos\left(t\,G_2(\xi)-\varphi(\xi)-x\cdot\xi\right)=$$
$$=\left[-(a^2|\xi|^2+\alpha(\xi))\sin\left(t\,G_2(\xi)-\varphi(\xi)-x\cdot\xi\right)-\right.$$
$$\left.-\beta(\xi)\cos\left(t\,G_2(\xi)-\varphi(\xi)-x\cdot\xi\right)\right]e^{-t\,G_1(\xi)}.$$

Now let us substitute the derivatives H_{tt} and $H_{x_jx_j}$ into Eq. (1),

$$H_{tt}(x,t;\xi)-a^2\sum_{j=1}^{n}H_{x_jx_j}(x,t;\xi)=$$
$$=[-(a^2|\xi|^2+\alpha(\xi))\sin\left(t\,G_2(\xi)-\varphi(\xi)-x\cdot\xi\right)-$$
$$-\beta(\xi)\cos\left(t\,G_2(\xi)-\varphi(\xi)-x\cdot\xi\right)+$$
$$+a^2\sum_{j=1}^{n}\xi_j^2\sin\left(t\,G_2(\xi)-\varphi(\xi)-x\cdot\xi\right)]e^{-t\,G_1(\xi)}=$$
$$=-[\alpha(\xi)\sin\left(t\,G_2(\xi)-\varphi(\xi)-x\cdot\xi\right)+$$
$$+\beta(\xi)\cos\left(t\,G_2(\xi)-\varphi(\xi)-x\cdot\xi\right)]e^{-t\,G_1(\xi)}=$$
$$=-\left[\sum_{j=1}^{n}b_j\cos\left(h_j\xi_j\right)\sin\left(t\,G_2(\xi)-\varphi(\xi)-x\cdot\xi\right)+\right.$$
$$\left.+\sum_{j=1}^{n}b_j\sin\left(h_j\xi_j\right)\cos\left(t\,G_2(\xi)-\varphi(\xi)-x\cdot\xi\right)\right]e^{-t\,G_1(\xi)}=$$
$$=-\sum_{j=1}^{n}b_j\sin\left(t\,G_2(\xi)-\varphi(\xi)-x\cdot\xi+h_j\xi_j\right)e^{-t\,G_1(\xi)}=$$
$$=-\sum_{j=1}^{n}b_j\sin\left(t\,G_2(\xi)-\varphi(\xi)-x_1\xi_1-\cdots-x_n\xi_n+h_j\xi_j\right)e^{-t\,G_1(\xi)}=$$
$$=-\sum_{j=1}^{n}b_j\sin\left(t\,G_2(\xi)-\varphi(\xi)-x_1\xi_1-\cdots-x_{j-1}\xi_{j-1}-\right.$$
$$\left.-(x_j-h_j)\xi_j-x_{j+1}\xi_{j+1}-\cdots-x_n\xi_n\right)e^{-t\,G_1(\xi)}=$$
$$=-\sum_{j=1}^{n}b_j\sin\left(t\,G_2(\xi)-\varphi(\xi)-\right.$$
$$\left.+(x_1,\ldots,x_{j-1},x_j-h_j,x_{j+1},\ldots,x_n)\cdot\xi\right)e^{-t\,G_1(\xi)}=$$
$$=-\sum_{j=1}^{n}b_jH(x_1,\ldots,x_{j-1},x_j-h_j,x_{j+1},\ldots,x_n,t;\xi).$$

A straightforward substitution into Eq. (1) shows that the function $H(x, t; \xi)$ satisfies this equation in the classical sense.

Note that the functions (4) and (5) are well defined for any $\xi \in \mathrm{R}^n$ under condition (8), because the radicand in formula (4) is always positive, and the denominator in the argument of the arctangent in (5) does not vanish. This means that the functions (9) and (10) are smooth solutions of the Eq. (1).

The proof of the theorem is complete.

Corollary 1 *Under condition (8), the family of functions*

$$G(x, t; A, B, \xi) := A\, e^{t\, G_1(\xi)} \sin\left(t\, G_2(\xi) + \varphi(\xi) + x \cdot \xi\right) + \\ + B\, e^{-t\, G_1(\xi)} \sin\left(t\, G_2(\xi) - \varphi(\xi) - x \cdot \xi\right), \tag{13}$$

where $\varphi(\xi)$ is given by (5) and $G_1(\xi)$ and $G_2(\xi)$ are given by (7), satisfies Eq. (1) in the classical sense for any real values of the parameters A, B, and ξ.

We represent the condition (8) in the form

$$\left(a^2\xi_1^2 + b_1 \cos\left(h_1\xi_1\right)\right) + \ldots + \left(a^2\xi_n^2 + b_n \cos\left(h_n\xi_n\right)\right) > 0.$$

Each of the n terms on the left side of this inequality will be positive if the conditions

$$0 < b_j h_j^2 \leq 2a^2, \quad j = \overline{1, n}.$$

For $\xi = \overrightarrow{0}$ the condition (8) will be satisfied if the coefficients at the nonlocal potentials satisfy the inequality

$$\sum_{j=1}^{n} b_j > 0.$$

Condition (8), holds for any shifts $h_1, \ldots, h_n$ and any values $\xi_1, \ldots, \xi_n$ if the coefficients and the shifts of the equation satisfy the conditions

$$\sum_{j=1}^{n} b_j > 0, \quad 0 < b_j h_j^2 \leq 2a^2, \quad j = \overline{1, n}.$$

These conditions are sufficient conditions that ensure the existence of a family of smooth solutions (13) to Eq. (1).

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