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Petrov Invariants for 1-D Control Hamiltonian Systems

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Abstract

In this paper we consider the action of symplectic feedback transformations on 1-D control Hamiltonian systems. We study differential invariants of the pseudogroup of feedback symplectic transformations, which we call Petrov invariants, and show that the algebra of invariants possesses a natural Poisson structure and central derivations. This structure allows us to classify regular 1-D control Hamiltonian systems.

Key Words: control Hamiltonian systems, differential invariants, Lie pseudogroups, symplectic feedback transformation, Poisson structures, Petrov invariant.

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1 Feedback Transformations and Control Hamiltonian Systems

A 1-D control Hamiltonian system with a Hamiltonian $H = H(q, p, u)$ is given by vector field

$$H_p\ \partial_q - H_q\ \partial_p, \tag{1.1}$$

where q and p are the phase variables, and u is a control parameter.

In control theory it is common to call transformations of the form

$$(q, p, u) \mapsto (Q(q, p), P(q, p), U(q, p, u)),$$

as *feedback transformations* (see [1, 3, 5, 8, 9]).

In our case they should preserve the class of Hamiltonian systems. Hence, it is easy to check, that they are of the following special form:

$$(q, p, u) \mapsto (Q(q, p), P(q, p), U(u)), \tag{1.2}$$

where $(q, p) \mapsto (Q(q, p), P(q, p))$ are symplectic transformations.

Such transformations we call *symplectic feedback transformations*.

We'll consider the problem of symplectic feedback equivalence of systems (1.1) with respect to transformations (1.2).

Remark that these transformations act on the Hamiltonians in the natural way:

$$\varphi^* : H(Q, P, U) \mapsto H(Q(q, p), P(q, p), U(u)).$$

2 Control Systems' Bundle

Let $M = \mathbb{R}^2$ be a phase space and let

$$\Omega = dp \wedge dq$$

be the structure 2-form on M.

Consider an extended phase space $B = M \times \mathbb{R}$ with coordinates q, p, u.

Infinitesimal symplectic feedback transformations are vector fields on the space B of the form

$$X_{H,\lambda} = X_H + Y_\lambda \tag{2.1}$$

where

$$X_H = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p},$$

and

$$Y_\lambda = \lambda(u)\frac{\partial}{\partial u},$$

and $H = H(p, q)$.

The Lie pseudogroup of symplectic feedback transformations we denote by G and the corresponding Lie algebra of symplectic feedback vector fields will be denoted by $\mathcal{G}$.

Let

$$\pi : B \times \mathbb{R} \to B, \qquad \pi : (q, p, u, h) \mapsto (q, p, u).$$

be one-dimensional trivial bundle over B.

Sections of this bundle can be viewed as a functions of the form $f(q, p, u)$, i.e. functions that define the control Hamiltonian systems.

For this reason, we call π as *control system bundle.*

Let $J^k(\pi)$ be the space of k-jets of sections of the bundle π.

Denote by q, p, u, h, h_σ the canonical coordinates on $J^k(\pi)$.

Here σ are multi-indexes of length $\leq k$:

$$\sigma = (\sigma_1, \sigma_2, \sigma_3), \quad |\sigma| = \sigma_1 + \sigma_2 + \sigma_3 \leq k.$$

Let $h = H(q, p, u)$ be a section of the bundle π. Then, in canonical coordinates, k-jet at a point $a \in B$ of this section has the form

$$[H]^k_a = \left(x(a), H(a), \cdots, \frac{\partial^{|\sigma|} H}{\partial x^\sigma}(a), \cdots \right),$$

where $|\sigma| \leq k$ and $x = (q, p, u)$.

Prolongations of a vector field X and a transformation φ into the spaces $J^k(\pi)$ will be denoted by $X^{(k)}$ and $\varphi^{(k)}$ respectively.

3 Petrov Differential Invariants

A smooth function J on k-jet space $J^k(\pi)$, which rational in fibrewise variables h_σ, we call *Petrov invariant of order* $\leq k$, if

$$\left(\varphi^{(k)}\right)^* (J) = J \tag{3.1}$$

for any symplectic feedback transformation φ, or

$$X^{(k)}_{H,\lambda}(J) = 0 \tag{3.2}$$

for any symplectic feedback vector field $X_{H,\lambda}$.

Remark that vector fields $X^{(k)}_{H,\lambda}$ generate a completely integrable distribution on $J^k(\pi)$ and rational first integrals of this distribution are Petrov differential invariants.

In a similar way, a function R on $J^k(\pi)$ is called a *relative Petrov invariant of order* $\leq k$, if

$$X^{(k)}(J) = \lambda_X J, \tag{3.3}$$

for any symplectic feedback vector field X and a weight 1-cocycle

$$\lambda : X \in \mathcal{G} \longmapsto \lambda_X \in C^\infty \left(J^k \pi \right), \tag{3.4}$$

on the Lie algebra $\mathcal{G}$.

A total derivation

$$\nabla = A \frac{d}{dq} + B \frac{d}{dp} + C \frac{d}{du}, \tag{3.5}$$

is called an *invariant derivation* if it commutes with any symplectic feedback vector field, i.e. if the following diagram

$$\begin{array}{ccc} C^\infty(J^\infty(\pi)) & \xrightarrow{\nabla} & C^\infty(J^\infty(\pi)) \\ X^{(\infty)} \Big\downarrow & & \Big\downarrow X^{(\infty)} \\ C^\infty(J^\infty(\pi)) & \xrightarrow{\nabla} & C^\infty(J^\infty(\pi)) \end{array}$$

commutes, for any vector field $X \in \mathcal{G}$.

Here A, B, and C are fibrewise rational smooth function on the space $J^\infty(\pi)$ and $\frac{d}{dx}$ are operators of the total derivatives in x (see [6]).

4 Dimensions of Jet Orbits

Splitting $B = M \times \mathbb{R}$ gives the decomposition

$$J_b^k(\pi) = \oplus_{s=0}^k J_a^{k-s}(M)$$

of the jet space at a point $b = (a, 0) \in B$, $a \in M$, in the following way.

Each function $f(q, p, u)$ can be presented in the following form

$$f = f_0(q,p) + u\ f_1(q,p) + \cdots + \frac{f_s(q,p)}{s!} u^s + \cdots + \frac{f_k(q,p)}{k!} u^k + u^{k+1}\ g(q,p,u),$$

where $f_0, ..., f_k, g$ are smooth function.

Therefore, for k-jets we get the following decomposition

$$[f]_b^k = [f_0]_a^k \oplus [f_1]_a^{k-1} \oplus \cdots \oplus [f_k]_a^0.$$

To find codimensions of G-orbits in $J^k(\pi)$ we remark that G acts in transitive way on B.

Therefore, these codimensions are equal to codimensions of the G_b-orbits in the fibre $J_b^k(\pi)$, where G_b is the stabilizer of the point b in G.

Let $\mathcal{O}(x_k) = G_b^{(k)}(x_k)$ be the orbit of $x_k = [f]_b^k$.

Then the tangent space to the orbit at the point x_k is generated by values of vector fields $X_{H,\lambda}^{(k)}$ at the point, where H has 2-nd order at the point a, and $\lambda(0) = 0$.

In other words,

$$H \in \mu_a^2, \lambda \in \mu_0,$$

where μ_a and μ_0 are the maximal ideals of the points $a \in M$ and $0 \in \mathbb{R}$.

The general prolongation formula (see, for example, [6]) shows that, in this case, value of $X_{H,\lambda}^{(k)}$ at the point x_k equals to

$$[X_H(f) + \lambda(u)\ \ f_u]_b^k.$$

Using the above decomposition we write s-component of this vector in the form

$$[X_H(f_s)]_a^{k-s} + \sum_{i=1}^{k} \binom{s}{i} \lambda_i\ [f_{s-i+1}]_a^{k-s},$$

where $\lambda_i = \lambda^{(i)}(0)$.

Consider the correspondence

$$(H, \lambda) \longmapsto [X_H(f) + \lambda(u)\ \ f_u]_b^k$$

as a linear operator

$$\kappa_k : J_a^{k+1,1}(M) \oplus J_0^{k,0}(\mathbb{R}) \to J_b^k(\pi).$$

Here we denoted by $J_a^{k+1,1}(M)$ the kernel of the projection $J_a^{k+1}(M) \to J_a^1(M)$, and by $J_0^{k,0}(\mathbb{R})$ the kernel of the projection $J_0^k(\mathbb{R}) \to J_0^0(\mathbb{R})$.

We say that the point $x_k \in J_b^k(\pi)$ is *regular*, if $f_1(a) \neq 0$ and vectors $X_{f_0}(a)$ and $X_{f_1}(a)$ are linear independent.

Theorem 4.1. *Let $x_k \in J_b^k(\pi)$ be a regular point. Then*

- $\dim \ker(\kappa_k) = 1$.
- *Codimension of the orbit $G_b^{(k)}(x_k)$ is equal to*

$$\frac{k(k+5)(k-2)}{6} + 2.$$

Proof. The kernel consist of solutions of the following linear system

$$E_s = [-X_{f_s}(H)]_a^{k-s} + \sum_{i=0}^{s} \binom{s}{i} \lambda_i\ [f_{s-i+1}]_a^{k-s} = 0$$

where $s = 0, ..., k$.

Taking 0-jets of E_s , and taking in account that $H \in \mu_a^2$, we get the following system

$$\sum_{i=0}^{s} \binom{s}{i} \lambda_i \, f_{s-i+1}(a) = 0,$$

which has the only trivial solution, if

$$f_1 \left(a \right) \neq 0.$$

Assuming that the last condition holds we get the following linear system for k-jet H :

$$E_s^0 = \left[X_{f_s} \left(H \right) \right]_a^{k-s} = 0,$$

where $s = 0, ..., k-1$.

Taking now 1-jets of E_s^0 we get the following system

$$\left[X_{f_s} \left(H \right) \right]_a^1 = 0,$$

where $s = 0, .., k-1$.

Let $\theta_2 = [H]_a^2 \in S^2 T_a^*$, and let denote by $\delta : S^l \left(T_a^* \right) \to S^{l-1} \left(T_a^* \right) \otimes T_a^*$ the Spencer δ-operator.

Then the last equations can be rewritten as follows

$$X_{f_s,a} \rfloor \delta \left(\theta_2 \right) = 0.$$

Therefore, if $k \geq 2$ and vectors $X_{f_s,a}$ are linear independent, we get $\delta \left(\theta_2 \right) = 0$, and $\theta_2 = 0$, or $H \in \mu_a^3$.

Then the projections of E_s^0 into 2-nd jets give us the next linear system

$$\left[X_{f_s} \left(H \right) \right]_a^2 = 0,$$

for $s = 0, ..., k-2$, or

$$X_{f_s,a} \rfloor \delta \left(\theta_3 \right) = 0,$$

where $\theta_3 = [H]_a^3 \in S^3 T_a^*$.

Assuming once more that $k \geq 3$, and that vectors $X_{f_s,a}$ are linear independent, we get $\delta \left(\theta_3 \right) = 0$, and $\theta_3 = 0$, or $H \in \mu_a^4$.

Continue in the same way we arrive to the condition $H \in \mu_a^{k+1}$ and to linear system

$$X_{f_0,a} \rfloor \delta \left(\theta_{k+1} \right) = 0,$$

$\theta_{k+1} = [H]_a^{k+1} \in S^{k+1} T_a^*$.

The last system has 1-dimensional solution space. □

Corollary 1. *Rational Petrov invariants of order $\leq k$ form a field. The transcendence degree of this field equals to*

$$\nu_k = \frac{k(k+5)(k-2)}{6} + 2.$$

Corollary 2. *There are ν_k independent Petrov invariants of order $\leq k$.*

The first values of ν_k given in the following table:

k	1	2	3	4	5
ν_k	1	2	6	12	25

5 Petrov Invariants of low order

In this section we describe Petrov invariants in order ≤ 3. In order ≤ 2 the result is rather obvious but in order 3 it was found by Ian Anderson's Differential Geometry package in Maple.

Indeed, we have obvious Petrov invariant of order 0,

$$J_0 = h.$$

Moreover, in order 1 function h_u and the total derivation

$$\frac{d}{du}$$

are relative invariants.

In order 2 the function

$$(h, h_u) = h_p h_{uq} - h_q h_{up}$$

is a relative invariant too.

Compare their weights we find the following Petrov invariants

$$\begin{aligned} J_0 &= h, \\ J_2 &= \frac{h_p h_{uq} - h_q h_{up}}{h_u} \end{aligned}$$

and invariant derivation

$$\nabla = \frac{1}{h_u} \frac{d}{du}.$$

To find invariants of order three we remark that the above corollary shows that in addition to invariants J_0, J_2 we have four invariants of pure order three.

Solving in Maple equation (3.2) for $k = 3$, we get:

$$
\begin{aligned}
J_{30} =& \frac{1}{h_u^3}(h_q h_u h_{puu} - h_p h_u h_{quu} - h_q h_{pu} h_{uu} + h_p h_{qu} h_{uu}),\\
J_{31} =& \frac{1}{h_u}(h_q^2 h_{ppu} - 2\, h_q h_p h_{qpu} + h_p^2 h_{qqu} - h_q h_{qu} h_{pp} + h_q h_{qp} h_{pu} -\\
&- h_p h_{pu} h_{qq} + h_p h_{qu} h_{qp}),\\
J_{32} =& \frac{1}{h_u^2}(h_q h_{qu} h_{ppu} - (h_q h_{pu} + h_p h_{qu}) h_{qpu} + h_p h_{pu} h_{qqu} - h_{pu}^2 h_{qq} +\\
&+ 2\, h_{pu} h_{qu} h_{qp} - h_{qu}^2 h_{pp}),\\
J_{33} =& \frac{1}{h_u^3}(h_{pu} h_{quu} - h_{qu} h_{puu}).
\end{aligned}
$$

Note also that the invariant J_{30} we can get from the invariant J_2 by differentiation: $J_{30} = \nabla(J_2)$.

These computations show that invariants up to order 3 are polynomials in h_σ, h_u^{-1}. For this reason, from now on we call Petrov invariants such differential invariants of the symplectic feedback pseudogroup, which polynomials in h_σ, h_u^{-1}.

To find Petrov invariants of higher order we'll need an additional structure on the algebra of invariants.

6 Poisson Algebra Structure

Let us consider the structure form Ω as a horizontal form on $J^\infty(\pi)$, and let's try to repeat the construction of the Hamiltonian vector fields.

Take a function $A \in C^\infty(J^\infty(\pi))$ and let's try to find a total derivation X_A such that $X_A \rfloor \Omega = \widehat{d}A$.

Because $\nabla \rfloor \Omega = 0$ one should correct the righthand side in such a way that it will annihilate derivation ∇.

Such correction leads us to the following result.

Theorem 6.1. *1. Let A be a smooth function on $J^\infty(\pi)$, $A \in C^\infty(J^\infty(\pi))$. Then relations*

$$
\begin{gathered}
X_A \rfloor \Omega = \widehat{d}A - \nabla(A)\widehat{d}h,\\
X_A(A) = 0,
\end{gathered}
$$

define a unique total derivation X_A on $J^\infty(\pi)$.

2. In canonical coordinates X_A has the following form:

$$
X_A = \left(\frac{dA}{dp} - \nabla(A)h_p\right)\frac{d}{dq} - \left(\frac{dA}{dq} - \nabla(A)h_q\right)\frac{d}{dp} + \left(\frac{dA}{dq}h_p - \frac{dA}{dp}h_q\right)\nabla.
$$

3.If A is a feedback differential invariant, then X_A is an invariant derivation.

Therefore, if A and B are Petrov invariants, then the function $X_A(B)$ is so also.

Let's introduce the following bracket on the algebra of Petrov invariants:

$$[A, B] = X_A(B). \tag{6.1}$$

This bracket can be rewritten as

$$[A, B] = (A, B) - \nabla(A)(h, B) + \nabla(B)(h, A),$$

where

$$(A, B) = \frac{dA}{dp}\frac{dB}{dq} - \frac{dA}{dq}\frac{dB}{dp}$$

is the prolongation of the classical Poisson bracket to $J^\infty(\pi)$.

Theorem 6.2. *1. Algebra of Petrov invariants is Poisson with respect to bracket (6.1).*

2. The operator ∇ is a derivation in this algebra:

$$\nabla[A, B] = [\nabla A, B] + [A, \nabla B].$$

3. The differential invariant J_0 is a Casimir function in the Poisson algebra, i.e. $[A, J_0] = 0$ for any Petrov invariant A.

7 Structure of the Petrov Invariant Algebra

Recall that a point $x_k = [f]_b^k \in J_b^k(\pi)$ is *regular* if $f_u(b) \neq 0$, and vectors $X_{f,b}$ and $X_{f_u,b}$ are linear independent.

Orbits $\mathcal{O}(x_k)$ of regular points we call *regular* .

The above discussion together with the final classification theorem (see below) shows that the following result holds.

Theorem 7.1. *Algebra of Petrov invariants, as a Poisson algebra, is generated by the invariants $J_0, J_2, J_{30}, J_{31}, J_{32}, J_{33}$, and invariant derivation ∇. This algebra separates regular orbits.*

8 Feedback classification

Consider a control Hamiltonian system given by a Hamiltonian $H(q, p, u)$, and denote by A_H the value of a Petrov invariant A on H.

We say that the control system is *regular* in a domain $D \subset B$, if there are two Petrov invariants, say A and B, such that functions

$$H = h_H, A_H, B_H$$

are independent in the domain, and the bracket

$$[A, B]_H \neq 0$$

in the domain.

Such invariants A and B we'll call *basic* for the system.

Lemma 8.1. *Let*

$$\begin{aligned} \widehat{d}A \wedge \ \widehat{d}B \wedge \widehat{d}h &\neq 0, \\ [A, B] &\neq 0 \end{aligned}$$

in a domain of $J^{\infty}(\pi)$.

Then in this domain we have the following representation of the structure form:

$$\Omega = \left(\frac{\nabla (B)}{[A, B]} \widehat{d}A - \frac{\nabla (A)}{[A, B]} \widehat{d}B \right) \wedge \widehat{d}h - \frac{1}{[A, B]} \widehat{d}A \wedge \ \widehat{d}B.$$

Proof. Let

$$\Omega = \left(\alpha \ \widehat{d}A + \beta \ \widehat{d}B \right) + \gamma \widehat{d}A \wedge \ \widehat{d}B$$

in the domain.

Then

$$\nabla \rfloor \Omega = (\alpha \ \nabla (A) + \beta \ \nabla (B)) \, \widehat{d}h - (\alpha + \gamma \nabla (B)) \, \widehat{d}A + (-\beta + \gamma \nabla (A)) \, \widehat{d}B = 0.$$

Therefore,

$$\alpha = \gamma \nabla (B), \ \beta = -\gamma \nabla (A).$$

On the other hand, we have

$$X_A \rfloor \Omega = \beta X_A (B) \, \widehat{d}h - \gamma X_A (B) \, \widehat{d}A = \widehat{d}A - \nabla (A) \, \widehat{d}h.$$

Therefore,

$$\alpha = \frac{\nabla (B)}{[A, B]}, \ \beta = -\frac{\nabla (A)}{[A, B]}, \ \gamma = -\frac{1}{[A, B]}.$$

□

Let now H be the Hamiltonian of a control system which is regular in a domain D.

Then functions

$$x \stackrel{\text{def}}{=} h_H,$$
$$y \stackrel{\text{def}}{=} A_H,$$
$$z \stackrel{\text{def}}{=} B_H,$$

for the basic Petrov invariants A and B can be viewed as coordinates in D.

Denote by P_0, P_1, P_2 the values of invariants $-\frac{1}{[A,B]}, \frac{\nabla(B)}{[A,B]}$ and $-\frac{\nabla(A)}{[A,B]}$ on H and call them *defining functions* for the system.

They are functions in (x, y, z) and $P_0 \neq 0$.

The above lemma shows that in coordinates (x, y, z) the structure form Ω and vector field ∇_H has the following form:

$$\begin{aligned} \Omega &= (P_1 dy + P_2 dz) \wedge dx + P_0 dy \wedge dz, \\ \nabla_H &= \partial_x + \frac{P_2}{P_0}\partial_y - \frac{P_1}{P_0}\partial_z. \end{aligned}$$

This gives us immediately the following classification of regular control systems.

Theorem 8.1. *Two regular control Hamiltonian systems are feedback equivalent if and only if they have the same basic invariants and the same defining functions.*

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