WHAT ARE THE POINCARÉ GAUGE FIELDS?

G. Sardanashvily

Physics Department, University of Moscow, 117 234 Moscow, USSR

Gauge fields of Poincaré translations fail to be identified with gravitational fields representing the Goldstone-type fields of spontaneously broken space-time symmetries.

1. INTRODUCTION

The Poincaré gauge was brought into play at the beginning of the 60th by Kibble, Sciama et al. for generalizing Utiyama's gauge version of gravity which had left open the question about the gauge status of gravitational fields. The main difficulty lay in the fact that gravitational fields were metric or tetrad fields, whereas gauge potentials represented connections of fibre bundles.

The Poincaré gauge solution of this dilemma is based on the coincidence of the tensor forms of tetrad fields h^a_μ and gauge fields A^a_μ of the translation group, and proclaims the identity of them. But is it in fact?

2. THE GAUGE IMAGE OF GRAVITATION THEORY

In the fibre bundle terms a metric gravitational field on an orientable space-time manifold X^4 is defined as a global section g of the fibre bundle B of pseudo-Euclidean bilinear forms in tangent spaces over X^4. This bundle B is associated with the tangent bundle $T(X^4)$ possessing the structure group $GL^+(4, R)$, and is isomorphic with the fibre bundle W in quotient spaces $GL^+(4, R)/SO(3, 1)$. The global section h of W, which is isomorphic with g, describes gravitational field in the tetrad form. The section h used to be written as a section of the principal $GL(4, R)$-bundle up to multiplication of h on the right by elements of the gauge Lorentz group.

The necessary and sufficient condition for gravitational field to exist on the manifold X^4 is the contraction of the structure group $GL^+(4, R)$ of the tangent bundle to the Lorentz group. It means the existence of some atlas $\Psi_g = \{U_i, \psi_i\}$ of $T(X)$, whose transition functions gluing the patches (U_i, ψ_i) of trivialization of $T(X)$ are elements of the gauge Lorentz group $SO(3, 1)(X)$. With respect to this atlas the metric gravitational field g is represented by the constant Minkowski metric field η, but h takes values in the centre of the quotient space $GL^+(4, R)/SO(3, 1)$. The field h can be represented by the family of matrix (tetrad) fields $\{h_i(x), x \in U_i\}$ acting in the typical fibre R^4 of $T(X)$ and describing the gauge transformation between a given atlas Ψ of $T(X)$ and the atlas Ψ_g. Changes of Ψ lead to the following gauge transformations of the tetrad fields h_i:

$$h_i(x) \to G(x)\, h_i(x)\,, \quad x \in U_i \tag{1}$$

To build the gauge version of gravitation theory we may base our ideas on Einstein's relativity and equivalence principles reformulated in the fibre bundle terms [1].

In these terms the relativity principle proves to be identic with the gauge principle of the covariance under the gauge group $GL(4, R)(X)$ of all transformations of atlases of the tangent bundle $T(X)$. This group contains the subgroup of holonomic transformations, when the choice of atlas $\Psi = \{U_i, \psi_i = d\varphi_i\}$ of $T(X)$ correlates with the choice of coordinate atlas $\Psi_X = \{U_i, \varphi_i\}$ of the manifold X^4, and this correlation is strictly retained under changes of the bundle and coordinate atlases.

Thus the gravitation theory can be build directly within the framework of gauge theory of external symmetries. As distinguished from the internal symmetry case, such a theory containes two kinds of gauge transformations, namely, the familiar atlas transformations of a matter field bundle, but also the atlas changes of the tangent bundle. The atlases of these bundles are equivalent, but not always the same, and their gauge transformations do not correlate in general and result in different conservation laws. Utiyama had remarked this fact and had brought tetrad gravitational fields into being just to secure the invariance under the tangent space gauge transformations, while the invariance under gauge transformations of matter fields was provided with inserting Lorentz gauge fields as in internal gauge models [2].

However the relativity principle fails to fix the Minkowski signature of metric fields, and consequently the equivalence principle must supplement it. The equivalence principle is formulated in the fibre bundle terms as the postulate of the existence of a reference frame, where Lorentz invariants can be defined everywhere on a manifold X^4, and they would be conserved under parallel translations.

This postulate holds, if the connection on the tangent and associated bundles can be reduced to the Lorentz gauge fields, that, in turn, entails the contraction of the structure group $GL^+(4, R)$ of these bundles to the Lorentz group, and consequently provides the existence of a gravitational field on X^4. In this fashion the equivalence principle establishes the situation of spontaneous breaking gauge external symmetries down to the Lorentz gauge group, and gravitational field figures as the sui generis Goldstone field corresponding to this breakdown [1, 3–5]. Thus just the Goldstone field treatment of gravitational field solves the dilemma of its gauge status.

Now we return to the title question of our paper.

3. THE NONCONVENTIONAL POINCARÉ GAUGES

The idea on the Poincaré gauge gravitation dominated in gauge gravitation researches in the 60–70th. Why?

In Special Relativity a space-time represents the affine Minkowski space, and the Poincaré group being the motion group of this space represents the fundamental dynamic group of Special Relativity, whose unitary representations are identified with the free particle states in Special Relativity. Of course, it motivated attempts to complete gauging internal and intrinsic spin symmetries with gauging the Poincaré group. However these attempts faced the specificity of gauging this group as the dynamic group.

This specificity lies in the fact that, in contrast with the internal and spin transformations varying field functions at a point, generators of dynamic symmetries are realized by differential operators

$$T_\mu = \partial/\partial x^\mu\,, \quad L^{\mathrm{orb}}_{\mu\nu} = \eta_{\mu\alpha}x^\alpha\partial/\partial x^\nu - \eta_{\nu\alpha}x^\alpha\partial/\partial x^\mu \tag{2}$$

which may be thought of, on the one hand, as operators of coordinate transformations and, on the other hand, as operators of transitions from point to point. Both of these interpretations are equivalent in a flat space, but differ from each other under gauging.

Authors of the first Poincaré gauge works (T. W. Kibble, D. Sciama and their followers) adhered to the coordinate interpretation of the Poincaré generators (2) [6]. They combined the gauging of the Lorentz intrinsic spin transformations, considered by Utiyama with coordinate translations $x^\mu \to x^\mu + a^\mu$. Localization of these translations $x \to x^\mu + a^\mu(x)$ reproduced the group of general coordinate transformations, which induced, in turn, the holonomic subgroup of the gauge group $GL(4, R)(X)$. And as like as in Utiyma's model just the requirement of invariance under these holonomic transformations, which had nothing to do with gauge translations, called into being gravitational field in the discussed model. Thus the gauge status of gravitational fields in Kibbles et al. approach is far from the conventional gauge potentials.

The procedure of gauging the Poincaré transformations (2) interpreted as point-to-point transitions was proposed by F. Hehl, P. von der Heyde et al. [7]. This procedure does not reduce to localization of group parameters only, but modificates also the generators of the Poincaré group by replacing ordinary derivatives in (2) with the covariant ones: $\partial_\mu \to D_\mu = \partial_\mu - A_\mu$, where A is a certain Lorentz connection. Hence the localization of a Poincaré transformation $p = \exp\left(t^\mu\partial_\mu + l^{\mu\nu}(L^{\mathrm{orb}}_{\mu\nu} + L^{\mathrm{sp}}_{\mu\nu})\right)$ takes the nonconventional form

$$p(x) = \exp\left(t^\mu(x)\, D_\mu + l^{\mu\nu}(x)\,(L^{\mathrm{orb}\prime}_{\mu\nu} + L^{\mathrm{sp}}_{\mu\nu})\right) \tag{3}$$

where $L^{\mathrm{orb}\prime}$ results from L^{orb} by the replacement $\partial_\mu \to D_\mu$.

The replacement $\partial_\mu \to D_\mu$ seems quite natural as generalization of translations in a flat space, but it violates the familar commutation relations of the Poincaré group, e.g., translation generators become noncommutative: $[D_\mu, D_\nu] \neq 0$. Obviously, transformations (3) fail to compose the conventional gauge Poincaré group $P(X)$.

At the same time the invariance of a matter field Lagrangian under transformations (3) reduces on extremal fields to the familiar invariance under gauge Lorentz spin transformations and holonomic gauge $GL(4, R)$-transformations, which are the same that we have faced in Utiyama's and Kibble's models.

Thus we see that both discussed Poincaré gauge versions being outside the conventional gauge scheme fail to provide gravitational field with the status of the gauge potential of the Poincaré translations, in spite of the initial declaration.

Indeed, let us suppose that the tetrad gravitational field h and the translation connection h^T being realized by the matrix fields h_i and h_i^T, respectively, in R^4 with respect to some atlas Ψ are identified. Let Ψ be Ψ_g. Then h_i being defined up to right Lorentz transformations can be represented by the unit matrix function $h_i = \mathrm{Id}\, R^4$ on all patches U_i. Then the translation connection h^T identified with h must reduce to the soldering form θ on X^4, where θ can be defined as the identic mapping of $T(X)$ on itself. But then, in virtue of the gauge transformation law of h^T, such a connection has to be identified with θ with respect to any atlas Ψ of $T(X)$, and thereby it fails to coincide with $h \neq \mathrm{Id}\, R^4$ in atlases $\Psi \neq \Psi_g$.

Remark that some authors [8] proposed to identify h with $(h^T + \theta)$, it does not change the main conclusion about the impossiblity to identify gravitational fields with gauge potentials of Poincaré translations.

This result returns us to the problem of physical treatment of Poincaré translations acting inside fibres of a bundle and their gauge fields.

In the discussed case of the affine frame bundle the Poincaré translations are translations of tangent vectors. Some authors [9] considered the realization of such translations on functions $f(x, v_x)$ depending not only on a space-time point x, but also on a tangent vector v_x, e.g. such functions possess sui generis "internal affine symmetry". For instance, the relevance of such functions for describing hadrons was discussed.

Alternatively to applying tangent vectors as arguments of field functions, another approach [12] uses them as values of field functions. Such functions are considered to take values in the space $V \times T$ of some nonlinear realization of the Poincaré group, where V is a space of representation of the Lorentz group, but $T = P/SO(3, 1)$ represents the typical fibre of $AT(X)$. Herewith the translation subgroup T of P acts only on Goldstone fields taking values in T. These fields can be removed by a certain translation gauge, but the translation connection h^T retains as some tensor field on X^4. However the physical relevance of h^T remains open to questions. Indeed, only if the translation connection form h^T reduces to the soldering form θ, the covariant derivative $D^L\theta$ represents the familiar geometric object, namely, the torsion form.

This fact motivates some authors to restrict their attention only to affine connection expressed by the soldering form θ [10]. But, in spite of some opinions, the soldering form itself is unable to define any torsion or gravitational fields. A linear connection having been established, the parallel transference of θ carried out by torsion components of this connection picks out the torsion part from the whole connection. Coefficients $h^a(x)$ of the soldering form $\theta = h^a t_a$, $h^a = h^a_\mu\, dx^\mu$, written with respect to a certain atlas Ψ also make sense of tetrad coefficients only if a gravitational field has been defined, and if Ψ is the atlas Ψ_g with regard to this field.

Thus we observe that the gauging of the Poincaré group as the dynamic group leads to nonconventional gauge models, while the standard Poincaré gauge leaves the question about the physical relevance of translation gauge fields open. And no

4. THE CONVENTIONAL POINCARÉ GAUGE

The conventional gauge technique can be applied for gauging the Poincaré group if one does not for a time consider its physical role as the dynamic group, and looks at it as at an abstract holonomy and structure group of some fibre bundle [8–12].

Because of our goal to match gravitational and translation gauge fields, we restrict ourselves in consideration of the affine tangent bundle $AT(X)$ or the associated principal bundle $A(X)$ in affine frames. These bundles are associated with the tangent bundle $T(X)$ and with the linear frame bundle $L(X)$, and consequently the structure affine group $GA(4, R)$ of $TA(X)$ and $A(X)$ contracts to the linear group $GL(4, R)$.

An affine connection 1-form A on the bundle $A(X)$ splits in two components $A = A^L + A^T$, where A^L is a linear connection, but $A^T = A^a_\mu T_a \, dx^\mu$ is some R^4-valid translation connection form, whose coefficients A^a_μ represent translation gauge fields. Owing to the contraction of the structure group of $A(X)$ to $GL(4, R)$ the global section S of the associated bundle in quotient spaces $GA(4, R)/GL(4, R)$ exists. Then one can expand the translation form A^T in two parts $A^T = A^S + h^T$, where A^S is evaluated from the condition $(D - A^S) S = 0$, $D = d - A^L$, and it reads: $(A^S)^a = (DS)^a$.

It is clear that just the component A^S is responsible for the inhomogeneous transformation law of the whole connection A^T under gauge translations, whereas h^T is invariant under translations and is a tensor under gauge linear transformations.

Moreover, there is always a certain translation gauge, where the part A^S of the translation connection A^T goes to zero, and A^T reduces to the homogeneous part h^T. For instance, the reduction $A^T = h^T$ occurs in all atlases with only linear transition functions, if one chooses S coinciding with the zero global function in these atlases.

Let us fix this translation gauge. Then one can make use of the known theorems establishing the one-to-one correspondence between general affine connections A on $A(X)$ and pairs (A^L, h^T) of linear connections A^L on $L(X)$ and R^4-valued forms h^T on X^4. This correspondence reads

$$A = \begin{pmatrix} A^L & h^T \\ 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} R^L & Dh^T \\ 0 & 0 \end{pmatrix} \tag{4}$$

where the general affine connection A and its curvature F are expressed by (5×5)-matrices acting on columns $\begin{pmatrix} r \\ 1 \end{pmatrix}$, $r \in R^4$, but R^L denotes the curvature form of the linear connection A^L.

The translation connection form h^T determines linear transformations $h^T(x) : T_x \to T_x$ of tangent spaces T_x at every point $x \in X$. With respect to some atlas of $T(X)$ it is represented by a family of matrix fields $h^T_i = \psi_i h^T \psi_i^{-1}$ acting in R^4 just as the tetrad gravitational field does. But the gauge transformation law of the connection form $h^T_i : h^T_i \to G h^T_i G^{-1}$, differs from the transformation law (1) of the tetrad gravitational field h. And just this difference destroys the hypothetic identity of h^T and h.

Poincaré gauge version justifies the identity of gravitational fields and translation gauge potentials. Moreover, one has the impression that the gauging of dynamic groups makes no sense in general.

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