Vol. 29 (1991) *REPORTS ON MATHEMATICAL PHYSICS* No. 1

ON FUNCTIONAL INTEGRALS IN QUANTUM FIELD THEORY

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(*Received September 11, 1989*)

In the framework of algebraic quantum field theory (QFT), generating functionals are defined as Fourier transforms of measures in infinite-dimensional linear topological spaces, without using any approximation procedure.

Functional integrals in terms of which generating functionals are expressed in QFT, are Feynman integrals as a rule. They are defined as formal limits of approximations by measures in finite-dimensional spaces [1–4]. Many questions, however, remain [4]. Does such a limit exist for a field system? Is it a measure? What are its properties, in contrast to finite-dimensional integrals? These problems become essential when describing non-perturbative phenomena, e.g. a mean field and a Higgs vacuum.

In the framework of algebraic QFT, we show that chronological forms on an algebra of real scalar fields are really defined by a generating function which represents the Fourier transform of some measure, without using any approximation procedure. Since chronological forms are commutative, we describe them as states on a corresponding commutative tensor algebra. The construction of states on such algebras is well-known [5].

According to algebraic QFT, a quantum field system can be characterized by a topological *-algebra A (with the unit 1_A) and a continuous positive form (state) f on A, i.e.

$$f(a^*a) \geqslant 0, \quad f(1_A) = 1, \ a \in A.$$

Note that only values of a form f have physical meaning, whereas the algebra A remains unknown to a certain extent because f is entirely defined by its kernel.

With reference to the field-particle dualism, realistic models are described by tensor algebras as a rule, and states are treated as vacuum expectation forms.

Let V be a real linear locally convex topological space endowed with some involution operation

$$*: \ v \to v^*, \ v, v^* \in V.$$

Then, the linear space

$$A_V = \bigoplus_0^\infty V_n, \quad V_0 = R, \quad V_{n>0} = \overset{n}{\otimes} V$$

is a Z-graded *-algebra with respect to the tensor product and the involution operation

$$(v_1 \dots v_n)^* = v_n^* \dots v_1^*.$$

The direct sum topology turns A_V into a topological *-algebra [6]. The sequence

$$\{v^{(i)} = (v_0^{(i)}, \dots, v_n^{(i)}, \dots,)\} \subset A_V, \ \{v_n^{(i)}\} \subset V_n,$$

converges to 0 with respect to this topology if any sequence $\{v_n^{(i)}\}$ converges to 0 with respect to the space V_n topology and there exists a number N, independent of i, such that $v_n^{(i)} = 0$ for all i and $n > N$.

A state f on A_V defines a collection of continuous forms $\{f_n\}$ on the spaces V_n.

In axiomatic QFT, A_V is the Borchers algebra, with $V = S(R^4)$ being the real nuclear Schwartz space of test functions $v(x)$ on R^4 [6, 7]. A state f on $A_{S(R^4)}$ is represented by a collection of distributions (tempérées) $W_n \in S'(R^{4n})$ such that

$$f_n(v_1, \dots, v_n) = \int W_n(x_1, \dots, x_n)\, v(x_1), \dots, v(x_n)\, d^4x_1, \dots, d^4x_n.$$

Hereafter, V' denotes the dual of V. If f obeys the Wightman axioms, W_n are n-point Wightman functions. For instance, free massive particles with infinite lifetime are described by distributions W_n^F which obey the Wick rules, and so are defined by distributions

$$\begin{aligned} W_1^F = 0, \quad W_2^F(x, x') = -iD^-(x-x'), \\ D^-(x) = i(2\pi)^{-3} \int d^4p\, e^{-ipx}\theta(p_0)\,\delta(p^2 - m^2), \end{aligned} \tag{1}$$

where θ is the step function and p^2 is the Minkowski scalar.

In QFT, particles, created at some moment and destructed at another one, are described by chronological forms f^c given by the expressions

$$W_n^c(x_1, \dots, x_n) = \sum_{(i_1, \dots, i_n)} \theta(x_{i_1}^0 - x_{i_2}^0), \dots, \theta(x_{i_{n-1}}^0 - x_{i_n}^0)\, W_n(x_{i_1}, \dots, x_{i_n}), \tag{2}$$

where $(i_1, \dots, i_n)$ is a rearrangement of numbers $1, \dots, n$. In general, expressions (2), however, fail to be distributions and do not define a positive form ($W^c \in S'(R)$ if $W \in S'(R_\infty)$, where R_∞ is the compactification of R [8]). Therefore, to take advantage of the algebraic methods, we go over to considering Euclidean forms F connected with chronological forms by transformation of the Wick rotation. Note that our approach differs from that of the constructive Euclidean theory which aims to reconstruct Wightman functions [9].

Since chronological forms, defined by the expression (2), are commutative, Euclidean forms can be described as states on commutative tensor algebras. Let us define a commutative tensor algebra B_V as the complexified quotient of A_V by the ideal whose generating set is

$$I = \{vv' - v'v,\ v,\ v' \in V\}.$$

This algebra can be represented as the enveloping algebra of the Lie algebra corresponding to the commutative Lie group G_V of translations on V [10]. Therefore, we can construct states on the algebra B_V as vector forms of its cyclic representations derived from strongly continuous unitary cyclic irreducible representations π of G_V [5, 10].

Such a representation of G_V is characterized by some continuous positive type function Z ($Z(0) = 1$) on V, i.e.

$$Z(v^i - v^j)\overline{\alpha}_i \alpha_j \geqslant 0$$

for any collections of complex numbers $\alpha_1, \ldots, \alpha_n$ and elements $v^1, \ldots, v^n \in V$ [5]. We restrict ourselves to functions v such that the function

$$R \ni \lambda \to Z(\lambda v) \in R$$

is analytic at 0 for any $v \in V$. Then the positive continuous form

$$F(v^1, \ldots, v^n) = \frac{1}{i^n} \frac{\partial}{\partial \lambda_1}, \ldots, \frac{\partial}{\partial \lambda_n} Z(\lambda_i v^i)|_{\lambda_i} = 0 \tag{3}$$

on the algebra B_V is defined by Z as a generating function [11].

The Bochner theorem asserts that any function Z of the above-mentioned type represents the Fourier transform

$$Z(v) = \int \exp(it)\, d\mu_v(t),\ t \in R$$

of some positive pro measure μ in the dual V' of V [5, 12]. Hereafter, we restrict our consideration to measures μ. In this case,

$$Z(v) = \int e^{i\langle u, v\rangle} d\mu(u),\ u \in V', \tag{4}$$

where $\langle\ ,\ \rangle$ denotes pairing between V' and V, and the corresponding representation π of G_V is realized by operators

$$\pi(g(v)):\ \varrho(u) \to e^{i\langle u, v\rangle} \varrho(u)$$

in the space $L^2(V', \mu)$ of quadratically μ-integrable functions $\varrho(u)$ on V', and

$$F(v^1, \ldots, v^n) = \int \langle u, v^1\rangle, \ldots, \langle u, v^n\rangle\, d\mu(u). \tag{5}$$

Note that representations π and π' of G_V are inequivalent if the measures μ and μ' in expression (4) are inequivalent.

In physical models, Z plays the role of a generating functional on a quantum field space V, and expression (4) gives its representation by a functional integral in a classical field space V'. For instance, free and quasi-free fields are described by Gaussian states F. Their generating functions are of the form

$$Z(v) = \exp\left(-\tfrac{1}{2}M(v, v)\right), \tag{6}$$

where $M(v, v')$ is a positive-definite Hermitian bilinear form on V continuous in each variable. In this case, the forms F_n obey the Wick rules where

$$F_2(v_1, v_2) = M(v_1, v_2), \quad F_1 = 0,$$

i.e. the Gaussian generating function (6) is entirely defined by the covariance form M. The generating function (6) is the Fourier transform of some Gaussian pro measure in V'. The Wiener measure exemplifies an infinite-dimensional Gaussian measure [12]. If V is a nuclear space, Gaussian pro measures in V' are measures [5].

Let V be $S(R^4)$ and F be a Gaussian state on $B_{S(R^4)}$ such that the covariance form is represented by a distribution $M(y, y') \in S'(R^8)$ which is the Green function of some positive-definite elliptic differential operator

$$L_y M(y, y') = \delta(y, y'). \tag{7}$$

Then, the Gaussian state F describes quasi-free Euclidean fields with $M(y, y')$ playing the role of their propagator. For instance, they are free Euclidean fields if

$$L_y = -\Delta_y + m^2, \quad M(y, y') = \int \frac{e^{-iq(y-y')}}{q^2+m^2} d^4q, \tag{8}$$

where q^2 is the Euclidean scalar.

Note that the measure μ in relation (4) fails to be reconstructed as a rule. The familiar expression

$$d\mu = e^{-S[\varphi]} \prod d\varphi \tag{9}$$

one uses in QFT is not a true measure in general.

The measure μ in V', however, is uniquely defined by the collection of all measures μ_N which are images of μ under canonical morphisms

$$\gamma\colon V' \to V'/E = R_N = (R^N)',$$

where $E \subset V'$ denotes a linear subspace of forms equal to zero on some finite-dimensional subspace $R^N \subset V$ [5, 12]. For instance, any vacuum expectation value $F(v^1, \ldots, v^n)$ admits a representation by functional integrals

$$F(v^1, \ldots, v^n) = \int \langle u, v^1\rangle, \ldots, \langle u, v^n\rangle \, d\mu_N$$

for any finite-dimensional subspace R^N which contains $v^1, \ldots, v^n$.

This fact enables us to formulate the following perturbation theory. Let F be a state on B_V and R^n be some finite-dimensional subspace of V. To calculate vacuum expectations of fields $v \in R^n$, one can replace the generating function (4) by the generating function

$$Z_N(\lambda_i e^i) = \int \exp(i\lambda_i u^i)\, d\mu_N(u^i)$$

on R^N, where $\{e^i\}$ is some basis of R^N and $\{u^i\}$ are coordinates with respect to the dual basis of R_N. If F is a Gaussian state,

$$d\mu_N = (2\pi \det[M^{ij}])^{-N/2} \exp\{-\tfrac{1}{2}(M^{-1})_{ij} u^i u^j\}\, d^N u, \tag{10}$$

where $M^{ij} = M(e^i, e^j)$ is the non-degenerate covariance matrix.

Let F' be a state on B_V such that the corresponding measure μ' is equivalent to μ. The measures μ'_N and μ_N are also equivalent, and

$$d\mu'_N = \varphi_N(u^i)\, d\mu_N,$$

where $\phi_N(u^i)$ is some positive finite μ_N-integrable function on R_N. If ϕ_N is an analytic function

$$\phi_N(u^i) = \sum_{k=0}^{\infty} a_{j_1,\ldots,j_k} u^{j_1} \ldots u^{j_k},\ j = 1, \ldots, N,$$

the familiar relations

$$\begin{gathered} Z'_N(\lambda_i e^i) = \phi_N\left(\frac{1}{i}\frac{\partial}{\partial \beta_j}\right) Z\left((\lambda_i + \beta_i) u^i\right)\big|_{\beta_j = 0}, \\ F'(e^{i_1}, \ldots, e^{i_n}) = \sum_{k=0}^{\infty} a_{j_1,\ldots,j_k} F(e^{i_1}, \ldots, e^{i_n} e^{j_1}, \ldots, e^{j_k}) \end{gathered} \tag{11}$$

of perturbation theory hold. As a consequence, any vacuum expectation $F'(v^1, \ldots, v^n)$ of some fields $v^1, \ldots, v^n \in V$ can be expanded in terms of Feynman diagrams which describe interaction between quasi-free fields $v^1, \ldots, v^n$. This expansion, however, is not unique because it depends on the chosen space $R^N \ni v^1, \ldots, v^n$. Therefore, the above-mentioned interaction is an effective interaction of some quasi-particles, and we do not know primary particles with interaction that could be responsible for all vacuum expectations $F'(v)$, $v \in B_V$. One can suggest the following generalization of relations (11).

Let $\{e_i, i \in I\}$ be an orthonormal basis of V with respect to some scalar form $\langle | \rangle$ on V. Let R^I denotes a linear space of real functions φ on the set I and R^I be provided with the simple convergence topology [12]. The collection of forms

$$\varepsilon^i\colon R^I \ni \varphi \to \varphi(i) \in R,\ i \in I$$

is an algebraic basis of the dual $(R^I)'$, isomorphic to the subspace $V_I \subset V$ of vectors v admitting finite decomposition $v = \lambda_i e^i$ with respect to the basis $\{e^i\}$ of V. Then, the restriction of the Gaussian generating function of F to $V_I \subset V$ is the Fourier transform of some Gaussian pro measure μ_I in R^I (which is a measure if I is a countable set) [12]. Then, vacuum expectations $F'(v^1, \ldots, v^n)$ for any $v^1, \ldots, v^n \in V_I$ can be given by expressions of type (11) (where $j = 1, \ldots, N, \ldots$), although one has to verify convergence of all expansions.

For example, if

$$M(e^i, e^j) = \tfrac{1}{2}\delta^{ij}$$

the corresponding measure μ_I in R^I is

$$\prod_{i \in I} (\pi^{-1/2} \exp\{-(u^i)^2\}\, du^i). \tag{12}$$

Expression (9) in QFT is the formal continuum generalization of measure (12), and vacuum expectations in perturbed QFT are expanded in plane wave interaction terms in general.

In contrast to the formal expression (9) of perturbed QFT, the true integral representation (4) of generating functionals enables us to take into account non-Gaussian and inequivalent Gaussian representations of B_V. We shall consider one of them which may be suitable for the description of a Higgs vacuum. In the sequel, V is assumed to be a nuclear space.

Gaussian measures in infinite-dimensional spaces fail to be quasi-invariant under translations as a rule. Let μ be a Gaussian measure in the dual V' of a nuclear space V, and let μ_a be the image of μ under translation

$$\gamma\colon V' \ni u \to u + a \in V', \; a \in V'.$$

Measures μ and μ_a are equivalent if and only if the vector $a \in V'$ belongs to the canonical image V^d of V in V' with respect to the scalar form $\langle | \rangle = M(,)$ where M is the covariance form corresponding to the measure μ [5]. Then the measures μ and μ_a *define equivalent representations of B_V if the vector a is given by the relation*

$$\langle a, V \rangle = \langle V | v_a \rangle$$

for a certain vector $v_a \in V$. The equivalence of these representations is realized by the unitary operator

$$\varrho(u) \to \exp\{-\langle u | v_a \rangle\}\, \varrho(u + a), \; \varrho(u) \in L^2(V', \mu).$$

This operator cannot be constructed if $a \in V' \setminus V^d$. In this case measures μ and μ_a define inequivalent representations of B_V.

In QFT, a Higgs vacuum is one of the types of vacua with broken symmetries if this breakdown is due to interaction of matter fields with a Higgs scalar field which contains some classical background part σ_0. In algebraic QFT, one can describe free Higgs fields $\hat{\sigma}$ similarly as matter fields v by an algebra B_Σ, where Σ is a nuclear space. Let $Z(\hat{\sigma})$ be a generating function of a Gaussian state on B_Σ. Then, introduction of a Higgs vacuum means translation on the space Σ' such that the Gaussian measure μ corresponding to Z, is replaced by the measure μ_{σ_0} possessing the Fourier transform

$$Z'(\hat{\sigma}) = e^{i\langle\sigma_0,\hat{\sigma}\rangle} Z(\hat{\sigma}).$$

The translation vector σ_0 must belong to the set $\Sigma'\backslash\Sigma^d$ in order that the measures μ and μ_{σ_0} be inequivalent. Some corollaries of such Higgs vacuum description are as follows. Hereafter, we call $\sigma_0 \in \Sigma'\backslash\Sigma^d$ a Higgs vacuum field and $\sigma \in \Sigma^d$ a Higgs disturbance field.

Since the measures μ and μ_{σ_0} are inequivalent, appearance of Higgs vacuum fails to be a perturbed process.

A Higgs vacuum field σ_0, responsible for appearance of Higgs vacuum is a classical field in the sense that $\sigma_0 \in \Sigma'\backslash\Sigma^d$, whereas Higgs disturbance fields $\sigma \in \Sigma^d$ are quantized fields because they possess quantum partners $\hat{\sigma} \in \Sigma$.

A Higgs field σ_0 and fields σ belong to different classes of functions. For instance, one usually chooses a constant Higgs vacuum field in QFT; if $\Sigma = S(R^4)$ a constant function belongs to $S'(R^4)\backslash S(R^4)$. At the same time, since Σ^d is a dense subset of Σ', the elements σ_0 and σ can be arbitrarily close to each other with respect to the topology in Σ'. But the covariance form M^d and some other functionals well defined at points σ become singular at points σ_0. The fact that σ_0 and σ belong to different classes of functions, in our opinion, indicates different physical origins of these fields.

Let the generating functions Z and Z' be restricted to some finite-dimensional subspace $R^N \subset \Sigma$. Then, there exists an element $\sigma_{0N} \in R_N$ such that

$$\langle\sigma_0, \hat{\sigma}\rangle = \langle\hat{\sigma}_{0N}|\hat{\sigma}\rangle$$

for any $\hat{\sigma} \in R^N$. As a consequence, the generating function Z'_N is of the form

$$Z'_N(\lambda_i\hat{\sigma}^i) = (2\pi \det [M^{ij}])^{-N/2} \int \exp(i\lambda_i\sigma^i) \times \exp\{-\tfrac{1}{2}(M^{-1})_{ij}(\sigma^i - \sigma^i_{0N})(\sigma^j - \sigma^j_{0N})\} \, d^N\sigma,$$

where M^{ij} is the covariance matrix of Z_N, σ^i denote coordinates in R_N, and σ^j_{0N} are coordinates of the vector $\sigma_{0N} \in R_N$. Thus, if the number of quantum fields $\hat{\sigma}$ is finite, their interaction with a classical Higgs vacuum field looks like the interaction with some quantum fields $\hat{\sigma}_{0N}$ by perturbation theory. Fields $\hat{\sigma}_{0N}$, however, are different for various collections $\hat{\sigma}^1, \ldots, \hat{\sigma}^n$.

Finally, we briefly consider operation of the Wick rotation. Let F be a Gaussian state on the algebra $B_{S(R^4)}$ of Euclidean fields and M the corresponding covariance form represented by a distribution $M(y, y')$. Let $M(y, y')$ be invariant under Euclidean time translations and its Fourier transform be a holomorphic function of $k_0 = q_0 + ip_0$ except for points with $q_0 = 0$. Then, using the double Laplace transformation, one can construct a transformation of M in the bilinear form M' on Minkowski space fields, which is a generalization of the familiar Wick rotation defined for the form (8) and fields holomorphic in the region $\operatorname{Re} q \operatorname{Im} q \geqslant 0$. The form M' can be used in order to define a formal Gaussian-like generating function, but in general $M'(x, x')$ fails to be a distribution and M' is not a positive-definite quadratic continuous form on $S(R^4)$. Therefore, chronological forms on Minkowski space fields, except for those in S-matrix limits, have, in our opinion, no physical meaning.

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