A precise definition of an inference (by the example of natural deduction systems for logics $I_{<\alpha,\beta>}$)

Abstract. In the paper, we reconsider a precise definition of a natural deduction inference given by V. Smirnov. In refining the definition, we argue that all the other *indirect* rules of inference in a system can be considered as special cases of the implication introduction rule in a sense that if one of those rules can be applied then the implication introduction rule can be applied, either, but not vice versa. As an example, we use logics $I_{<\alpha,\beta>}$, $\alpha, \beta \in \{0, 1, 2, 3,\dots \omega\}$, such that $I_{<0,0>}$ is propositional classical logic, presented by V. Popov. He uses these logics, in particular, a Hilbert-style calculus $HI_{<\alpha,\beta>}$, $\alpha, \beta \in \{0, 1, 2, 3,\dots \omega\}$, for each logic in question, in order to construct examples of effects of Glivenko theorem's generalization. Here we, first, propose a subordinated natural deduction system $NI_{<\alpha,\beta>}$, $\alpha, \beta \in \{0, 1, 2, 3,\dots \omega\}$, for each logic in question, with a precise definition of a $NI_{<\alpha,\beta>}$-inference. Moreover, we, comparatively, analyze precise and traditional definitions. Second, we prove that, for each $\alpha, \beta \in \{0, 1, 2, 3,\dots \omega\}$, a Hilbert-style calculus $HI_{<\alpha,\beta>}$ and a natural deduction system $NI_{<\alpha,\beta>}$ are equipollent, that is, a formula A is provable in $HI_{<\alpha,\beta>}$ iff A is provable in $NI_{<\alpha,\beta>}$.

Абстракт. В статье мы вновь обращаемся к точному определению натурального вывода, которое дал В.А. Смирнов. При уточнении определения мы утверждаем, что все остальные *непрямые* правила вывода в системе могут рассматриваться как частные случаи правила введения импликации в том смысле, что если применимо одно из этих правил, то также применимо и правило введения импликации, но не наоборот. В качестве примера мы используем логики $I_{<\alpha,\beta>}$, $\alpha, \beta \in \{0, 1, 2, 3,\dots \omega\}$ такие, что $I_{<0,0>}$ - это классическая логика высказываний, которые предложил В.М. Попов. Он использует данные логики, в частности, исчисление Гильберта $HI_{<\alpha,\beta>}$, $\alpha, \beta \in \{0, 1, 2, 3,\dots \omega\}$, для каждой указанной логики, для того, чтобы построить примеры действия обобщенной теоремы Гливенко. Здесь мы, во-первых, предлагаем систему субординатного натурального вывода $NI_{<\alpha,\beta>}$, $\alpha, \beta \in \{0, 1, 2, 3,\dots \omega\}$, для каждой указанной логики, и даем точное определение $NI_{<\alpha,\beta>}$-вывода. Более того, мы проводим сравнительный анализ точного и традиционного определений. Во-вторых, мы показываем, для каждых $\alpha, \beta \in \{0, 1, 2, 3,\dots \omega\}$, эквиполентность исчисления Гильберта $HI_{<\alpha,\beta>}$ и системы натурального вывода $NI_{<\alpha,\beta>}$, то есть, что формула А доказуема в $HI_{<\alpha,\beta>}$ т.т.т. А доказуема в $NI_{<\alpha,\beta>}$.

Keywords: precise definition of inference, indirect rule, implication introduction rule, natural deduction, quasi-elemental formula, subordinated sequence

Ключевые слова: точное определение вывода, непрямое правило, правило введения импликации, натуральный вывод, квазиэлементарная формула, субординатная последовательность

Introduction

In [9], V. Popov presents logics $I_{<\alpha,\beta>}$ and Hilbert-style calculi $HI_{<\alpha,\beta>}$, $\alpha, \beta \in \{0, 1, 2, 3,\dots \omega\}$, for these logics, such that $I_{<0,0>}$ is propositional classical logic. He uses them in order to construct examples of effects of a generalization of Glivenko theorem. So, the purpose of the present paper is to present, within the framework of [7-8], a subordinated natural deduction (abbreviated passim as 'ND') calculus $NI_{<\alpha,\beta>}$, for each logic in question, with the precise definition of an $NI_{<\alpha,\beta>}$-inference, following the works of V. Smirnov [11, 13]. We, also, show the equipollentness between a Hilbert-style calculus $HI_{<\alpha,\beta>}$ and a ND system $NI_{<\alpha,\beta>}$, for each $\alpha, \beta \in \{0, 1, 2, 3,\dots \omega\}$, that is, a formula A is provable in $HI_{<\alpha,\beta>}$ iff A is provable in $NI_{<\alpha,\beta>}$.

Following [9], we fix a standard propositional language *L* over an alphabet $\{p, p_1, p_2,\dots, (,), \&, \vee, \supset, \neg\}$. A notion of a formula of language *L* is defined as usual. (Passim by 'a formula' we mean 'a formula of language *L*'.) A formula is said to be *quasi-elemental* iff no logical connective $\&, \vee, \supset$ occurs in it ([9]). A length of a formula A is said to be the number of all occurrences of the logical connectives in *L* in A. Letters A, B, C, D, E with lower indexes run over arbitrary formulae. Letters Γ, Δ with upper and lower indexes run over arbitrary finite sets of formulae. Letters α and β run over $\{0, 1, 2, 3,\dots \omega\}$ passim.

In [9], V. Popov presents a Hilbert-style calculus $HI_{<\alpha,\beta>}$. The language of the calculus is the language *L* mentioned above. We follow (and, for more details, refer the reader to) [9] in describing a Hilbert-style calculus $HI_{<\alpha,\beta>}$. A formula is an axiom of $HI_{<\alpha,\beta>}$ iff it is one of the following forms: (I) $(A\supset B)\supset((B\supset C)\supset(A\supset C))$, (II) $A\supset(A\vee B)$, (III) $B\supset(A\vee B)$, (IV) $(A\supset C)\supset((B\supset C)\supset((A\vee B)\supset C))$, (V) $(A\&B)\supset A$, (VI) $(A\&B)\supset B$, (VII) $(C\supset A)\supset((C\supset B)\supset(C\supset(A\&B)))$, (VIII) $(A\supset(B\supset C))\supset((A\&B)\supset C)$, (IX) $((A\&B)\supset C)\supset(A\supset(B\supset C))$, (X) $(((A\supset B)\supset A)\supset A)$, (XI,$\alpha$) $\neg D\supset(D\supset A)$, where D is a formula which is not a quasi-elemental formula of a length less than α, (XII,β) $(E\supset\neg(A\supset A))\supset\neg E$, where E is a formula which is not a quasi-elemental formula of a length less than β. *Modus ponens* is the only inference rule of the calculus.

Definitions of an inference in $HI_{<\alpha,\beta>}$ (abbreviated as $HI_{<\alpha,\beta>}$-inference) and a proof in $HI_{<\alpha,\beta>}$ are given in the standard way for a Hilbert-style calculus. Notions of the length of an inference and the length of a proof as well as the notion of a theorem are defined as usual.

In [9], the following fact is particularly highlighted: $I_{<0,0>}$ is propositional classical logic, where $I_{<0,0>}$ is the set of formulae provable in $HI_{<0,0>}$. This fact implies both schemata $A\supset(B\supset A)$ and $(A\supset(B\supset C))\supset((A\supset B)\supset(A\supset C))$ are theorems of $HI_{<0,0>}$ and, therefore, of each Hilbert-style calculus $HI_{<\alpha,\beta>}$, $\alpha, \beta \in \{0, 1, 2, 3,\dots \omega\}$. So, we, non-constructively, point out the standard deduction theorem holds for each calculus in question.

The paper is organized as follows. Section 1 presents a ND system $NI_{<\alpha,\beta>}$ with both precise and traditional definitions of an $NI_{<\alpha,\beta>}$-inference. In Section 2, the Hilbert-style calculus $NI_{<\alpha,\beta>}$ and the ND system $NI_{<\alpha,\beta>}$ are shown to be equipollent. The final section concludes the work and outlines the future research.

1. ND systems $NI_{<\alpha,\beta>}$

Let us set up a subordinated ND system $NI_{<\alpha,\beta>}$ and give a precise definition of a $NI_{<\alpha,\beta>}$-inference. The language of the system is, again, the language *L* mentioned above. There are two kinds of rules in the system. Here is the list of the rules of the first kind (sometimes called *direct*). The rules of the second kind (sometimes called *indirect*) are defined *with* the precise definition of an inference below.

The direct $NI_{<\alpha,\beta>}$-rules:

A&B	A&B	A, B
------$\&_{el1}$	-------$\&_{el2}$	------$\&_{in}$
A	B	A&B
A	B	$A\supset B$, A
-----$\vee_{in1}$	------$\vee_{in2}$	----------$\supset_{el}$
$A\vee B$	$A\vee B$	B

D, $\neg$D
---------$\neg_{in1(\alpha)}$, where D is a formula which is not a quasi-elemental formula of a length less than α.
A

The necessity of a precise definition of $NI_{<\alpha,\beta>}$-inference is illustrated with V. Smirnov's thesis: "… By natural deduction systems we shall refer to logistic systems with a special notion of an inference. *In these systems, an inference is more complex object than just a sequence of formulae or a tree-like of formulae*. Due to this property of natural deduction systems, a definite object entitled a formal inference corresponds to both direct and indirect ways of argument" [11, p. 96, both the translation and the *italics* are ours].

In defining both a $NI_{<\alpha,\beta>}$-inference и its length, we, with modifications, follow V. Smirnov [11, p. 116-118], [13, p. 245]. Letters η и ι with indexes denote $NI_{<\alpha,\beta>}$-inferences, a letter γ with indexes denotes *parts* of $NI_{<\alpha,\beta>}$-inferences, and a letter h denotes the length of an inference.[1]

An precise definition of $NI_{<\alpha,\beta>}$-*inference* and a definition a *height* of $NI_{<\alpha,\beta>}$-inference

1. A is an inference ι of A from a set of premises {A}, and $h(\iota) = 1$.

[1] In every case, the precise definition specifies which part of a $NI_{<\alpha,\beta>}$-inference is under consideration. The reason we introduce a special letter to run over parts of inference is that, in general, as we will see, a part of an inference is not an inference.

2. If η is an inference from Γ and A is a formula, then $\begin{array}{c}\eta\\ A\end{array}$ is an inference ι of A from $\{A\}\cup\Gamma$, and $h(\iota) = h(\eta)+1$.

3. If η is an inference from Γ, η contains $A_1, \ldots, A_k$ (k = 1, 2) and B is inferred from $A_1, \ldots, A_k$ via one of the rules $\&_{el1}$, $\&_{el2}$, $\&_{in}$, $\vee_{in1}$, $\vee_{in2}$, $\supset_{el}$ and $\neg_{in1(\alpha)}$, then $\begin{array}{c}\eta\\ B\end{array}$ is an inference ι of B from Γ, and $h(\iota) = h(\eta)+1$.

4. If η is an inference of B from $\{A\}\cup\Gamma$ and η is $\begin{array}{c}\gamma\\ \gamma_1\end{array}$, where γ_1 is a part of η, starting from the last premise A in η until B itself,[2] then $\begin{array}{l}\quad\gamma\\ |\gamma 1\\ A\supset B\end{array}$ is an inference ι of $A\supset B$ from Γ, and $h(\iota) = h(\eta)+1$.

5. If η is an inference of B from $\{A\}\cup\Gamma$ and η is $\begin{array}{c}\gamma\\ \gamma_1\end{array}$, where B is C, A is $C\supset D$, γ_1 is a part of η, starting from the last premise $C\supset D$ in η until C itself, then $\begin{array}{l}\ \ \gamma\\ |\gamma 1\\ \ \ C\end{array}$ is an inference ι of C from Γ, and $h(\iota) = h(\eta)+1$.

6. If η is an inference of B from $\{A\}\cup\Gamma$ and η is $\begin{array}{c}\gamma\\ \gamma_1\end{array}$, where B is $\neg(A\supset A)$, A is E, where E is a formula which is not a quasi-elemental formula of a length less than β and γ_1 is a part of η, starting from the last premise E in η until $\neg(A\supset A)$ itself, then $\begin{array}{l}\ \ \gamma\\ |\gamma 1\\ \neg E\end{array}$ is an inference ι of $\neg E$ from Γ, and $h(\iota) = h(\eta)+1$.

7. If η is an inference of B from $\{D_1\}\cup\{D_2\}\cup\Gamma$ and η is $\begin{array}{c}\gamma\\ \gamma 1\\ \gamma 2\end{array}$, where B is C and γ contains $D_1\vee D_2$, γ_1 is a part of η, starting from the last premise D_1 in η until C, γ_2 is a part of η, is a part of η, starting from the last premise D_2 in η until C *itself*, then $\begin{array}{l}\ \ \gamma\\ |\gamma 1\\ |\gamma 2\\ \ \ C\end{array}$ is an inference ι of C from Γ, and $h(\iota) = h(\eta)+1$.[3]

The *core of modifications* is as follows. An essential modification deals with V. Smirnov's suggestion that *any discarded* part of a $NI_{<\alpha,\beta>}$-inference is a $NI_{<\alpha,\beta>}$-inference. (A discarded part of an inference is marked with a horizontal line from the left.) This is not the case if a part of a $NI_{<\alpha,\beta>}$-inference contains a formula that is not a premise and is inferred from the formulae which this part of a $NI_{<\alpha,\beta>}$-inference does not contain. For example, in the clause 4, a part γ_1 may contain a formula that is inferred from some formula contained in a part γ (and γ may be an inference, itself). So, γ_1 is not an inference while $\begin{array}{c}\gamma\\ \gamma_1\end{array}$ is. Sometimes, V. Smirnov applies a notion of an *auxiliary inference* (or a *subderivation*) to such sequences of formulae as γ_1. The name of this notion obviously reflects the idea that such an inference plays a secondary role, and can be considered only with respect to the 'key' inference. However, we can't find it satisfactory that an auxiliary inference is shown not to be a kind of an inference. At last, minor modifications deal with evaluating a height of $NI_{<\alpha,\beta>}$-inference in $\supset_{el}$ and $\vee_{el}$ rules as well as with evaluating the height of a $NI_{<\alpha,\beta>}$-inference that now cannot be equal to 0.

Clause 4 (5, 6, and 7, respectfully) of the above definition is a formulation of an *indirect* rule of $\supset_{in}$ ($\supset_P$, $\neg_{in2(\beta)}$, and $\vee_{el}$, respectfully). We pay attention (and exemplify it below) to the fact that clauses 5-7 are special cases of clause 4. (In case of clause 7, the situation is a little bit more complex than in the other cases because it allows simultaneously discarding two parts of an inference, not one part. It is the reason why we choose clause 7 in the example below.) By the fact that a rule, say, $\supset_P$, is a special case of a rule $\supset_{in}$ we mean that if one can apply $\supset_p$ in the inference then one can apply $\supset_{in}$, either, but not vice versa. To be sure, we don't mean $\supset_p$ is derivable via $\supset_{in}$.

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[2] It is the last occurrence of B in η that is under consideration. In what follows, we will omit this specification everywhere, except clause 7.

[3] Clause 7 may have alternative formulations: 1) γ_3 occurs between γ_1 and γ_2; 2) γ_1 reorders with γ_2; 3) $D_1\vee D_2$ occurs below a part γ_1 etc. This analysis goes beyond the scope of the paper.

There different notation formats for a subordinated inference in ND systems [11, p. 119-126]. We will use so called Jaskowski-Quine notation in [2].[4]

Let us consider the following sequence of formulae:

1. $A \supset C$ – premise.
2. $B \supset C$ – premise.
3. $A \vee B$ – premise.
4. A – premise.
5. C – $\supset_{el}$: 1, 4
6. B – premise.
7. C – $\supset_{el}$: 2, 6

In accordance to clause 4, we have an inference of C from premises $A \supset C$, $B \supset C$, $A \vee B$, A, and B. Thus, we are legitimate to proceed with an inference of $B \supset C$ from premises $A \supset C$, $B \supset C$, $A \vee B$, and A:

1. $A \supset C$ – premise.
2. $B \supset C$ – premise.
3. $A \vee B$ – premise.
4. A – premise.
5. C – $\supset_{el}$: 1, 4

|6. B – premise.

|7. C – $\supset_{el}$: 2, 6

8. $B \supset C$ – $\supset_{in}$: 7

On the other hand, in accordance to clause 7, an inference of C from premises $A \supset C$, $B \supset C$, $A \vee B$, A, and B contains a part, starting from the last premise A until C (steps 4-5), and a part, starting from the last premise B until C (steps 6-7), as well as it contains $A \vee B$ which contains in no parts mentioned above. Thus, we are legitimate to proceed with an inference of C from premises $A \supset C$, $B \supset C$, and $A \vee B$:

1. $A \supset C$ – premise.
2. $B \supset C$ – premise.
3. $A \vee B$ – premise.

|4. A – premise.

|5. C – $\supset_{el}$: 1, 4

|6. B – premise.

|7. C – $\supset_{el}$: 2, 6

8. C – $\vee_{in}$: 3, 5, 7

As a result, we see the complexity of a notion of an inference in ND systems leads to the fact that *a sequence of formulae turns out to be different inferences of the same formula from different set of premises*. Discussing this fact (which is impossible for the other conventional proof systems like Hilbert-style calculus, sequent-style calculus and tree-like ND system) and its consequences is not a topic of the paper. We are fully aware, however, that the fact that a *precise definition of an inference leads to some ambivalence* seems to be absurd. But we strongly believe that the reason of this fact is caused by the nature of indirect argument, itself, which have been being under suspicion in the development of logic.[5]

On the other hand, the difference between direct and indirect rules has become more evident. A direct rule is applicable provided an inference contains formula (formulae) which is (are) *above the line* in a formulation of this rule. One can apply a direct rule to any formula; it is not necessary for the formula to be the last one in this inference. For example, in applying $\&_{el1}$ or some other direct rule, A&B (the one that is above the line) is not necessary the last formula of the inference, i.e., it is not necessary that this inference is an inference of A&B from (possibly, empty) Γ.

The situation is not the same in case of indirect rules. An indirect rule is applicable, too, provided there is an inference of the formula which is above the line in the formulation of this rule. The crucial difference is that it applies to the last formula in an inference only. (Note, at any moment, there is only

[4] In the literature, a subordinated inference is sometimes called a *linear-type* ND or a *Fitch-style* ND [10]. A subordinated inference differs from a tree-like inference presented by G. Gentzen [5], where, roughly, no formula is used more than once in the inference as a premise.

[5] It is well-known that intuitionists have been criticizing the *reductio ad absurdum*, a type of indirect argument.

one formula that is the last one in an inference.) For example, in applying $\supset_{in}$, a formula B (the one that is above the line) is, necessarily, the last one in an inference, i.e., it is a must that there is an inference of B from a non-empty set of premises Γ, where A is a member of Γ. In the example above, we see that one has different options in applying indirect rules in the same way one has different options in applying direct rules. However, *it is impossible for a sequence of formulae to be an inference of two formulae*: such a possibility is allowed by so called traditional formulation of some indirect rules (in the next subsection it holds for the traditional formulation of $\vee_{el}$). Sometimes negation introduction rule is formulated, roughly, as follows: if there is an inference of both formulae A and ¬A from the last premise C then there is an inference of a formula ¬C [1, p. 140], or 'to be applied, an indirect rule $\neg_{in}$ requires two auxiliary inferences Γ, A |– B and Γ, A |– ¬B' [12, p. 66]. In general, one can't determine another inference ('auxiliary inference') 'inside' a given inference. Let us, again, say that discussing it goes beyond the scope of this paper.

In the end of this section, let us present so called the *traditional* formulations of both indirect rules and of an inference.[6] In the rules below, a formula A (A⊃B or E) is the *last* premise. In $\neg_{in2(\beta)}$, a formula E is, additionally, a formula which is not a quasi-elemental formula of a length less than β. In this subsection, by 'inference' we mean '$NI_{<\alpha,\beta>}$-inference'.

[A] B	[A⊃B] A	[E] ¬(B⊃B)
-------$\supset_{in}$	------------$\supset_{P}$	---------------$\neg_{in2(\beta)}$
A⊃B	A	¬E

An inference is said to be a non-empty finite linearly ordered sequence of formulae $C_1, C_2, \ldots, C_k$, satisfying the following conditions:[7]

- Each C_i is either a premise or is inferred from the previous formulae via a rule;
- In applying $\supset_{in}$, each formula, starting from the last premise A until A⊃B, the result of this application, exclusively, is discarded from an inference;
- In applying $\supset_{P}$, each formula, starting from the last premise A⊃B until A, the result of this application, exclusively, is discarded from an inference;
- In applying $\neg_{in2(\beta)}$,each formula, starting from the last premise E until ¬E, the result of this application, exclusively, is discarded from an inference.

Given an inference $C_1, C_2, \ldots, C_k$ with $A_1, A_2, \ldots, A_n$ being non-discarded premises and with the last formula C_k being graphically identical to B, we say this is an inference of B from premises $A_1, A_2, \ldots, A_n$. If a set of formulae Γ contains $A_1, A_2, \ldots, A_n$ and there is an inference of B from premises $A_1, A_2, \ldots, A_n$ then we say there is an inference of B from a set of formulae Γ [2, p. 129-130].

2. Metatheory of a ND system

We proof the following *Theorem*: $\Gamma \mid\!-_{HI<\alpha,\beta>} A \Leftrightarrow \Gamma \mid\!-_{NI<\alpha,\beta>} A$, for each $\alpha, \beta \in \{0, 1, 2, 3, \ldots \omega\}$.

Proof ⇒. Proof is by the method of complete induction on a height s of an arbitrary $HI_{<\alpha,\beta>}$-inference of A from Γ.[8]

The scheme of complete induction is as follows: $(P(1) \,\&\, \forall x(\forall y((y < x) \supset P(y)) \supset P(x))) \supset \forall x P(x)$.

Let P(s) denote a sentence "if there is a $HI_{<\alpha,\beta>}$-inference of a height s of A from Γ then there is a $NI_{<\alpha,\beta>}$-inference of A from Γ".

Then the scheme looks as follows: ((if there is a $HI_{<\alpha,\beta>}$-inference of a height 1 of A from Γ then there is a $NI_{<\alpha,\beta>}$-inference of A from Γ) & ∀s(∀t((t < s) ⊃ (if there is a $HI_{<\alpha,\beta>}$-inference of a height t of A from Γ then there is a $NI_{<\alpha,\beta>}$-inference of A from Γ)) ⊃ (if there is a $HI_{<\alpha,\beta>}$-inference of a height s of A

[6] For the sake of simplicity and without loss of generality, we don't present a traditional formulation of $\vee_{el}$ and refer the reader to, for example, [6]. Note, sometimes, the traditional formulation of the indirect rules includes the derivability symbol '|–' [1].

[7] Here is (of course, incomplete) list of (text)books reproducing the traditional formulation one way or another: [1-4, 6, 14-15]. On the other hand, we are fully aware that textbooks' authors are, mostly, driven by pedagogy trying to 'not go deep into theoretical subtleties of all kinds' and following the principle 'to tell the truth and only the truth, but not all the truth' [2, p. 11, 12].

[8] We recall the standard definition of a length of an inference in a Hilbert-style calculus.

from Γ then there is a NI$_{<\alpha,\beta>}$-inference of A from Γ))) $\supset \forall$s(if there is a HI$_{<\alpha,\beta>}$-inference of a height s of A from Γ then there is a NI$_{<\alpha,\beta>}$-inference of A from Γ).

The *base case* is trivial according to the definitions of inferences in both HI$_{<\alpha,\beta>}$ and NI$_{<\alpha,\beta>}$.

We prove the *inductive step*: $\forall$s($\forall$t((t < s) $\supset$ (if there is a HI$_{<\alpha,\beta>}$-inference of a height t of A from Γ then there is a NI$_{<\alpha,\beta>}$-inference of A from Γ)) $\supset$ (if there is a HI$_{<\alpha,\beta>}$-inference of a height s of A from Γ then there is a NI$_{<\alpha,\beta>}$-inference of A from Γ)).

For modus ponens is an inference rule in both HI$_{<\alpha,\beta>}$ and NI$_{<\alpha,\beta>}$, it is enough to show that every HI$_{<\alpha,\beta>}$-axiom is provable in NI$_{<\alpha,\beta>}$. We confine ourselves to proving two specific HI$_{<\alpha,\beta>}$-axioms: axiom (XI,α) $\neg D\supset(D\supset A)$, where D is a formula which is not a quasi-elemental formula of a length less than α, and axiom (XII,β) $(E\supset\neg(A \supset A))\supset\neg E$, where E is a formula which is not a quasi-elemental formula of a length less than β

$|\!-_{NI<\alpha,\beta>}$ $\neg D\supset(D\supset A)$
|1. $\neg D$ – premise
||2. D – premise
||3. A – $_{in1(\alpha)}$: 1, 2
|4. $D\supset A$ – $\supset_{in}$: 3
5. $\neg D\supset(D\supset A)$ – $\supset_{in}$: 4

$|\!-_{NI<\alpha,\beta>}$ $(E\supset\neg(A \supset A))\supset\neg E$
|1. $E\supset\neg(A \supset A)$ – premise
||2. E – premise
||3. $\neg(A \supset A)$ – $\supset_{el}$: 1, 2
|4. $\neg E$ – $_{in2(\beta)}$: 3
5. $(E\supset\neg(A \supset A))\supset\neg E$ – $\supset_{in}$: 4

Proof $\Leftarrow$. Proof is by the method of complete induction on a height n of an arbitrary NI$_{<\alpha,\beta>}$-inference of A from Γ.

The scheme of complete induction is as follows: (Q(1) & $\forall$x($\forall$y((y < x) $\supset$ Q(y)) $\supset$ Q(x))) $\supset$ $\forall$xQ(x).

Let Q(n) denote a sentence "if there is a NI$_{<\alpha,\beta>}$-inference of a height n of A from Γ then there is a HI$_{<\alpha,\beta>}$-inference of A from Γ".

Then the scheme looks as follows: ((if there is a NI$_{<\alpha,\beta>}$-inference of a height 1 of A from Γ then there is a HI$_{<\alpha,\beta>}$-inference of A from Γ) & $\forall$n($\forall$q((q < n) $\supset$ (if there is a NI$_{<\alpha,\beta>}$-inference of a height q of A from Γ then there is a HI$_{<\alpha,\beta>}$-inference of A from Γ)) $\supset$ (if there is a NI$_{<\alpha,\beta>}$-inference of a height n of A from Γ then there is a HI$_{<\alpha,\beta>}$-inference of A from Γ))) $\supset$ $\forall$n(if there is a NI$_{<\alpha,\beta>}$-inference of a height n of A from Γ then there is a HI$_{<\alpha,\beta>}$-inference of A from Γ).

The *base case*: h(η) = 1.

According to clause 1 of the definition of a NI$_{<\alpha,\beta>}$-inference, a NI$_{<\alpha,\beta>}$-inference η of a height 1 of A from a set за premises Γ looks as follows: A is an inference from {A}:

1. A – premise.

This inference corresponds to the following HI$_{<\alpha,\beta>}$-inference of A from a set за premises {A}:

1. A – premise.

We prove the *inductive step*: $\forall$n($\forall$q((q < n) $\supset$ (if there is a NI$_{<\alpha,\beta>}$-inference of a height q of A from Γ then there is a HI$_{<\alpha,\beta>}$-inference of A from Γ)) $\supset$ (if there is a NI$_{<\alpha,\beta>}$-inference of a height n of A from Γ then there is a HI$_{<\alpha,\beta>}$-inference of A from Γ)).

According to clauses 2-7 of the definition of a NI$_{<\alpha,\beta>}$-inference, a NI$_{<\alpha,\beta>}$-inference η of a height n of A from a set за premises Γ looks as one of the six following cases:

Case 1 (2nd clause of the definition of a NI$_{<\alpha,\beta>}$-inference): $\frac{\eta'}{A}$, where η' is an inference from a set за premises Γ' and Γ is {A}$\cup$Γ'.

Γ'
…
n. A – premise.

For $h(\eta') < h(\eta)$,[9] one can, by the inductive hypothesis, build up a $HI_{<\alpha,\beta>}$-inference from a set за premises Γ'. Then a $HI_{<\alpha,\beta>}$-inference of A from a set за premises Γ looks as follows:

Γ'

…

n'. A – premise.

Case 2 (3rd clause of the definition of a $NI_{<\alpha,\beta>}$-inference): $\frac{\eta'}{A}$, where η' is an inference of C from a set за premises Γ', η' contains $A_1, \ldots, A_k$; A is inferred from $A_1, \ldots, A_k$ via one of the rules $\&_{el1}$, $\&_{el2}$, $\&_{in}$, $\vee_{in1}$, $\vee_{in2}$, $\supset_{el}$, and $\neg_{in1(\alpha)}$.

Subcase 2.1.: η' contains ¬D and D; A is inferred from ¬D and D via $\neg_{in1(\alpha)}$, where j < n-1 and m < n-1.

Γ

…

j. ¬D

…

m. D

…

n-1. C

n. A - $\neg_{in1(\alpha)}$: j, m

Let η' be an $NI_{<\alpha,\beta>}$-inference of C from Γ, η'_1 be an $NI_{<\alpha,\beta>}$-inference of ¬D from Γ, and η'_2 be an $NI_{<\alpha,\beta>}$-inference of D from Γ, where $h(\eta'_1) < h(\eta')$ and $h(\eta'_2) < h(\eta')$, by the definition. The fact that $h(\eta') < h(\eta)$, implies that $h(\eta'_1) < h(\eta)$ and $h(\eta'_2) < h(\eta)$, and, by the inductive hypothesis, one can build up the following $HI_{<\alpha,\beta>}$-inferences: a $HI_{<\alpha,\beta>}$-inference of ¬D from Γ, a $HI_{<\alpha,\beta>}$-inference of D from Γ, and a $HI_{<\alpha,\beta>}$-inference of C from Γ. Then a $HI_{<\alpha,\beta>}$-inference of A from a set за premises Γ looks as follows:

Γ

…

j'. ¬D

…

m'. D

…

n'-1. C

n'. $\neg D\supset(D\supset A)$ - $HI_{<\alpha,\beta>}$-axiom (XI,α)

n'+1. A – modus ponens: j', m', n' (two times)

Subcase 2.2.: η' contains A&B; A is inferred from A&B via $\&_{el1}$, where m < n-1.

Γ

…

m. A&B

…

n-1. C

n. A - $\&_{el1}$: m

Let η' be an $NI_{<\alpha,\beta>}$-inference of C from Γ and η'_1 be an $NI_{<\alpha,\beta>}$-inference of A&B from Γ, where $h(\eta'_1) < h(\eta')$, by the definition. The fact that $h(\eta') < h(\eta)$, implies that $h(\eta'_1) < h(\eta)$ and, by the inductive hypothesis, one can build up the following $HI_{<\alpha,\beta>}$-inferences: a $HI_{<\alpha,\beta>}$-inference of A&B from Γ, a $HI_{<\alpha,\beta>}$-inference of C from Γ. Then a $HI_{<\alpha,\beta>}$-inference of A from a set за premises Γ looks as follows:

Γ

…

m'. A&B

…

n'-1. C

n'. $(A\&B)\supset A$ - $HI_{<\alpha,\beta>}$-axiom (V)

[9] By the definition, $h(\frac{\eta'}{A}) = h(\eta')+1$.

n'+1. A - modus ponens: m', n'

Subcase 2.3., where η' contains B&A; A is inferred from B&A via $\&_{el2}$, is treated analogously to subcase 2.2.

Subcase 2.4.: η' contains B and D; A is B&D and is inferred from B and D via $\&_{in}$, where f < m, j < n-1, and m < n-1.

Γ'

…

j. B

…

m. D

…

n-1. C

n. B&D - $\&_{in}$: j, m

Let η' be an $NI_{<\alpha,\beta>}$-inference of C from Γ, η'_1 be an $NI_{<\alpha,\beta>}$-inference of B from Γ, and η'_2 be an $NI_{<\alpha,\beta>}$-inference of D from Γ, where $h(\eta'_1) < h(\eta')$ and $h(\eta'_2) < h(\eta')$, by the definition. The fact that $h(\eta') < h(\eta)$, implies that $h(\eta'_1) < h(\eta)$ and $h(\eta'_2) < h(\eta)$, and, by the inductive hypothesis, one can build up the following $HI_{<\alpha,\beta>}$-inferences: a $HI_{<\alpha,\beta>}$-inference of B from Γ, a $HI_{<\alpha,\beta>}$-inference of D from Γ, and a $HI_{<\alpha,\beta>}$-inference of C from Γ. Then a $HI_{<\alpha,\beta>}$-inference of B&D from Γ looks as follows:

Γ

…

f'. A_1 – any $HI_{<\alpha,\beta>}$-axiom

f'+1. $B\supset(A_1\supset B)$ - $HI_{<\alpha,\beta>}$-theorem

f'+2. $D\supset(A_1\supset D)$ - $HI_{<\alpha,\beta>}$-theorem

…

j'. B

j'+1. $A_1\supset B$ - modus ponens: f'+1, j'

…

m'. D

m'+1. $A_1\supset D$ - modus ponens: f'+2, m'

…

n'-1. C

n'. $(A_1\supset B)\supset((A_1\supset D)\supset(A_1\supset(B\&D)))$ - $HI_{<\alpha,\beta>}$-axiom (VII)

n'+1. B&D - modus ponens: j+1', m+1', f', n' (three times)

Subcase 2.5.: η' contains B; A is $B\vee D$ and is inferred from B via $\vee_{in1}$, where m < n-1.

Γ

…

m. B

…

n-1. C

n. $B\vee D$ - $\vee_{in1}$: m

Let η' be an $NI_{<\alpha,\beta>}$-inference of C from Γ and η'_1 be an $NI_{<\alpha,\beta>}$-inference of B from Γ, where $h(\eta'_1) < h(\eta')$, by the definition. The fact that $h(\eta') < h(\eta)$, implies that $h(\eta'_1) < h(\eta)$ and, by the inductive hypothesis, one can build up the following $HI_{<\alpha,\beta>}$-inferences: a $HI_{<\alpha,\beta>}$-inference of B from Γ, a $HI_{<\alpha,\beta>}$-inference of C from Γ. Then a $HI_{<\alpha,\beta>}$-inference of $B\vee D$ from a set за premises Γ looks as follows:

Γ

…

m'. B

…

n'-1. C

n'. $B\supset(B\vee D)$ - $HI_{<\alpha,\beta>}$-axiom (II)

n'+1. $B\vee D$ - modus ponens: m', n'

Subcase 2.6., where η' contains D; A is $B\vee D$ and is inferred from D via $\vee_{in2}$, is treated analogously to subcase 2.5.

Subcase 2.7.: η' contains B⊃A and B; A is inferred from B⊃A and B via $\supset_{el}$, where j < n-1, and m < n-1.

Γ
…
j. B⊃A
…
m. B
…
n-1. C
n. A - $\supset_{el}$: j, m

Let η' be an $NI_{<\alpha,\beta>}$-inference of C from Γ, η'_1 be an $NI_{<\alpha,\beta>}$-inference of B⊃A from Γ, and η'_2 be an $NI_{<\alpha,\beta>}$-inference of B from Γ, where $h(\eta'_1) < h(\eta')$ and $h(\eta'_2) < h(\eta')$, by the definition. The fact that $h(\eta') < h(\eta)$, implies that $h(\eta'_1) < h(\eta)$ and $h(\eta'_2) < h(\eta)$, and, by the inductive hypothesis, one can build up the following $HI_{<\alpha,\beta>}$-inferences: a $HI_{<\alpha,\beta>}$-inference of B⊃A from Γ, a $HI_{<\alpha,\beta>}$-inference of B from Γ, and a $HI_{<\alpha,\beta>}$-inference of C from Γ. Then a $HI_{<\alpha,\beta>}$-inference of A from Γ looks as follows:

Γ
…
j'. B⊃A
…
m'. B
…
n'-1. C
n'. A - modus ponens: j', m'

Case 3 (4th clause of the definition of a $NI_{<\alpha,\beta>}$-inference). A is B⊃C and a $NI_{<\alpha,\beta>}$-inference η of a height n of B⊃C from Γ looks as follows: $\begin{array}{c}\gamma\\ |\gamma 1\\ B\supset C\end{array}$, where $\begin{array}{c}\gamma\\ \gamma 1\end{array}$ is a $NI_{<\alpha,\beta>}$-inference ι of C from {B}∪Γ, γ_1 is a part of ι, starting from the last premise B in ι until C, itself, and m < n-1.

Γ
…
|m. B – premise
|…
|n-1. C
n. B⊃C - $\supset_{in}$: n-1

For $h(\iota) < h(\eta)$, one can, by the inductive hypothesis,[10] build up a $HI_{<\alpha,\beta>}$-inference of C from {B}∪Γ. Then a $HI_{<\alpha,\beta>}$-inference of B⊃C from Γ looks as follows:

Γ
…
m'. B – premise
…
n'-1. C
n'. B⊃C – deduction theorem: m', n'-1

Case 4 (5th clause of the definition of a $NI_{<\alpha,\beta>}$-inference). A $NI_{<\alpha,\beta>}$-inference η of a height n of A from Γ looks as follows: $\begin{array}{c}\gamma\\ |\gamma 1\\ A\end{array}$, where $\begin{array}{c}\gamma\\ \gamma 1\end{array}$ is a $NI_{<\alpha,\beta>}$-inference ι of A from {A⊃B}∪Γ, γ_1 is a part of ι, starting from the last premise A⊃B in ι until A, itself, and m < n-1.

Γ
…
|m. A⊃B – premise

[10] Here and in the cases below, we stress the fact that we proceed from one inference to another inference, not from a *part* of an inference to another inference. So, the inductive hypothesis of the theorem is applicable.

|…

|n-1. A

n. A - $\supset_P$: n-1

For $h(\iota) < h(\eta)$, one can, by the inductive hypothesis, build up a $HI_{<\alpha,\beta>}$-inference of A from $\{A\supset B\}\cup\Gamma$. Then a $HI_{<\alpha,\beta>}$-inference of A from Γ looks as follows:

Γ

…

m′. $A\supset B$ – premise

…

n′-1. A

n′. $(A\supset B)\supset A$ – deduction theorem: m′, n′-1

n′+1. $((A\supset B)\supset A)\supset A$ - $HI_{<\alpha,\beta>}$-axiom (X)

n′+2. A – modus ponens: n′, n′+1

Case 5 (6th clause of the definition of a $NI_{<\alpha,\beta>}$-inference). A is $\neg E$, where E is a formula which is not a quasi-elemental formula of a length less than β, and a $NI_{<\alpha,\beta>}$-inference η of a height n of $\neg E$ from Γ looks as follows: $\begin{array}{c}\gamma\\ |\gamma_1\\ \neg E\end{array}$, where $\begin{array}{c}\gamma\\ \gamma_1\end{array}$ is a $NI_{<\alpha,\beta>}$-inference ι of $\neg(A\supset A)$ from $\{E\}\cup\Gamma$, γ_1 is a part of ι, starting from the last premise E in ι until $\neg(A\supset A)$, itself, and m < n-1.

Γ

…

|m. E – premise

|…

|n-1. $\neg(A\supset A)$

n. $\neg E$ - $\neg_{in2(\beta)}$: n-1

For $h(\iota) < h(\eta)$, one can, by the inductive hypothesis, build up a $HI_{<\alpha,\beta>}$-inference of $\neg(A\supset A)$ from $\{E\}\cup\Gamma$. Then a $HI_{<\alpha,\beta>}$-inference of A from Γ looks as follows:

Γ

…

m′. E – premise

…

n′-1. $\neg(A\supset A)$

n′. $E\supset\neg(A\supset A)$ – deduction theorem: m′, n′-1

n′+1. $(E\supset\neg(A\supset A))\supset\neg E$ - $HI_{<\alpha,\beta>}$-axiom (XII,β)

n′+2. A – modus ponens: n′, n′+1

Case 6 (7th clause of the definition of a $NI_{<\alpha,\beta>}$-inference). a $NI_{<\alpha,\beta>}$-inference η of a height n of A from Γ looks as follows: $\begin{array}{c}\gamma\\ |\gamma_1\\ |\gamma_2\end{array}$, where $\begin{array}{c}\gamma\\ \gamma_1\\ \gamma_2\\ A\end{array}$ is a $NI_{<\alpha,\beta>}$-inference ι of A $\{D\}\cup\{B\}\cup\Gamma$, γ contains $D\vee B$, γ_1 is a part of η, starting from a premise D in η until A, γ_2 is a part of η, starting from the last premise B in η until A, itself, and f < g, g < j, j < n-1.[11]

Γ

…

f. $D\vee B$

…

|g. D – premise

|…

|j. A

|j+1. B – premise

|…

|n-1. A

[11] On alternatives of this case see the footnote to the 7th clause of the definition of an $NI_{<\alpha,\beta>}$-inference.

n. A - $\vee_{el}$: f, j, n-1

First, let us consider a $NI_{<\alpha,\beta>}$-inference ι of A from $\{D\}\cup\{B\}\cup\Gamma$.

Γ

…

f. D$\vee$B

…

g. D – premise

…

j. A

j+1. B – premise

…

n-1. A

By the construction, ι contains the following $NI_{<\alpha,\beta>}$-inferences: ι_1 of D$\vee$B from Γ, ι_2 of A from $\{D\}\cup\Gamma$, and ι_3 of A from $\{B\}\cup\{D\}\cup\Gamma$.

For $h(\iota_i) < h(\iota)$,[12] for each i from {1, 2, 3}, one can build up a $HI_{<\alpha,\beta>}$-inference of D$\vee$B from Γ, a $HI_{<\alpha,\beta>}$-inference of A from $\{D\}\cup\Gamma$, and a $HI_{<\alpha,\beta>}$-inference of A from $\{B\}\cup\{D\}\cup\Gamma$. Then a $HI_{<\alpha,\beta>}$-inference of A from $\{D\}\cup\{B\}\cup\Gamma$ looks as follows:

Γ

…

f'. D$\vee$B

…

g'. D – premise

…

j'. A

j'+1. B – premise

…

n'-1. A

So, a $HI_{<\alpha,\beta>}$-inference of A from Γ looks as follows:

Γ

…

f'. D$\vee$B

…

g'. D – premise

…

j'. A

j'+1. B – premise

…

n'-1. A

n'. B$\supset$A – deduction theorem: j'+1, n'-1

Γ

…

f''. D$\vee$B

…

g''. D – premise

…

j''. A

j''+1. D$\supset$A – deduction theorem: g'', j''

j''+2. (D$\supset$A)$\supset$((B$\supset$A)$\supset$((D$\vee$B)$\supset$A)) - $HI_{<\alpha,\beta>}$-axiom (IV)

j''+3. A – $\supset_{el}$: f'', j''+1, j''+2 (three times)

[12] Unlike the other cases, this case requires the inductive hypothesis hold true for a $NI_{<\alpha,\beta>}$-inference of *any* length less than the length of ι, *not only* for a $NI_{<\alpha,\beta>}$-inference of a length *h(ι)-1*.

The Theorem implies a *Corollary*: for each $\alpha, \beta \in \{0, 1, 2, 3, \ldots \omega\}$, a Hilbert-style calculus $HI_{<\alpha,\beta>}$ and a ND system $NI_{<\alpha,\beta>}$ are equipollent, i.e., A is a $HI_{<\alpha,\beta>}$-theorem iff A is a $NI_{<\alpha,\beta>}$-theorem.

Final remarks

In the paper, for each logic, $I_{<\alpha,\beta>}$, $\alpha, \beta \in \{0, 1, 2, 3, \ldots \omega\}$, such that $I_{<0,0>}$ is propositional classical logic [9], we, continuing the series of works [7-8], present a subordinated ND system $NI_{<\alpha,\beta>}$. Moreover, each ND system has a precise definition of an inference which is a modification of V. Smirnov's approach. Our approach highlights a view on the implication introduction rule as the *genus* for the other indirect rules. Using a Hilbert-style calculus $HI_{<\alpha,\beta>}$, for each logic in question, presented by V. Popov [9], we show that a formula A is provable in $HI_{<\alpha,\beta>}$ iff it is provable in $NI_{<\alpha,\beta>}$. In the future, we point out studying consequences of the precise definition with an application to complexity problems. Last, not least, we see forward to formulating proof searching procedures for these ND systems in the fashion of [3-4].

Corrections

The paper "Natural deduction in a paracomplete setting" by A. Bolotov and V. Shangin to have been published in this Journal's 20th volume needs two corrections. First, the 23rd entry in the references list should be replaced with "Popov V. and Solotschenkov A. Semantics of propositional paracomplete Nelson logic // Integrated scientific journal. V. 8. 2012. P. 31-32. (in Russian).". Second, the truth-table definitions for the connectives of logic PComp in the 2nd section must be added with the following footnote: A. Avron had told V. Popov about these definitions at the World Congress on Paraconsistency (Ghent, 1997) and then V. Popov told one of the paper's authors about these definitions.

Acknowledgments

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Точное определение натурального вывода (на примере систем натурального вывода для логик $I_{<\alpha,\beta>}$)
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An precise definition of a natural deduction inference (by the example of natural deduction systems for logics $I_{<\alpha,\beta>}$)