# FDE and its Relatives, Part II: Axiomatization

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- Algebraic semantics for **FDE** relatives
  **D. V. Zaitsev, ’FDE and its Relatives’**
  (plenary talk)

- Dunn-Belnap logic or simply **FDE** operates with well-known four truth-values from the set $\mathbf{4} = \{\mathbf{T}, \mathbf{B}, \mathbf{N}, \mathbf{F}\}$.
- We can use semilattices to provide a semantics for recently discovered **FDE** relatives: **Exactly True Logic** (**ETL**) and **Non-Falsity Logic** (**NFL**).

We consider a family of logics over the same propositional language $Lang$, which we define in the Backus-Naur form:

$$A := p \mid {\sim}A \mid A \wedge A \mid A \vee A$$

We use $Form$ to denote the set of all formulas of $Lang$, and $Var$ represents the set of all propositional variables of $Lang$.

$\mathcal{V}_{fde} = \langle\{\mathbf{T}, \mathbf{B}, \mathbf{N}, \mathbf{F}\}, \{\mathbf{T}, \mathbf{B}\}, \{f_\wedge, f_\vee, f_\neg\}\rangle$, where $f_i \in \{f_\wedge, f_\vee, f_\neg\}$ being an $n$-ary function if $c_i \in \{\wedge, \vee, \neg\}$ is also an $n$-ary connective.

- If we consider $\{\mathbf{T}\}$ as the set of designated values, then we obtain $\mathcal{V}_{etl}$.
- If we consider $\{\mathbf{T}, \mathbf{B}, \mathbf{N}\}$ as the set of designated values, then we obtain $\mathcal{V}_{nfl}$.

Any of these *valuation systems* might be equipped with an assignment function $a$, that maps $Var$ into $\{\mathbf{T}, \mathbf{B}, \mathbf{N}, \mathbf{F}\}$. One can extend this function to all formulas, following conditions below:

$\forall p \in Var, \upsilon(p) = a(p)$

$\forall c_i \in C, \upsilon(c_i(A_1, \dots, A_n)) = f_{c_i}(\upsilon(A_1), \dots, \upsilon(A_n))$, with $C$ – a set of logical connectives of $Lang$.

Then, we can define three consequence relations:

- **FDE**-consequence:
  $\Gamma \models_{fde} A \Leftrightarrow$ for any valuation $\upsilon$, if $\upsilon[\Gamma] \subseteq \{\mathbf{T}, \mathbf{B}\}$, then $h(A) \in \{\mathbf{T}, \mathbf{B}\}$.
- **ETL**-consequence:
  $\Gamma \models_{etl} A \Leftrightarrow$ for any valuation $\upsilon$, if $\upsilon[\Gamma] \subseteq \{\mathbf{T}\}$, then $h(A) \in \{\mathbf{T}\}$.
- **NFL**-consequence:
  $\Gamma \models_{nfl} A \Leftrightarrow$ for any valuation $\upsilon$, if $\upsilon[\Gamma] \subseteq \{\mathbf{T}, \mathbf{B}, \mathbf{N}\}$, then $h(A) \in \{\mathbf{T}, \mathbf{B}, \mathbf{N}\}$.

These relations could easily be reduced from expressions of the *set-formula* type to expressions of the *formula-formula* type.

# Some semantical facts

| **Logics** | **Fallacies** |
|---|---|
| **F**irst **D**egree **E**ntailments: | $A \wedge \neg A \not\models_{fde} B$ (absurdity)<br>$B \not\models_{fde} A \vee \neg A$ (triviality)<br>$A \wedge \neg A \not\models_{fde} B \vee \neg B$ (safety) |
| **E**xactly **T**rue **L**ogic: | $A \wedge {\sim}A \models_{etl} B$ (absurdity)<br>$A \wedge {\sim}A \models_{etl} B \vee {\sim}B$ (safety)<br>$B \not\models_{etl} A \vee {\sim}A$ (triviality) |
| **N**on **F**alsity **L**ogic: | $B \models_{nfl} A \vee {\sim}A$ (triviality)<br>$A \wedge {\sim}A \models_{nfl} B \vee {\sim}B$ (safety)<br>$A \wedge {\sim}A \not\models_{nfl} B$ (absurdity) |

**Exactly True Logic**:

$$A \wedge {\sim}A \models_{etl} C \qquad\qquad B \wedge {\sim}B \models_{etl} C$$

but

$$(A \wedge {\sim}A) \vee (B \wedge {\sim}B) \not\models_{etl} C$$

**Non Falsity Logic**:

$$A \models_{nfl} (B \vee {\sim}B) \qquad\qquad A \models_{nfl} (C \vee {\sim}C)$$

but

$$A \not\models_{nfl} (B \vee {\sim}B) \wedge (C \vee {\sim}C)$$

# Some semantical facts. Counterexample

**Exactly True Logic**:

$$A \wedge \sim A \models_{etl} B$$

but

$$\sim B \not\models_{etl} \sim(A \wedge \sim A)$$

**Non Falsity Logic**:

$$A \models_{nfl} B \vee \sim B$$

but

$$\sim(B \vee \sim B) \not\models_{nfl} \sim A$$

- **Exactly True Logic** (**ETL**)
  2013 Pietz, A., Rivieccio, U. ’Nothing but the Truth’
  *Journal of Philosophical Logic*

- **Non-Falsity Logic** (**NFL**)
  Shramko, Y., Zaitsev, D., Belikov, A. ’First Degree Entailment and its Children’ (to appear in *Studia Logica*)

**Definition**

Denote by $\mathbf{PR_{etl}}$ the sentential logic defined through the following set of rules (and no axioms), where $p, q, r \in Var$:

$$\frac{p \wedge q}{p}\,(R1) \qquad \frac{p \wedge q}{q}\,(R2) \qquad \frac{p \quad q}{p \wedge q}\,(R3) \qquad \frac{p}{p \vee q}\,(R4)$$

$$\frac{p \vee q}{q \vee p}\,(R5) \qquad \frac{p \vee p}{p}\,(R6) \qquad \frac{p \vee (q \vee r)}{(p \vee q) \vee r}\,(R7) \qquad \frac{p \vee (q \wedge r)}{(p \vee q) \wedge (p \vee r)}\,(R8)$$

$$\frac{(p \vee q) \wedge (p \vee r)}{p \vee (q \wedge r)}\,(R9) \qquad \frac{p \vee r}{\sim\sim p \vee r}\,(R10) \qquad \frac{\sim\sim p \vee r}{p \vee r}\,(R11)$$

$$\frac{\sim(p \vee q) \vee r}{(\sim p \wedge \sim q) \vee r}\,(R12) \qquad \frac{(\sim p \wedge \sim q) \vee r}{\sim(p \vee q) \vee r}\,(R13) \qquad \frac{\sim(p \wedge q) \vee r}{(\sim p \vee \sim q) \vee r}\,(R14)$$

$$\frac{(\sim p \vee \sim q) \vee r}{\sim(p \wedge q) \vee r}\,(R15) \qquad \frac{p \wedge (\sim p \vee q)}{q}\,(R16)$$

## Problems

- **Provide natural ’Hilbert-Style’ axiomatizations for ETL and NFL**

# Problems

- **Provide natural ’Hilbert-Style’ axiomatizations for ETL and NFL**
- **Clarify relationships between FDE and its relatives**

A pair $\langle Lang; \vdash_{fde} \rangle$ is called logical system **FDE**, where $Lang$ – propositional language defined above; and $\vdash_{fde}$ is reflexive relation satisfying following principles and rules:

$$a1.\ A \wedge B \vdash_{fde} A \quad a2.\ A \wedge B \vdash_{fde} B \quad a3.\ A \vdash_{fde} A \vee B$$

$$a4.\ B \vdash_{fde} A \vee B \quad a5.\ A \vdash_{fde} {\sim}{\sim}A \quad a6.\ {\sim}{\sim}A \vdash_{fde} A$$

$$a7.\ A \wedge (B \vee C) \vdash_{fde} (A \wedge B) \vee (A \wedge C)$$
$$a8.\ {\sim}(A \wedge B) \vdash_{fde} {\sim}A \vee {\sim}B \quad a9.\ {\sim}A \vee {\sim}B \vdash_{fde} {\sim}(A \wedge B)$$
$$a10.\ {\sim}(A \vee B) \vdash_{fde} {\sim}A \wedge {\sim}B \quad a11.\ {\sim}A \wedge {\sim}B \vdash_{fde} {\sim}(A \vee B)$$

$$r1.\ A \vdash_{fde} B;\ B \vdash_{fde} C \ /\ A \vdash_{fde} C$$
$$r2.\ A \vdash_{fde} B;\ A \vdash_{fde} C \ /\ A \vdash_{fde} B \wedge C$$
$$r3.\ A \vdash_{fde} C;\ B \vdash_{fde} C \ /\ A \vee B \vdash_{fde} C$$

## Binary consequence proof system for ETL

A pair $\langle Lang; \vdash_{etl} \rangle$ is called logical system $\mathbf{L}_{etl}$, where $Lang$ – propositional language defined above; and $\vdash_{etl}$ is reflexive relation satisfying following principles and rules:

$$a1.\ A \wedge B \vdash_{etl} A \quad a2.\ A \wedge B \vdash_{etl} B \quad a3.\ A \vdash_{etl} A \vee B$$

$$a4.\ B \vdash_{etl} A \vee B \quad a5.\ A \vdash_{etl} {\sim}{\sim}A \quad a6.\ {\sim}{\sim}A \vdash_{etl} A$$

$$a7.\ A \wedge (B \vee C) \vdash_{etl} (A \wedge B) \vee (A \wedge C)$$
$$a8.\ {\sim}(A \wedge B) \vdash_{etl} {\sim}A \vee {\sim}B \quad a9.\ {\sim}A \vee {\sim}B \vdash_{etl} {\sim}(A \wedge B)$$
$$a10.\ {\sim}(A \vee B) \vdash_{etl} {\sim}A \wedge {\sim}B \quad a11.\ {\sim}A \wedge {\sim}B \vdash_{etl} {\sim}(A \vee B) \quad a12.\ {\sim}A \wedge (A \vee B) \vdash_{etl} B$$

$$r1.\ A \vdash_{etl} B;\ B \vdash_{etl} C\ /\ A \vdash_{etl} C$$
$$r2.\ A \vdash_{etl} B;\ A \vdash_{etl} C\ /\ A \vdash_{etl} B \wedge C$$

# Binary consequence proof system for ETL

A pair $\langle Lang; \vdash_{etl} \rangle$ is called logical system $\mathbf{L}_{etl}$, where $Lang$ – propositional language defined above; and $\vdash_{etl}$ is reflexive relation satisfying following principles and rules:

$$a1.\ A \wedge B \vdash_{etl} A \quad a2.\ A \wedge B \vdash_{etl} B \quad a3.\ A \vdash_{etl} A \vee B$$

$$a4.\ B \vdash_{etl} A \vee B \quad a5.\ A \vdash_{etl} {\sim\sim}A \quad a6.\ {\sim\sim}A \vdash_{etl} A$$

$$a7.\ A \wedge (B \vee C) \vdash_{etl} (A \wedge B) \vee (A \wedge C)$$
$$a8.\ {\sim}(A \wedge B) \vdash_{etl} {\sim}A \vee {\sim}B \quad a9.\ {\sim}A \vee {\sim}B \vdash_{etl} {\sim}(A \wedge B)$$
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$$r1.\ A \vdash_{etl} B;\ B \vdash_{etl} C\ /\ A \vdash_{etl} C$$
$$r2.\ A \vdash_{etl} B;\ A \vdash_{etl} C\ /\ A \vdash_{etl} B \wedge C$$

$$r3.\ A \vdash_{etl} C;\ {\sim}C \vdash_{etl} {\sim}A;\ B \vdash_{etl} C;\ {\sim}C \vdash_{etl} {\sim}B /\ A \vee B \vdash_{etl} C$$

# Binary consequence proof system for NFL

A pair $\langle Lang; \vdash_{nfl} \rangle$ is called logical system $\mathbf{L}_{nfl}$, where $Lang$ – propositional language defined above; and $\vdash_{nfl}$ is reflexive relation satisfying following principles and rules:

$$a1.\ A \wedge B \vdash_{nfl} A \quad a2.\ A \wedge B \vdash_{nfl} B \quad a3.\ A \vdash_{nfl} A \vee B$$

$$a4.\ B \vdash_{nfl} A \vee B \quad a5.\ A \vdash_{nfl} {\sim\sim}A \quad a6.\ {\sim\sim}A \vdash_{nfl} A$$

$$a7.\ A \wedge (B \vee C) \vdash_{nfl} (A \wedge B) \vee (A \wedge C)$$

$$a8.\ {\sim}(A \wedge B) \vdash_{nfl} {\sim}A \vee {\sim}B \quad a9.\ {\sim}A \vee {\sim}B \vdash_{nfl} {\sim}(A \wedge B)$$

$$a10.\ {\sim}(A \vee B) \vdash_{nfl} {\sim}A \wedge {\sim}B \quad a11.\ {\sim}A \wedge {\sim}B \vdash_{nfl} {\sim}(A \vee B)$$

$$a12.\ B \vdash_{nfl} A \vee ({\sim}A \wedge B)$$

$$r1.\ A \vdash_{nfl} B;\ B \vdash_{nfl} C \ /\ A \vdash_{nfl} C$$

$$r3.\ A \vdash_{nfl} C;\ B \vdash_{nfl} C \ /\ A \vee B \vdash_{nfl} C$$

## Binary consequence proof system for NFL

A pair $\langle Lang; \vdash_{nfl} \rangle$ is called logical system $\mathbf{L}_{nfl}$, where $Lang$ – propositional language defined above; and $\vdash_{nfl}$ is reflexive relation satisfying following principles and rules:

$$a1.\ A \wedge B \vdash_{nfl} A \quad a2.\ A \wedge B \vdash_{nfl} B \quad a3.\ A \vdash_{nfl} A \vee B$$

$$a4.\ B \vdash_{nfl} A \vee B \quad a5.\ A \vdash_{nfl} {\sim}{\sim}A \quad a6.\ {\sim}{\sim}A \vdash_{nfl} A$$

$$a7.\ A \wedge (B \vee C) \vdash_{nfl} (A \wedge B) \vee (A \wedge C)$$

$$a8.\ {\sim}(A \wedge B) \vdash_{nfl} {\sim}A \vee {\sim}B \quad a9.\ {\sim}A \vee {\sim}B \vdash_{nfl} {\sim}(A \wedge B)$$

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$$a12.\ B \vdash_{nfl} A \vee ({\sim}A \wedge B)$$

$$r1.\ A \vdash_{nfl} B;\ B \vdash_{nfl} C\ /\ A \vdash_{nfl} C$$

$$r3.\ A \vdash_{nfl} C;\ B \vdash_{nfl} C\ /\ A \vee B \vdash_{nfl} C$$

$$r2.\ A \vdash_{nfl} B;\ {\sim}B \vdash_{nfl} {\sim}A;\ A \vdash_{nfl} C; {\sim}C \vdash_{nfl} {\sim}A\ /\ A \vdash_{nfl} B \wedge C$$

## Contrapositive Filtration

- Why these rules? Prof. Yaroslav Shramko remarked that if we replace standard disjunction elimination only with DS in **ETL**, then how to infer theorems like this one: $A \vee B \vdash_{etl} B \vee A$? The dual question holds in **NFL** case, how to infer $A \wedge B \vdash_{nfl} B \wedge A$, if we replace standard conjunction introduction rule with DDS?
- Usually, in **FDE** case, we use mentioned rules to infer $A \vee B \vdash B \vee A$ and $A \wedge B \vdash B \wedge A$, but in **ETL** and **NFL** we cannot use them. Nevertheless, it is obvious that $A \vee B \models_{etl} B \vee A$ and $A \wedge B \models_{nfl} B \wedge A$.

# Contrapositive Filtration

**Exactly True Logic**:

$$A \models_{etl} B \vee A \qquad B \models_{etl} B \vee A$$

and

$$A \vee B \models_{etl} B \vee A$$

**Non Falsity Logic**:

$$A \wedge B \models_{nfl} B \qquad A \wedge B \models_{nfl} A$$

and

$$A \wedge B \models_{nfl} B \wedge A$$

# Contrapositive Filtration

- The idea is in that we could use these rules if it were not semantical counterexamples for disjunction elimination (in **ETL** semantics) and conjunction introduction (in **NFL** semantics).
- The following relations holds: $\models_{fde} \Rightarrow \models_{etl}$ and $\models_{fde} \Rightarrow \models_{nfl}$.
- Then, we need some enriched version of disjunction elimination rule for **ETL**, and some enriched version of conjunction introduction rule for **NFL**. Since, $\models_{fde} \Rightarrow \models_{etl}$ and $\models_{fde} \Rightarrow \models_{nfl}$, and the fact that contraposition rule is *admissible* in **FDE**, i. e. for every $A \vdash_{fde} B$ we have $\sim B \vdash_{fde} \sim A$, we formulate disjunction elimination and conjunction introduction in the following (*filtrated*) form:

**’Filtration’ Rules**:

$$r2.\ A \vdash_{nfl} B;\ {\sim}B \vdash_{nfl} {\sim}A;\ A \vdash_{nfl} C; {\sim}C \vdash_{nfl} {\sim}A \ /\ A \vdash_{nfl} B \wedge C$$

$$r3.\ A \vdash_{etl} C;\ {\sim}C \vdash_{etl} {\sim}A;\ B \vdash_{etl} C;\ {\sim}C \vdash_{etl} {\sim}B/\ A \vee B \vdash_{etl} C$$

- Figuratively speaking, these rules filtrate premisses, reducing them to those whose contrapositive images are derivable in **ETL** and **NFL**. Thus they black out *bad* premisses, for instance, $A \wedge {\sim}A \vdash B$, ${\sim}A \wedge (A \vee B) \vdash B$, and $B \vdash A \vee {\sim}A$, $B \vdash A \vee ({\sim}A \wedge B)$, whose contrapositive images are not derivable!!!

The following theorems are proved.

**Theorem**

**ETL**, *Soundness and completeness:* $A \vdash_{etl} B \Leftrightarrow A \models_{etl} B$

**Theorem**

**NFL**, *Soundness and completeness:* $A \vdash_{nfl} B \Leftrightarrow A \models_{nfl} B$

[5] Dunn, J. M., (2000) *«Partiality and its dual»*. Studia Logica 66(1), pp.5-40.

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Extentions of **FDE**:

**FDE** + ($A \wedge \sim A \vdash B$) = $\mathbf{K_3}$

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Extentions of **FDE**:

$\mathbf{FDE} + (A \wedge {\sim}A \vdash B) = \mathbf{K_3}$ $\mathbf{FDE} + (B \vdash A \vee {\sim}A) = \mathbf{LP}$

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$\mathbf{FDE} + (A \wedge {\sim}A \vdash B \vee {\sim}B) = \mathbf{RM_{fde}}$

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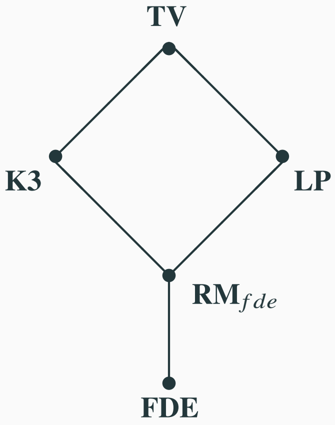
Extentions of **FDE**:

$\mathbf{FDE} + (A \wedge {\sim}A \vdash B) = \mathbf{K_3}$ $\quad$ $\mathbf{FDE} + (B \vdash A \vee {\sim}A) = \mathbf{LP}$

$\mathbf{FDE} + (A \wedge {\sim}A \vdash B \vee {\sim}B) = \mathbf{RM_{fde}}$

$\mathbf{FDE} + (A \wedge {\sim}A \vdash B) + (B \vdash A \vee {\sim}A) = \mathbf{TV}$

# ’KITE’, ’BULAVA’, what else?



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$$a1.\ A \wedge B \vdash_{\mathbf{FDE}_f} A \quad a2.\ A \wedge B \vdash_{\mathbf{FDE}_f} B \quad a3.\ A \vdash_{\mathbf{FDE}_f} A \vee B$$
$$a4.\ B \vdash_{\mathbf{FDE}_f} A \vee B \quad a5.\ A \vdash_{\mathbf{FDE}_f} {\sim}{\sim}A \quad a6.\ {\sim}{\sim}A \vdash_{\mathbf{FDE}_f} A$$
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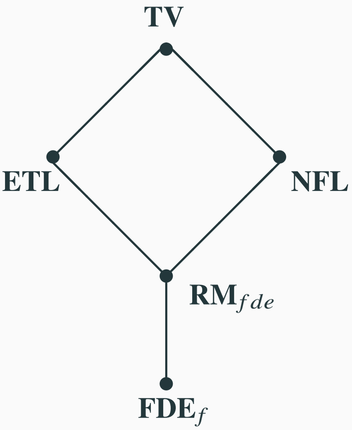
$$r1.\ A \vdash_{\mathbf{FDE}_f} B;\ B \vdash_{\mathbf{FDE}_f} C \,/\, A \vdash_{\mathbf{FDE}_f} C$$

$$r2.\ A \vdash_{\mathbf{FDE}_f} B;\ {\sim}B \vdash_{\mathbf{FDE}_f} {\sim}A;\ A \vdash_{\mathbf{FDE}_f} C; {\sim}C \vdash_{\mathbf{FDE}_f} {\sim}A \,/\, A \vdash_{\mathbf{FDE}_f} B \wedge C$$

$$r3.\ A \vdash_{\mathbf{FDE}_f} C;\ {\sim}C \vdash_{\mathbf{FDE}_f} {\sim}A;\ B \vdash_{\mathbf{FDE}_f} C;\ {\sim}C \vdash_{\mathbf{FDE}_f} {\sim}B /\ A \vee B \vdash_{\mathbf{FDE}_f} C$$

’BULAVA’



THANK YOU!