ROMAI J., v.19, no.1(2023), 53–62

GEOMETRIC SUBPROGRESSION STABILIZER IN COMMON METRIC

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Abstract A series of such metric spaces is constructed (subgeometric sequences of real numbers), for which the multiplication of the metric by any positive number not equal to one, gives a space at an infinite Gromov–Hausdorff distance from the original space.

Keywords: metric space, Gromov–Hausdorff distance, group action.
2020 MSC: 51F99.

1. INTRODUCTION

The work is devoted to the geometry of the Gromov–Hausdorff distance [1, 2, 3, 4] defined on the class of all non-empty metric spaces and is closely related to the works [7, 9], the concepts and results of which we use without detailed explanation.

M. Gromov in his "Metric structures for Riemannian and non-Riemannian spaces" [3] made a short remark: "One can also make a moduli space of isometry classes of non-compact spaces X lying within a finite Hausdorff distance from a given X_0, e.g. $X_0 = \mathbb{R}^n$. Such moduli spaces are also complete and contractible."

This observation was not proved in [3] because it probably seemed obvious. In [7, Theorem 4] the completeness of moduli spaces (clouds) is proved and it is stated, that a natural attempt to prove the contractibility of a cloud poses the problem of describing the stabilizer and the center of the cloud. Let us give the basic definitions.

Let (X, ϱ) be an arbitrary metric space and $0 < r \le \infty$ be a real number. As is customary in metric geometry, instead of $\varrho(x, y)$ we write $|xy|$ as a rule. If A and B are non-empty subsets of X, then we put

$$|AB| = |BA| = \inf\{|ab| : a \in A,\, b \in B\}.$$

Next, we define the closed r-neighborhood of the set A by setting $B_r(A) = \{x \in X : |xA| \le r\}$.

Finally, the *Hausdorff distance* between A and B is the value

$$d_H(A, B) = \inf\{r : A \subset B_r(B), \quad B \subset B_r(A)\}.$$

The Hausdorff distance is a generalized pseudometric. The word "generalized" means that the distance may take infinite value, as in the case of the straight line $\mathbb{R}$ and any of its points. The prefix "pseudo" means that the distance may take zero value between different subsets, as in the case of a set and its dense subset. It is obvious that the Hausdorff distance satisfies all the axioms of generalized pseudometrics: it is non-negative, symmetric and satisfies the triangle inequality. Nevertheless, on the set consisting of all non-empty bounded closed subsets of a metric space X, the Hausdorff distance is a metric.

The *Gromov–Hausdorff distance* between non-empty metric spaces X and Y is the value

$$d_{GH}(X,Y) = \inf\bigl\{d_H(X',Y')\colon X',Y' \subset Z,\ X' \approx X,\ Y' \approx Y\bigr\},$$

where for the metric spaces X and X' the expression $X \approx X'$ means that these spaces are isometric. *The Gromov–Hausdorff distance is a generalized pseudometric vanishing on each pair of isometric spaces* [4]. There are a countable discrete complete bounded metric space X and a countable locally compact complete bounded metric space with exactly one non-isolated point Y such that $d_{GH}(X,Y) = 0$.

Compact metric spaces form the set $\mathcal{G}H_c$, on which the Gromov–Hausdorff distance is a metric. The class $\mathcal{G}H_b$ of all bounded metric spaces no longer forms a set. But within the framework of von Neumann–Bernays–Gödel set theory we can say that the Gromov–Hausdorff distance is a pseudometric on $\mathcal{G}H_b$. We will denote the class of all metric spaces by $\mathcal{G}H$, and the class of all metric spaces located at a finite distance from of a given metric space X will be called *the cloud of the space* X and denoted by $[X]$. By Δ_1 we denote a one-point metric space. It is clear that $[\Delta_1] = \mathcal{G}H_b$. For a metric space (X, ϱ) and a positive number $\lambda > 0$, λX means the "similar space" $(X, \lambda\varrho)$, i.e. the set X, the distances on which are multiplied by λ.

The transformation $H_\lambda\colon \mathcal{G}H \to \mathcal{G}H$, $H_\lambda : X \mapsto \lambda X$ for $\lambda > 0$ we call *similarity with the coefficient* λ.

The diameter of a metric space is defined as

$$\operatorname{diam} X = \sup\{|xy|\colon x, y \in X\}.$$

Theorem 1.1 ([4]). *For any metric spaces X and Y,*

(1) $2d_{GH}(\Delta_1, X) = \operatorname{diam} X$*;*

(2) $2d_{GH}(X,Y) \le \max\{\operatorname{diam} X, \operatorname{diam} Y\}$*;*

(3) if the diameter of X or Y is finite, then $\bigl|\operatorname{diam} X - \operatorname{diam} Y\bigr| \le 2d_{GH}(X,Y)$.

(4) if the diameter of X is finite, then for any $\lambda > 0$ and $\mu > 0$ we have $d_{GH}(\lambda X, \mu X) = \frac{1}{2}|\lambda - \mu| \operatorname{diam} X$*, whence it immediately follows that the curve $\gamma(t) := tX$ is shortest between any of its points, and the length of such a segment of the curve is equal to the distance between its ends.*

(5) for any $\lambda > 0$, we have $d_{GH}(\lambda X, \lambda Y) = \lambda\, d_{GH}(X, Y)$.

Property (5) implies that similarities are well defined on the classes $\mathcal{GH}_c$, $\mathcal{GH}_b$, and $\mathcal{GH}$.

Returning to the contractibility of the cloud, we note that formulas (4) and (5) illustrate the existence of a canonical contraction of the Gromov–Hausdorff space of all compact metric spaces to the one-point space Δ_1. It is also possible to give a strict meaning to the statement that similarity carries out contraction of the cloud of all bounded metric spaces to the one-point space Δ_1.

Formula (5) means that the similarity is continuous in space, but formula (4) in all other clouds does not guarantee continuity with respect to the contraction parameter λ.

There are constructed [5, Corollary 5.9], [10] examples of spaces X such that the spaces X and λX lie in the same cloud if and only if $\lambda = 1$. This means that, in the general case, the similarity cannot contract the cloud by itself. Therefore, the author believes that the statement about the contractibility of any cloud (even in the case of cloud $[\mathbb{R}^n]$, mentioned by Gromov) is currently a hypothesis. Recently the author proved that the cloud of any space "with large metric gaps" (the spaces considered in this work are as follows) is contractible [11, Theorem 1.2].

Since for an unbounded metric space (X, ϱ) the "similar" space $\lambda X = (X, \lambda\varrho)$ can be at infinite Gromov–Hausdorff distance from the original space X [5, 6, 7, 8, 9, 10], then the cloud stabilizer becomes important:

$$\mathrm{St}[X] = \big\{\lambda \in \mathbb{R}_+ : d_{GH}(X, \lambda X) < \infty\big\} = \Big\{\lambda \in \mathbb{R}_+ : [\lambda X] = [X]\Big\}.$$

The cloud stabilizer does not depend on the representative X taken from the cloud and is a subgroup in the multiplicative group of positive numbers $(\mathbb{R}_+, \times)$.

In this plan, already subsets of the half-line (of non-negative numbers with the standard metric of the modulus of the difference) give many interesting and varied examples.

For example, in [5] it is shown that in the case of a countable subset X whose points go to infinity very quickly (for example, for geometric hyperprogression $X = \big\{p^{n^\alpha}\big\}_{n=1}^{\infty}$, $p > 1$, $\alpha > 1$) and the standard line metric, $\mathrm{St}[X] = \{1\}$. In [10] a similar result is proved for an arbitrary normalized metric on a geometric hyperprogression, i.e. for a metric for which only distances to the zero point are induced from the straight line.

In [5, 6, 8] it is shown that for a geometric progression $X_p = \big\{p^n\big\}_{n=1}^{\infty}$, $p > 1$, and the standard line metric, $\mathrm{St}[X_p] = G_p = \big\{p^n\big\}_{n=-\infty}^{\infty}$. In [9] it is shown that for an arbitrary normalized metric on a geometric progression any subgroup of the group G_p can be the stabilizer. One could get the feeling that the value $\alpha = 1$ is some kind of watershed in the nature of the stabilizer.

In this work, we show that for a geometric subprogression $X = \{p^{n^\alpha}\}_{n=1}^{\infty}$, $p > 1$, $0 < \alpha < 1$, and an arbitrary normalized metric, the equality $\mathrm{St}[X] = \{1\}$ holds. Thus, we can say that for $\alpha = 1$ there is not a "water divide", but "an archipelago of islands in a sea of trivial stabilizer".

The normed vector spaces form the "Himalayas" with maximum stabilizer – the whole multiplicative group of positive numbers. We especially note that for every nontrivial proper subgroup $H \subset (\mathbb{R}_+, \times)$, one of the following is valid:

a) H is closed - in which case, $H = G_p$ for some $p > 1$;

b) H is not closed - in which case, H is everywhere dense.

There are exactly $2^{\mathfrak{c}} = 2^{2^{\aleph_0}}$ of proper dense subgroups, but the author does not know of any example of a cloud with such a stabilizer.

2. BASIC CONCEPTS

Let X and Y be arbitrary sets. A multi-valued mapping $R\colon X \to Y$ is uniquely determined by its graph, for which we keep the notation

$$R = \{(x, y)\colon\ y \in R(x)\}.$$

It is clear that the graphs of set-valued mappings are exactly subsets of $R \subset X \times Y$ such that for any point $x \in X$ there exists a point $y \in Y$ such that $(x, y) \in R$. Such a set $R \subset X \times Y$ will also be called a *complete relation.* To simplify the notation for a point from $R(x)$, we will also use the notation y_x. In metric geometry, a surjective set-valued mapping is called a *correspondence.* For a R correspondence, the R^{-1} inverse plot is a subset of the product $Y \times X$, so we will denote it by R^*. The set of all correspondences X in Y is denoted by $\mathcal{R}(X, Y)$. To avoid confusion, we always denote the points of the second space as y even though it is also denoted by X.

For a correspondence $R \subset X \times Y$ of metric spaces (X, ϱ_X) and (Y, ϱ_Y), define its *distortion* as

$$\operatorname{dis} R = \sup\Big\{\big|\varrho_X(x, x') - \varrho_Y(y, y')\big| : (x, y),\ (x', y') \in R\Big\}. \tag{1}$$

It is convenient to estimate the Gromov–Hausdorff distance in terms of distortion of correspondences [4]

Theorem 2.1. *For any metric spaces X and Y the following equality holds:*

$$d_{GH}(X, Y) = \frac{1}{2} \inf\{\operatorname{dis} R : R \in \mathcal{R}(X, Y)\}.$$

In what follows, we will assume that $X, Y \subset [0, \infty)$ and $0 \in X$, $0 \in Y$. The point 0 in the set X will be denoted by 0_X. Since we will consider different metrics on these sets, then, if necessary, we will use the notation $\operatorname{dis}_{\varrho_X, \varrho_Y} R$.

We are interested in *normalized* metrics, i.e. such metrics ϱ on the set X that

$$\varrho(x, 0_X) = x \text{ for any point } x \in X. \tag{2}$$

It follows from the triangle inequality that for any two points $x, x' \in X$ we have:

$$x - x' = \varrho(x, 0_X) - \varrho(x', 0_X) \leq \varrho(x, x') \leq \varrho(x, 0_X) + \varrho(0_X, x') = x + x'. \tag{3}$$

Both extreme cases provide interesting examples. The case of left equality (for all $x > x'$) corresponds to the fact that the metric is taken from the line on the set X. The case of right equality (for all $x \neq x'$) corresponds to the discrete hedgehog $\hat{X}$ [7]. Intermediate "linear" metrics also provide important examples. For any $-1 \leq \alpha \leq 1$ we define on the set of non-negative numbers, and hence on any set X we consider, the metric

$$\varrho_\alpha(x, x') = x + \alpha x' = \frac{1-\alpha}{2}|x - x'| + \frac{1+\alpha}{2}(x + x') \text{ when } x' < x. \tag{4}$$

It is clear that the formula (4) defines a metric on the set of non-negative real numbers. The inequalities (3) can be formulated as the assertion that for any normalized metric ϱ the inequalities $\varrho_{-1} \leq \varrho \leq \varrho_1$ are valid.

Let $\varphi\colon \{0\} \cup \mathbb{N} \to [0, \infty)$, $\varphi(0) = 0$, be a strictly increasing function. Consider on the number line the subset

$$X_\varphi = \big\{x_n = \varphi(n)\colon n \in \{0\} \cup \mathbb{N}\} \subset [0, \infty)\big\}. \tag{5}$$

The set of all normalized metrics on X_φ denoted by $\mathcal{M}_\varphi$ or $\mathcal{M}$ for a fixed function φ.

The function φ will be called *sparse*, if the difference

$$\Delta_\varphi(n) = \varphi(n) - \varphi(n-1), n \geq 1,$$

monotonically (from some rank n_0) increases to infinity, i.e.

$$\Delta_\varphi(n+1) \geq \Delta_\varphi(n) \quad \text{for} \quad n \geq n_0 \quad \text{and} \quad \Delta_\varphi(n) \overset{n\to\infty}{\longrightarrow} \infty. \tag{6}$$

Sparse functions are remarkable in that their correspondences with finite distortion have a simple structure.

Theorem 2.2. *Let φ and ψ be sparse functions and $R \in \mathcal{R}(X_{\varphi,\varrho}, X_{\psi,\rho})$ is a correspondence such that* $\operatorname{dis} R < M$. *Then for some n_0 and an integer k for all $n \geq n_0$ the equality $R(x_n) = \{y_{n+k}\}$ holds.*

This is where our main result comes from.

Theorem 2.3. *For numbers $p > 1$, $0 < \alpha < 1$ and any normalized metric $\varrho \in \mathcal{M}_\varphi$ on a sparse set X_φ, where $\varphi(n) = p^{n^\alpha}$, we have the equality*

$$\mathrm{St}\big[(X_\varphi, \varrho)\big] = \{1\}.$$

On the one hand, the theorem 2.2 gives a strong necessary condition for the correspondence of a finite distortion. On the other hand, this necessary condition is based only on comparing the distances to the zero point, therefore, it cannot be sufficient for the finiteness of the distortion of this correspondence. Sufficiency holds for metrics with some condition of "translation invariance". For a geometric progression, invariant normalized metrics are important. In our case, the analog is the class of the following metrics. Let's say that the metric $\varrho \in \mathcal{M}_\varphi$ is *invariant* ($\varrho \in \mathcal{IM}_\varphi$), if there exists a function $\alpha\colon \mathbb{N} \to [-1, 1]$ such that

$$\varrho(x_m, x_n) = x_m + \alpha(m-n)x_n \quad \text{for} \quad n < m. \tag{7}$$

In [9, Proposition 3.5], there is a description of such functions α that define a metric on X_φ by the formula (7). The following result [9, Theorem 2.14] contains all linear metrics of (4).

Theorem 2.4. *Any function $\alpha\colon \mathbb{N} \to [a, b]$, where $-1 \le a \le b \le 1$ and $b \le 1 + 2a$, by the formula (7) defines an invariant normalized metric $\varrho_\alpha \in \mathcal{IM}_\varphi$.*

Theorem 2.5. *Let φ and ψ be strictly increasing functions such that $|\psi(n+k) - \varphi(n)| < K$ for some fixed $k \in \mathbb{Z}$, $K > 0$ and all sufficiently large n ($n \ge n_0$). Then*

$$d_{GH}\big((X_\varphi, \varrho_\alpha), (X_\psi, \varrho_\alpha)\big) < \infty$$

for any function α from theorem 2.4.

Corollary 2.1. *Let φ and ψ be sparse functions, and α and β be the functions from theorem 2.4. Then the following conditions are equivalent:*

1) $d_{GH}\big((X_\varphi, \varrho_\alpha), (X_\psi, \varrho_\beta)\big) < \infty$;

2) $\alpha = \beta$ and $d_{GH}\big((X_\varphi, \varrho_{-1}), (X_\psi, \varrho_{-1})\big) < \infty$;

3) $\alpha = \beta$ and $|\psi(n+k) - \varphi(n)| < K$ for some fixed $k \in \mathbb{Z}$, $K > 0$ and all sufficiently large n ($n \ge n_0$).

Example 2.1. *For any strictly increasing functions φ and ψ the implications* 3) $\Longrightarrow$ 1), 2) *are valid for any number $-1 \le \alpha \le 1$. For the functions $\varphi(n) = 3^{\left[\frac{n+1}{2}\right]} + (-1)^n$ and $\psi(n) = 3^n$ condition* 2) *is satisfied, but for $\alpha > -1$ conditions* 1) *and* 3) *are not true.*

However, the reason lies not so much in the non-sparseness of the functions φ and ψ, but in the fact that that the case $\alpha = -1$ is exceptional and different *from the general function $\alpha\colon \mathbb{N} \to [-1+\varepsilon, 1]$, whose values are separated from the number -1.*

3. PROOFS

In [9, Proposition 1.1] the assertion is proved, which we present in full for the convenience of the reader.

Proposition 3.1. *If for the complete relation R of the spaces X_{ϱ_X} and Y_{ϱ_Y} the inequality $\operatorname{dis} R < M$ is true, then for every point $x \in X$ and every point $y_x \in R(x)$ the inequality $|x - y_x| < K$ is true, where $K = M + y_0$.*

For any sparse function φ, for any number $M > 0$, there exists a number $n > 1$, such that

$$\varphi(n) - \varphi(n-1) \geq M.$$

We denote the smallest such number by $n_\varphi(M)$.

Proposition 3.2. *If for a sparse function $\psi\colon \mathbb{N} \to \mathbb{R}_+$ and a complete relation R of the spaces X_{ϱ_X} and $X_{\psi,\rho}$ the inequality $\operatorname{dis} R < M$ is true, then for any point $x \in X$ from $y \in R(x)$ and $y \geq y_{n_\psi(M)} = \psi(n_\psi(M))$ the equality $R(x) = \{y\}$ follows.*

Proof. Let $y' \in R(x)$. Then

$$|y' - y| \leq \rho(y', y) = \rho(y', y) - \varrho(x, x) \leq \operatorname{dis} R < M.$$

The condition $y \geq \psi(n_\psi(M))$ and the definition of the number $n_\psi(M)$ imply the equality $y' = y$. ∎

Proposition 3.3. *If for a sparse function $\varphi\colon \mathbb{N} \to \mathbb{R}_+$ and a complete relation R of the spaces X_{φ,ϱ_X} and X_ρ the inequality $\operatorname{dis} R < M$ is true, then for any point $x \in X$ from $x \geq x_{n_\varphi(M)} = \varphi(n_\varphi(M))$ and $R(x) \cap R(x') \neq \emptyset$ follows $x = x'$.*

Proof. Let $y \in R(x) \cap R(x')$. Then

$$|x' - x| \leq \varrho(x', x) = \varrho(x', x) - \rho(y, y) \leq \operatorname{dis} R < M.$$

The condition $x \geq \varphi(n_\varphi(M))$ and the definition of the number $n_\varphi(M)$ imply the equality $x' = x$. ∎

Proof of the theorem 2.2. Since the metric spaces $X_{\varphi,\varrho}$ and $X_{\psi,\rho}$ are unbounded, then there are numbers $n_0 \geq n_\varphi(2M + 2y_0) \geq n_\varphi(M)$ and $m_0 \geq n_\psi(2M + 2x_0) \geq n_\psi(M)$ such that $y_{m_0} \in R(x_{n_0})$. Here $y_0 \in R(0)$ and $x_0 \in R^*(0)$, i.e. $0 \in R(x_0)$.

Since $m_0 \geq n_\psi(M)$, then according to the proposition 3.2 $R(x_{n_0}) = \{y_{m_0}\}$; from $n_0 \geq n_\varphi(M)$ according to the proposition 3.3 it follows $R^*(y_{m_0}) = \{x_{n_0}\}$.

According to Proposition 3.1 $|x_{n_0} - y_{m_0}| < M + y_0$, so $y_{m_0} < x_{n_0} + M + y_0$. Similarly $|x_{n_0+1} - y_{x_{n_0+1}}| < M + y_0$, so $x_{n_0+1} - M - y_0 < y_{x_{n_0+1}}$.

It follows from the inequality $n_0 \geq n(2M + 2y_0)$ that

$$x_{n_0+1} - x_{n_0} > x_{n_0} - x_{n_0-1} \geq 2M + 2y_0.$$

So $y_{x_{n_0+1}} > x_{n_0+1} - M - y_0 > x_{n_0} + M + y_0$. Therefore, $y_{x_{n_0+1}} > y_{m_0} = y_{x_{n_0}}$.

Thus, we have proved that $y_{x_{n_2}} > y_{x_{n_1}} \geq y_{m_0}$ follows from $n_2 > n_1 \geq n_0$.

A similar property is also true for the inverse (symmetric) correspondence R^*. $m_2 > m_1 \geq m_0$ implies $x_{y_{m_2}} > x_{y_{m_1}} \geq x_{n_0}$.

If $y_{x_{n_0+1}} > y_{m_0+1}$, then for the point $x_{y_{m_0+1}}$ from the proven monotonicity property the inequality $x_{n_0} < x_{y_{m_0+1}} < x_{n_0+1}$.

The resulting contradiction shows that $y_{x_{n_0+1}} = y_{m_0+1}$. We prove by induction that $y_{x_{n_0+i}} = y_{m_0+i}$ for every $i \geq 1$. It is clear that $k = m_0 - n_0$ is the desired one. $\square$

Proposition 3.4. *For $p > 1$ and $0 < \alpha < 1$, the following properties hold for the function $\varphi(x) = p^{x^\alpha}$:*

1) The function $\varphi(n)$ is sparse.

2) For any integer k the equality $\lim_{x\to\infty} \frac{\varphi(x+k)}{\varphi(x)} = 1$ is true.

Proof. Consider the function $\varphi(x) = p^{x^\alpha}$, $x > 0$. It is easy to calculate that $\varphi'(x) = \alpha x^{\alpha-1}\varphi(x)\ln p = \alpha \ln p \frac{p^{x^\alpha}}{x^{1-\alpha}} > 0$.

1) Therefore, the sequence $\{\varphi(n) = p^{n^\alpha}\}$ is strictly increasing. An increase of the sequence $\Delta_\varphi(n)$ is equivalent to the convexity of the sequence $\{\varphi(n)\}$, i.e. to the condition

$$\varphi(n+1) \leq \frac{\varphi(n+2) + \varphi(n)}{2} \text{ for each } n. \tag{8}$$

It is easy to calculate that $\varphi''(x) = \alpha x^{\alpha-2}\big(\alpha x^\alpha \ln p + \alpha - 1\big)\varphi(x)\ln p$. For sufficiently large x the second derivative is positive $\varphi''(x) > 0$, therefore the inequality (8) is true for all sufficiently large n.

Let us show that $\varphi'(x) \to \infty$ as $x \to \infty$. This will be done via L'Hopital's rule applied to the related exponent, by means of the substitution $x^\alpha = t$:

$$\lim_{x\to\infty} \frac{p^{x^\alpha}}{x^{1-\alpha}} = \lim_{t\to\infty} \frac{p^t}{t^{\frac{1-\alpha}{\alpha}}} = \lim_{t\to\infty} \frac{\alpha \ln p}{1-\alpha} \cdot \frac{p^t}{t^{\frac{1-2\alpha}{\alpha}}} = \ldots =$$

$$= \lim_{t\to\infty} \frac{\alpha^k \ln p}{\Pi_{i=1}^k (1 - i\alpha)} \cdot \frac{p^t}{t^{\frac{1-(k+1)\alpha}{\alpha}}} = \infty,$$

where k is a number such that $1-k\alpha > 0$ and $1-(k+1)\alpha \le 0$. By Lagrange's theorem, $\Delta_\varphi(n+1) = \varphi(n+1)-\varphi(n) = \varphi'(n+\theta_n) \xrightarrow{n\to\infty} \infty$, where $0 < \theta_n < 1$.

2) Consider the function $f(x) = x^\alpha$. It is easy to calculate that $f'(x) = \alpha x^{\alpha-1} = \frac{\alpha}{x^{1-\alpha}} \xrightarrow{x\to\infty} 0$. By Lagrange's theorem,

$$\Delta_f(n+k) = f(n+k) - f(n) = f'(n+k\theta_n)k \xrightarrow{n\to\infty} 0,$$

where $0 < \theta_n < 1$.
Hence $\lim_{n\to\infty} \frac{\varphi(n+k)}{\varphi(n)} = \lim_{n\to\infty} p^{(n+k)^\alpha - n^\alpha} = p^{\lim_{n\to\infty} \Delta_f(n+k)} = p^0 = 1$. ▮

Proof of Theorem 2.3. The sparseness of the φ function is proved in the Proposition 3.4.

Let $\varrho \in \mathcal{M}_{X_\varphi}$ and $\lambda \in \mathrm{St}\big[(X_\varphi, \varrho)\big]$. According to [9, Proposition 1.4], the $\lambda\varrho$ metric on X_φ can be identified with the normalized metric on the sparse set $X_{\lambda\varphi}$ given by the function $\lambda\varphi$. Let R be a correspondence between sets X_φ and $X_{\lambda\varphi}$ such that $\operatorname{dis} R < \infty$. According to the theorem 2.2 there exist natural n_0 and integer k such that that $R(x_n) = \{y_{n+k}\}$ for all $n \ge n_0$. According to the proposition 3.1 there exists a number $K > 0$, that $|y_{n+k} - x_n| < K$ for all $n \ge n_0$. The latter means that

$$|\lambda p^{(n+k)^\alpha} - p^{n^\alpha}| < K \text{ for all } n \ge n_0.$$

The inequality can be written as

$$|\lambda - p^{n^\alpha-(n+k)^\alpha}| < \frac{K}{p^{(n+k)^\alpha}} \text{ for all } n \ge n_0.$$

The left and right sides of the last inequality have limits as $n \to \infty$. Obviously, the limit of the right-hand side is the number 0. According to the proposition 3.4 the limit of the left side is equal to $|\lambda - 1|$. From the limit inequality $|\lambda-1| \le 0$ the required equality $\lambda = 1$ follows.

Proof of Theorem 2.5. The correspondence $R \in \mathcal{R}(X_\varphi, X_\psi)$ is given by the formula $R(x_n) = \{y_{n+k}\}$ for $n \ge n_0$ and $R(x_n) = \{0_Y, y_1, \dots, y_{n+k-1}\}$ for $n < n_0$. Let us estimate $\operatorname{dis} R$.

For numbers $m > n \ge n_0$, the following estimate is true

$$\big||x_m x_n|-|y_{m+k}y_{n+k}|\big| = \big|x_m+\alpha(m-n)x_n-y_{m+k}-\alpha(m+k-n-k)y_{n+k}\big| \le \big|x_m-y_{m+k}\big|+\big|\alpha(m-n)(x_n-y$$

For numbers $m \ge n_0 > n$, the estimate is true $\big||x_m x_n| - |y_{m+k}y_{n+k}|\big| = \big|x_m + a_n x_n - y_{m+k} - a_{n+k}y_{n+k}\big| \le \big|x_m - y_{m+k}\big| + x_n + y_{n+k} \le K + \varphi(n_0 - 1) + \varphi(n_0 + k - 1)$, where a_n, a_{n+k} are some numbers between -1 and 1.

For numbers $n_0 > m, n$, the estimate is true $\big||x_m x_n| - |y_{m+k}y_{n+k}|\big| \le x_m + x_n+y_{m+k}+y_{n+k} \le 2\varphi(n_0-1)+2\varphi(n_0+k-1)$. □

Proof of the theorem 2.1. $1) \Longrightarrow 3)$. Let $R \in \mathcal{R}(X_\varphi, X_\psi)$ be a correspondence such that $\operatorname{dis} R < M$. According to the theorem 2.2 for some n_0 and an integer k for all $n \geq n_0$ $R(x_n) = \{y_{n+k}\}$. According to the proposition 3.1 there are integers n_0 and $K > 0$ such that for every $n \geq n_0$ the inequality $|x_n - y_{n+k}| < K$ is true.

Let r be an arbitrary natural number. For every $n \geq 1$, the following inequality holds: $|\beta(r)-\alpha(r)|x_n \leq |\beta(r)-\alpha(r)|x_n+K-|\beta(r)||y_{n+k}-x_n|+K-|y_{n+r+k}-x_{n+r}| \leq 2K+\big|(\beta(r)-\alpha(r))x_n+\beta(r)(y_{n+k}-x_n)+(y_{n+r+k}-x_{n+r})\big| = 2K + \big|(y_{n+r+k} + \beta(r)y_{n+k}) - (x_{n+r} + \alpha(r))x_n)\big| \leq 3K$. Since the numbers x_n tend to infinity, it follows from the above inequality that $|\beta(r) - \alpha(r)| = 1$. The latter means that α and β functions coincide.

The implications $3) \Longrightarrow 1),\ 2)$ are contained in Theorem 2.5.

The implication $2) \Longrightarrow 3)$ is contained in Theorem 2.2. $\square$

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