

NONLINEAR OPTICAL PHENOMENA

PACS numbers: 42.65.–k; 42.65.Wi; 42.65.Tg; 05.45.–a
DOI: 10.1070/QE2012v042n12ABEH014974

Chirped elliptically polarised cnoidal waves and polarisation 'chaos' in an isotropic medium with spatial dispersion of cubic nonlinearity

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Abstract. **It is found that chirped elliptically polarised cnoidal waves can propagate and aperiodic regimes, resembling polarisation chaos, can emerge in an isotropic medium with local and nonlocal components of cubic nonlinearity and second-order frequency dispersion. In the particular case of the formation of the waveguides of the same profile for two circularly polarised components of the light field relevant analytical solutions are derived and the frequencies of chirped components are shown to vary in concord with periodic changes of their intensities. In this case, the nature of the changes in the polarisation state during the light wave propagation depends on the values of nonlinear phase shifts of circularly polarised components of the field during the period and is sensitive to changes in the initial conditions.**

Keywords: *cubic nonlinearity, spatial dispersion, nonlinear Schrödinger equation, elliptical polarisation, cnoidal waves, chirp, polarisation chaos.*

1. Introduction

Propagation of elliptically polarised electromagnetic waves in an isotropic medium with cubic nonlinearity and second-order frequency dispersion in the absence of diffraction is described by a system of two nonlinear Schrödinger equations (NSEs). If the polarisation of the medium has local and nonlocal (associated with spatial dispersion [1–4]) components, this system of NSEs takes the form [5–9]:

$$\frac{\partial A_\pm}{\partial z} - \frac{\mathrm{i}k_2}{2}\frac{\partial^2 A_\pm}{\partial t^2} + \mathrm{i}[\mp\rho_0 + (\sigma_1/2\mp\rho_1)|A_\pm|^2 + (\sigma_1/2+\sigma_2)|A_\mp|^2]A_\pm = 0, \qquad (1)$$

where $A_\pm(z, t)$ are the slowly varying complex amplitudes of circularly polarised components of the light field; t is the time in the intrinsic (travelling) coordinate system; $k_2 = \mathrm{d}^2k/\mathrm{d}\omega^2$ is the constant characterising the frequency dispersion of the medium; ω is the frequency of the propagating wave; and k is the modulus of its wave vector directed along the z axis.

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Received 28 August 2012; revision received 22 October 2012
Kvantovaya Elektronika **42** (12) 1118 – 1122 (2012)
Translated by I.A. Ulitkin

In equation (1) the constants $\sigma_1 = 4\pi\omega^2\chi_{xyxy}/(kc^2)$ and $\sigma_2 = 2\pi\omega^2\chi_{xxyy}/(kc^2)$ are related to the components of a local cubic nonlinearity tensor, $\hat{\chi}^{(3)}(\omega; -\omega,\omega,\omega)$, that is symmetric with respect to replacement of the last two indices. The constants $\rho_{0,1} = 2\pi\omega^2\gamma_{0,1}/c^2$ are proportional to pseudoscalar constants of linear and nonlinear gyration, γ_0 and γ_1. They, respectively, define nonzero tensor components of nonlocal linear and nonlinear susceptibilities, $\hat{\gamma}^{(1)}$ and $\hat{\gamma}^{(3)}$, with which the contributions $\hat{\gamma}^{(1)}\nabla\boldsymbol{E}$ and $\hat{\gamma}^{(3)}\boldsymbol{E}\boldsymbol{E}\nabla\boldsymbol{E}$ to the polarisation of the medium are associated [1–4].

At arbitrary relationships between $\sigma_{1,2}$ and $\rho_{0,1}$, system (1) is nonintegrable [10–14], and it is impossible to find the boundaries of the stability of nonlinear polarisation modes (nonlinear eigenpolarisations according to the terminology of paper [14]). In this case, consideration is restricted to the search and detailed analysis of the families of particular solutions of (1), the form of which in some cases allows some conclusions about the nature of radiation propagation in a medium with a nonlocal nonlinear optical response. A number of its numerical [5–7] and analytical [8–13] particular solutions is known, which were obtained by imposing some additional constraints. For example, Makarov and Petrov [8], assuming a linear relationship of the amplitudes of two circularly polarised components of the light field, found the solutions of system (1) in the form of soliton pairs. Under the condition of the formation of waveguides of the same profile for both components of the light field $A_\pm(z, t)$ the authors of paper [9] found a family of solutions to system (1), which includes nine types of elliptically polarised cnoidal waves whose amplitudes are proportional to the Jacobi elliptic functions [15] and the phases are independent of time and vary linearly with increasing z.

In this paper we find the particular (corresponding to the formation of the waveguides of the same type for two circularly polarised components of the light field) solutions of system (1), in which, in contrast to the solutions presented in [9], the phases of the components $A_\pm(z, t)$ are not only linearly dependent on z, but also change nonlinearly with t:

$$A_\pm(z,t) = R_\pm(t)\exp\{\mathrm{i}[\phi_\pm(t) + \kappa_\pm z]\}. \qquad (2)$$

Here $\kappa_\pm$ are the linear additions to the propagation constants (constants of separation of variables), and $R_\pm(t)$ and $\phi_\pm(t)$ are real functions. It will be shown that in this particular case, the intensities of the circularly polarised components $R_\pm^2$ and nonlinear additions $\omega_\pm(t) = \mathrm{d}\phi_\pm/\mathrm{d}t$ to the frequency ω (chirp) are expressed in terms of the Jacobi elliptic functions and consistently oscillate. The evolution of the polarisation state of waves is also unusual. Depending on the initial conditions, in a nonlinear medium both the solutions with strictly periodic changes in the polarisation state (chirped elliptically polarised cnoidal waves) and the solutions resembling the polarisation chaos can be formed. Note that the solutions of NSEs of type (2) in the integrable case have been already analysed in [16–19] when considering parametric processes in quadratic nonlinearity.

2. Integrals of the problem and particular analytical solutions

By substituting (2) into (1) and performing the standard procedure of separation of variables, with $k_2 \neq 0$ we derive a system of four ordinary differential equations:

$$\frac{d^2R_\pm}{dt^2} - R_\pm\left(\frac{d\phi_\pm}{dt}\right)^2 - \frac{2}{k_2}[\kappa_\pm \mp \rho_0 + (\sigma_1/2 \mp \rho_1)R_\pm^2 + (\sigma_1/2 + \sigma_2)R_\mp^2]R_\pm = 0, \quad (3a)$$

$$2\frac{dR_\pm}{dt}\frac{d\phi_\pm}{dt} + R_\pm\frac{d^2\phi_\pm}{dt^2} = 0. \quad (3b)$$

Note that equations (3b) are valid in all cases where the nonlinear addition to the refractive index of two circularly polarised components of the field in (1) does not depend on the phases $\phi_\pm$, being arbitrary functions of intensities $|A_\pm|^2$. Indeed, system (3) coincides with the equations describing the mechanical motion of a system consisting of two material points of unit mass, with the radius vectors $\boldsymbol{R}_\pm(t) = \{R_\pm(t)\cos\phi_\pm(t), R_\pm(t)\sin\phi_\pm(t)\}$, in the field of central forces. From the point of view of the analogy, relations (3b) reflect the law of conservation of angular momenta of these points and hold true in all cases when the potential energy of the system is an arbitrary function of $R_\pm^2$.

Equations (3b) are easily integrated, which, as in [16–19], defines two integrals of the problem

$$R_\pm^2(t)[d\phi_\pm(t)/dt] \equiv R_\pm^2(t)\omega_\pm(t) = R_\pm^2(0)\omega_\pm(0) = R_{\pm0}^2\omega_{\pm0}, \quad (4)$$

which, within the above-mentioned analogy, are the angular momenta of the material points. It follows from (4) that if the relations $R_\pm(t_0) = 0$ and $(dR_\pm/dt)|_{t=t_0} \neq 0$ are fulfilled at least for one value of t_0, at all points of time t_1, when the $R_\pm(t_1) \neq 0$, the derivative $(d\phi_\pm/dt)|_{t=t_1} \equiv 0$, and hence, $\phi_\pm = \text{const}$. In this case, the phases $\phi_\pm$ can change abruptly just at the points corresponding to zeros of $R_\pm$. Therefore, the solutions of (3) we are interested in, where $\omega_\pm(t) \neq 0$, can exist only if $|R_\pm(t)| > 0$ for all t. In this case, the phases $\phi_\pm$ and $\omega_\pm$ are found using the integrals (4) after solving (3a):

$$\phi_\pm(t) = \phi_\pm(0) + R_{\pm0}^2(d\phi_\pm/dt)|_{t=0}\int_0^t R_\pm^{-2}(\tau)d\tau,$$
$$\omega_\pm(t) = R_{\pm0}^2\omega_{\pm0}R_\pm^{-2}(t). \quad (5)$$

The condition for the formation of the waveguides of the same type for two circularly polarised components of the field, $A_\pm(z, t)$, is equivalent to the requirement of linear coupling $\delta_+|A_+|^2 + \delta_-|A_-|^2 = \delta_0$ between their intensities, in which the constants $\delta_{0,\pm}$ are subject to further determination (see [9]). Due to this, the amplitudes $R_\pm(t)$ become dependent, and system (3a) is transformed into a pair of formally independent equations, which in view of (4) can be written as

$$\frac{d^2R_\pm}{dt^2} - \frac{R_{\pm0}^4\omega_{\pm0}^2}{R_\pm^3} - \frac{\sigma_1 + 2\sigma_2}{k_2}\times\left[\frac{2(\kappa_\pm \mp \rho_0)}{\sigma_1 + 2\sigma_2} + \frac{\delta_0}{\delta_\mp} + \left(\frac{\sigma_1 \mp 2\rho_1}{\sigma_1 + 2\sigma_2} - \frac{\delta_\pm}{\delta_\mp}\right)R_\pm^2\right]R_\pm = 0. \quad (6)$$

Now, using the general form of the solution of equation (6) given in [16–19], we find δ_0/δ_+, δ_-/δ_+ and the amplitudes

$$R_\pm(t) = R_{\pm0}[1 + n_\pm \mathrm{sn}^2(\nu t, \mu)]^{1/2}, \quad (7)$$

where

$$n_\pm = n_\pm(\mu, \nu) = \mu^2\nu^2k_2(\sigma_2 \mp \rho_1)/[R_{\pm0}^2(\rho_1^2 + \sigma_1\sigma_2 + \sigma_2^2)].$$

In this case, the actual real factor ν and modulus μ of the Jacobi elliptic sine [15], which set the same type of the waveguides for two circularly polarised components of the field, are determined by the initial conditions with the help of the relations

$$\left.\frac{d\phi_\pm}{dt}\right|_{t=0} \equiv \omega_{\pm0} = \pm\nu\left[\frac{(\mu^2 + n_\pm)(1 + n_\pm)}{n_\pm}\right]^{1/2}. \quad (8)$$

Note that the values of ν and μ must be such that $R_\pm^2(t)$ and $\omega_{\pm0}^2$ defined by them were positive. This is possible if at least one of the inequalities

$$0 < n_\pm(\mu, \nu) < \infty, \quad -1 < n_\pm(\mu, \nu) < -\mu^2, \quad (9)$$

allowing one to obtain the desired constraints on the values of ν and μ, is fulfilled. The values of $R_{\pm0}$ are always defined with accuracy up to a sign, which, by analogy with [16–19], leads to the existence of two ('positive' and 'negative') branches of the solutions of (7).

Using (5), (7) and (8), the phases of the components $A_\pm(z, t)$ can be expressed through the elliptic integral of the third kind [15]

$$\phi_\pm(t) - \phi_\pm(0) = \frac{\omega_{\pm0}}{\nu}\int_0^{\nu t}\frac{d\tau}{1 + n_\pm \mathrm{sn}^2(\tau, \mu)} = \frac{\omega_{\pm0}}{\nu}\Pi(\nu t, n_\pm, \mu), \quad (10)$$

where $n_\pm$ is the elliptic characteristic. Nonlinear additions to the frequencies are expressed through the Jacobi elliptic functions:

$$\omega_\pm(t) = \frac{\omega_{\pm0}}{1 + n_\pm \mathrm{sn}^2(\nu t, \mu)}. \quad (11)$$

The condition $\omega_\pm(t) \neq 0$, corresponding to a nonzero chirp of a cnoidal wave, removes degeneration of solutions with respect to ν and μ, which took place in [9]. In (2), additions to the propagation constants $\kappa_\pm$ are expressed by the formulas

$$\kappa_\pm = \pm\rho_0 - R_{\mp0}^2(\sigma_1/2 + \sigma_2) -$$

$$-\frac{2k_2\nu^2(1+\mu^2)+3R_{\pm0}^2(\sigma_1\mp2\rho_1)}{4}$$

$$-\frac{(\sigma_2\pm\rho_1)(\sigma_1+2\sigma_2)R_{\pm0}^2}{4(\sigma_2\mp\rho_1)}. \qquad (12)$$

The solutions $R_\pm(t)$, whose ratios $R_+(t)/R_-(t)$ are independent of time, as in [9] will be called degenerate and denoted by a superscript d in parentheses. It follows from the above formulas that the amplitudes $R_{\pm0}^{(\mathrm{d})}$, phases $\phi_\pm^{(\mathrm{d})}(t)$, and additions to the frequency $\omega_\pm^{(\mathrm{d})}(t)$ of these solutions are related by the expressions

$$[R_{+0}^{(\mathrm{d})}/R_{-0}^{(\mathrm{d})}]^2=(\sigma_2-\rho_1)/(\sigma_2+\rho_1),$$

$$\phi_+^{(\mathrm{d})}(t)-\phi_+^{(\mathrm{d})}(0)=\phi_-^{(\mathrm{d})}(t)-\phi_-^{(\mathrm{d})}(0),\;\; \omega_+^{(\mathrm{d})}(t)=\omega_-^{(\mathrm{d})}(t). \qquad (13)$$

It is easy to verify that elliptically polarised cnoidal waves, which were discussed in paper [9], are special limit cases of the above solutions for $\omega_\pm(t)\equiv0$, corresponding to boundaries [following from inequalities (9)] of admissible values of $n_\pm$. The solutions $R_\pm(t)=\nu[-k_2(\sigma_2\mp\rho_1)/(\rho_1^2+\sigma_1\sigma_2+\sigma_2^2)]^{1/2}\mathrm{dn}(\nu t,\mu)$ which do not change the sign, are formed from the positive branch of formula (7) at $n_\pm=-\mu^2$. Sign-changing solutions $R_\pm(t)=\nu\mu[k_2(\sigma_2\mp\rho_1)/(\rho_1^2+\sigma_1\sigma_2+\sigma_2^2)]^{1/2}\times\mathrm{sn}(\nu t,\mu)$ and $R_\pm(t)=\nu\mu[-k_2(\sigma_2\mp\rho_1)/(\rho_1^2+\sigma_1\sigma_2+\sigma_2^2)]^{1/2}\mathrm{cn}(\nu t,\mu)$ are obtained from (7) at $n_\pm\to\infty$ and $n_\pm=-1$, respectively. And in the last two cases, one must sew (see [18]) the positive and negative branches of (7) at those point of times when $R_\pm(t)=0$ and the phase jumps occur. Note also the possible existence of 'hybrid' solutions, in which one of the circularly polarised components of the field is chirped, and the other is not.

3. Chirped cnoidal elliptically polarised waves and polarisation 'chaos'

Based on the character of the variation of the amplitude of two circularly polarised field components in time and their asymptotics [9], as well as on the connectivity of boundaries of their existence regions, all the solutions of (7)–(10) can be divided into three groups.

The first group includes those in which the amplitudes of both field components grow at point $t=0$. Solutions of this type exist in the cases when the signs of $k_2(\rho_1^2+\sigma_1\sigma_2+\sigma_2^2)$ and $(\sigma_2\mp\rho_1)$ are the same and, therefore, $n_\pm>0$. When $\omega_\pm(t)\equiv0$, they undergo a transition to the solutions of ss type according to the classification given in [9]. The second group consists of those solutions in which the amplitudes of both field components at point $t=0$ start to decrease. They exist in the cases where the signs of $k_2(\rho_1^2+\sigma_1\sigma_2+\sigma_2^2)$ and $(\sigma_2\mp\rho_1)$ are opposite and, therefore, $n_\pm<0$. When $\omega_\pm(t)\equiv0$, the solutions of this group become the solutions of cc, cd and dd types (classification given in [9]). In the cases when $k_2(\rho_1^2+\sigma_1\sigma_2+\sigma_2^2)$ and $(\sigma_2-\rho_1)$ are positive and the parameter $(\sigma_2+\rho_1)$ is negative, or vice versa, the amplitude of one of the field components at point $t=0$ begins to increase, and the amplitude of the second one – to decrease, thereby forming the third group of solutions. When $\omega_\pm(t)\equiv0$, they undergo a transition to the solution of sc and sd types (classification given in [9]). Note that the solutions of the three groups listed above have the soliton asymptotics corresponding to the passage to the limit of the Jacobi elliptic functions in the hyperbolic functions at $\mu\to1$. The degeneracy is only possible for the solutions of the first and second groups.

In the optical range the characteristic scale of manifestation of the nonlocal response to the external light field in gyrotropic media is significantly smaller than the wavelength, so that $|\rho_1|\ll|\sigma_{1,2}|$. Consequently, the solutions of the third group in such media are unlikely to be realised. Figure 1 illustrates the typical behaviour of the dependences of the normalised moduli $r_\pm=|R_\pm|\nu^{-1}|\sigma_1/k_2|^{1/2}$, phases $\phi_\pm$, and nonlinear additions $\omega_\pm(t)=\mathrm{d}\phi_\pm/\mathrm{d}t$ to the frequency ω on the dimensionless time νt, corresponding to the solutions of the first of the three groups listed above.

The evolution of the polarisation state of chirped cnoidal waves corresponding to the found solutions is convenient to

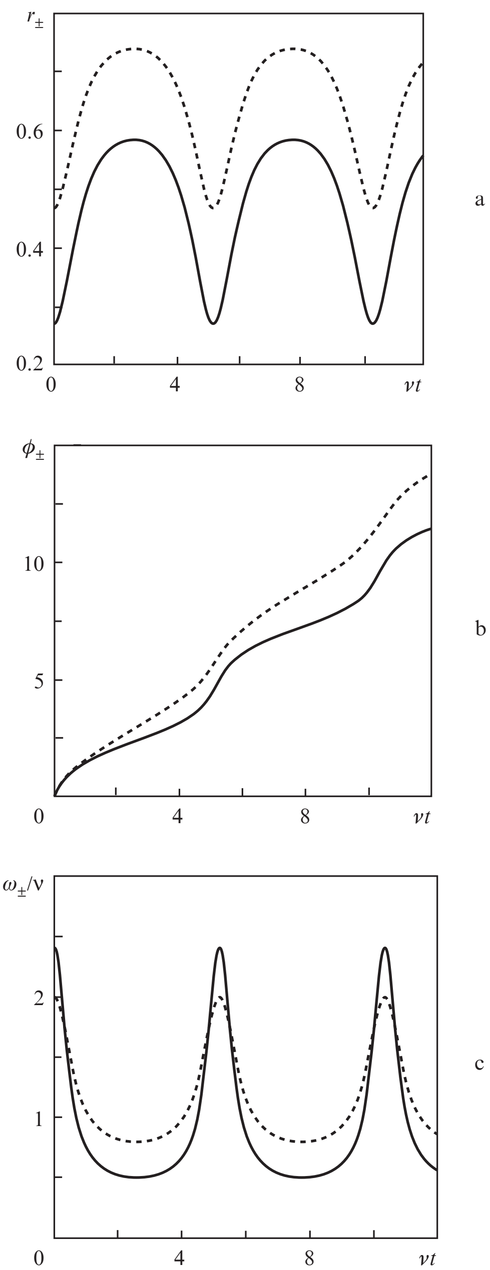

Figure 1. Dependences of $r_\pm$ (a), $\phi_\pm$ (b) and $\omega_\pm/\nu$ (c) on the dimensionless time νt at $z=0$, $r_+(0)=0.27$, $r_-(0)=0.47$, $\rho_1/\sigma_1=0.2$, $\sigma_2/\sigma_1=2$ and $\mu=0.95$.

describe using the Stokes parameters [20] coupled with the complex amplitudes $A_{\pm}$ by the relations:

$$S_0(t) = (|A_+|^2 + |A_-|^2)/2, \quad S_1(t) = \mathrm{Re}\{A_+A_-^*\},$$
$$S_2(t) = \mathrm{Im}\{A_+A_-^*\}, \quad S_3(t) = (|A_-|^2 - |A_+|^2)/2. \tag{14}$$

In this case, the parameters $s_{x,y,z} = S_{1,2,3}/S_0$ are Cartesian coordinates of the end of the unit vector $\boldsymbol{s}$, which moves, as t changes, on the surface of the so-called Poincare sphere [20].

The Stokes parameters are uniquely related to the polarisation characteristics, which were used previously in paper [5–9]. In this case, S_0 specifies the instantaneous intensity of the light field

$$S_0 = (R_{+0}^2 + R_{-0}^2)/2$$
$$+ [v^2\mu^2 k_2\sigma_2/(\rho_1^2 + \sigma_1\sigma_2 + \sigma_2^2)]\,\mathrm{sn}^2(vt,\mu), \tag{15}$$

the s_z component describes the degree of ellipticity of the polarisation ellipse, M,

$$s_z = -M = \frac{|A_-|^2 - |A_+|^2}{|A_+|^2 + |A_-|^2}$$
$$= -\frac{(R_{+0}^2 - R_{-0}^2)(\rho_1^2 + \sigma_1\sigma_2 + \sigma_2^2) - 2v^2\mu^2 k_2\rho_1 \mathrm{sn}^2(vt,\mu)}{(R_{+0}^2 + R_{-0}^2)(\rho_1^2 + \sigma_1\sigma_2 + \sigma_2^2) + 2v^2\mu^2 k_2\sigma_2 \mathrm{sn}^2(vt,\mu)}, \tag{16}$$

and the longitude

$$\Phi = \arctan\left(\frac{s_y}{s_x}\right) = 2\rho_0 z + \frac{(3\rho_1^2 + \sigma_1\sigma_2 - \sigma_2^2)z}{4}$$
$$\times\left[\frac{R_{+0}^2}{\sigma_2 - \rho_1} - \frac{R_{-0}^2}{\sigma_2 + \rho_1}\right] + \Delta\Phi(t) \tag{17}$$

is twice the angle of rotation of the main axis of the polarisation ellipse $(\mathrm{Arg}\{A_+A_-^*\}/2)$. In (17), $\Delta\Phi(t) = (\omega_{+0}/v)\times \Pi(vt,n_+,\mu) - (\omega_{-0}/v)\Pi(vt,n_-,\mu)$.

It is easy to see that, as in [9], the dependence of the longitude on the coordinate z reduces to a renormalisation of the constant ρ_0 of linear gyration due to the nonlinearity. For fixed z the change of Φ in time is due only to the dependence $\Delta\Phi(t)$. In this case, the end of the vector $\boldsymbol{s}$ moves along the surface of a spherical shell, the lower and upper boundaries of which are determined by the extrema s_z. The dimensionless period vT of this motion is twice the complete elliptic integrals of the first kind $K(\mu)$. During the time T the angle Φ increases by $\Delta\Phi(t = 2K/v)$.

If $p\Delta\Phi(t = 2K/v) = 2q\pi$, where p and q are integers, then the orientation of the end of the vector $\boldsymbol{s}$ in space and, therefore, the polarisation state of the light wave will change periodically. In all other cases, the end of the vector $\boldsymbol{s}$ will pass over time through any point of the surface of the specified layer. The change in the polarisation state of the light wave will seem chaotic, but the overall situation will be similar to the evolution of a strange attractor, which eventually fills a region of its phase space. The typical character of the evolution of the polarisation state for the solutions of the first group on the Poincare sphere is demonstrated in Fig. 2. The figure shows periodic [corresponding to a chirped elliptically polarised cnoidal wave (Figs 2a, c)] and aperiodic [corresponding to polarisation 'chaos' (Figs 2b, d)] trajectories of the movement of the end of the normalised Stokes vector $\boldsymbol{s}$ on the Poincare sphere. We emphasise that the term 'chaos' is used here only for the sake of brevity, as its correct application requires a detailed study of the correlation properties of the aperiodic solutions obtained. The emergence of 'loops' on the trajectories (Figs 2c, d) are due to the possibility of a nonmonotonic dependence $\Delta\Phi(t)$ (the formation of local extrema) for certain values of the parameters of the problem (Fig. 3).

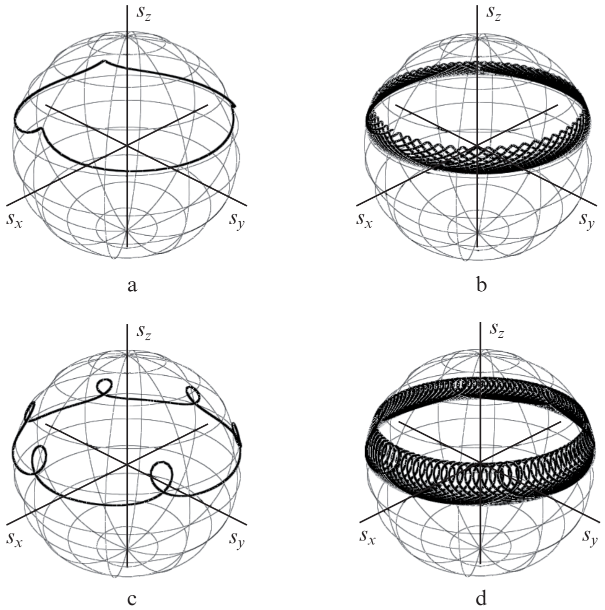

Figure 2. Periodic motion of the end of the Stokes vector on the surface of the Poincare sphere at $r_+(0) = 0.47$, $r_-(0) = 0.82$ ($p/q = 3$) (a), $r_+(0) = 0.27$, $r_-(0) = 0.47$ ($p/q = 6$) (c) and transition to its aperiodic motion for irrational values of p/q (b and d); the other parameters are the same as in Fig. 1.

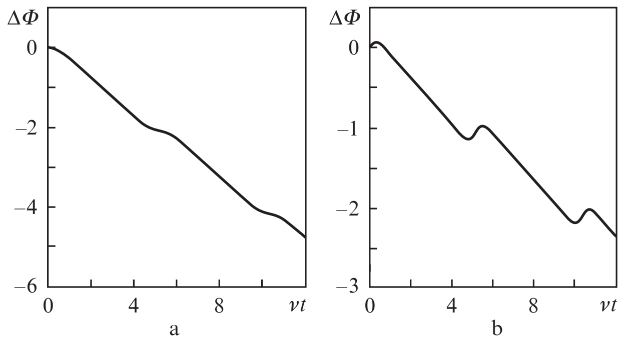

Figure 3. Dependence $\Delta\Phi(t)$ at $r_+(0) = 0.47$, $r_-(0) = 0.82$ (a) and $r_+(0) = 0.27$, $r_-(0) = 0.47$(b); the other parameters are the same as in Fig. 1.

4. Conclusions

It is shown that in an isotropic medium with local and nonlocal cubic nonlinearity and second-order frequency dispersion, chirped elliptically polarised cnoidal waves can propagate and the regimes that resemble polarisation chaos can appear.

The analytical solutions of the system of two nonlinear Schrödinger equations, corresponding to these two situations, are found and analysed in a particular case when in a nonlinear medium nonlinear waveguides of the same type are formed for the two circularly polarised components of the light field. It is found that the frequencies of both components vary in concord with a periodic change of their moduli, and the evolution of the polarisation state of chirped nonlinear waves during their propagation can radically change with changing the initial conditions.

Acknowledgements. The authors thank I.A. Perezhogin and K.V. Rudenko for useful discussions.

This work was supported by the Russian Foundation for Basic Research (Grant No. 11-02-00653-a), the President Grant for Government Support of the Leading Scientific Schools of the Russian Federation (No. NSH-6897.2012.2) and the Federal Program of the Ministry of Education and Science of the Russian Federation (Grant No. 8393).

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