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General Fractional Economic Dynamics with Memory †

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† This article is dedicated to the 140-th anniversary of the first publication on general fractional calculus, written by Nikolai Ya. Sonin.

Abstract: For the first time, a self-consistent mathematical approach to describe economic processes with a general form of a memory function is proposed. In this approach, power-type memory is a special case of such general memory. The memory is described by pairs of memory functions that satisfy the Sonin and Luchko conditions. We propose using general fractional calculus (GFC) as a mathematical language that allows us to describe a general form of memory in economic processes. The existence of memory (non-locality in time) means that the process depends on the history of changes to this process in the past. Using GFC, exactly solvable economic models of natural growth with a general form of memory are proposed. Equations of natural growth with general memory are equations with general fractional derivatives and general fractional integrals for which the fundamental theorems of GFC are satisfied. Exact solutions for these equations of models of natural growth with general memory are derived. The properties of dynamic maps with a general form of memory are described in the general form and do not depend on the choice of specific types of memory functions. Examples of these solutions for various types of memory functions are suggested.

Keywords: economic dynamics; fractional calculus; memory; general fractional calculus; fractional differential equation; growth models

MSC: 26A33; 34A08; 91B02; 91B55; 91B62

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1. Introduction

Processes and systems in economics can be described using various mathematical methods [1–4]. These methods include standard differential and integral calculus, differential and difference equations, variational calculus, matrix algebra, computational methods, and others. To describe economic dynamics with continuous time, differential equations with derivatives of integer order are actively used [1–4]. However, differential equations, which contain only integer-order derivatives, cannot be used to describe processes with memory, i.e., processes for which the variable $Y(t)$ depends on change $Y(s)$ for $s < t$. This is due to the fact that differential equations of integer order describe changes of $Y(t)$ and its derivatives in an infinitesimally small neighborhood of t.

The first mathematical theory of processes with memory was proposed in physics by Boltzmann one hundred and fifty years ago [5–9] for construction models of isotropic viscoelastic media. Then, mathematical descriptions of processes with memory were considered by Vito Volterra in 1928 and 1930 [10–14]. A more complete description of the concept of memory was considered in papers [15–20], which are dedicated to continuum mechanics with memory, and in paper [15] for economics.

The first economics result that proved the importance of memory in economic data was described sixty years ago in the Granger papers [21–23]). These works proved that many spectral densities, which are derived from economic time series, have a similar form.

Therefore in economics, the memory effect was discovered sixty years ago. In 2003, the Nobel memorial prize was awarded to Granger "for methods of analyzing economic time series with common trends (cointegration)".

In economics, the first mathematical formulation of processes with memory is associated with works of Granger and Joyeux [24–26] published in 1980. To describe economic processes with memory, Granger [24] in 1980 proposed the fractional ARIMA models, which are also called ARFIMA(p, d, q).

In paper [24], Granger and Joyeux proposed so-called fractional integrating and differencing to generalize the ARIMA models (see [26–36]). The fractional difference operators of Granger and Joyeux were proposed in 1980 and have been used in economics up to the present time. In fact, these fractional operators of Granger and Joyeux are the Grunwald–Letnikov fractional differences that have been used for more than one hundred and fifty-five years [37,38].

The approach that was proposed by Granger and Joyeux is the most commonly used method among economists [33–36]. This approach is restricted to only one type of fractional finite difference (Grunwald–Letnikov fractional differences). In addition, this Granger–Joyeux approach is used without an explicit connection with the fractional calculus. This calculus includes not only various fractional finite differences but also fractional derivatives and integrals of non-integer order.

Fractional calculus is a mathematical theory of integral and integro-differential operators of arbitrary (integer and non-integer) orders. This theory is called calculus because these operators satisfy some analogues of the fundamental theorems of standard calculus. Operators that satisfy the fundamental theorems of fractional calculus are called fractional integrals (FIs) and fractional derivatives (FDs). These fundamental theorems connect the integral and differential operators of non-integer orders [39–45]. The history of fractional calculus was first described in 1868 by Letnikov [46] and then by other authors in [39,47–54]. Note that a lot of standard properties and rules of integer-order derivatives and integrals are violated for fractional derivatives and integrals [55–65]. For example, the standard forms of the chain rule, product rule, and semigroup property are violated for fractional derivatives in general.

Fractional calculus has been actively used in recent decades to describe non-standard properties of systems and processes with nonlocality in space and time in various subjects of physics [66–75], economics [76,77], biology [78], and other sciences.

The Granger–Joyeux approach to describe memory has significant drawbacks since it is limited only to the Grunwald–Letnikov fractional differences and discrete-time models. From the beginning of the 21st century, this approach, which is based on fractional calculus, has been used in economics and finance [53,76,77].

Recent work by various scientists on the use of fractional calculus in economics is contained in the editorial volume [76] and in the list of articles in a review [53] devoted to the history of applications of fractional calculus in economics. Note that the concepts and models of economic dynamics with memory are developed in detail in book [77], which was published in 2021.

To describe a wider type of memory, one can use the integral and integro-differential operators with kernels belonging to a wider class of functions. In order to have mathematically self-consistent descriptions, these operators should satisfy some general analogs of the fundamental theorems of fractional calculus. This approach is realized in the so-called general fractional calculus (GFC) that is based on the concepts of kernel pairs and operators, which were suggested by Sonin 140 years ago [79,80]. Note that the terms general fractional calculus, general fractional integrals, and general fractional derivatives were suggested by Kochubei in 2011.

Although the first published work (the Sonin article [79]) on general fractional calculus (GFC) appeared exactly 140 years ago (in 1884), it can be said that GFC and its application is a new, young area of research in mathematics, physics, and other sciences. Note that currently the number of published works on GFC does not exceed one hundred, and, more precisely, there are about sixty such publications. In the proposed paper, almost all of the works on general fractional calculus and its applications are presented in the References. Let us note three main types of GFC that use three different sets of Sonin kernels: (K) Kochubei GFC, (H) Hanyga GFC, and (L) Luchko GFC. For a wide class of applications in the most convenient form of GFC is the Luchko GFC. In this regard, all the works written over 140 years, which are devoted to the general fractional calculus, can be divided into the following groups of works.

- Sonin's works [79–81]. This year is the 140-th anniversary of the first publication on general fractional calculus, written by Nikolai Ya. Sonin.
- Luchko's works devoted to the formulation of his form of GFC in 2021 [82–85], and its development and generalization [86–97], some of which were written with co-authors. The works include the general fractional operational calculus [84,89,96], the GFC on a finite interval proposed by Al-Refai and Luchko paper [93], the GFC of operators of distributed order suggested by Al-Refai and Luchko paper [94] and other.
- Works, in which the Luchko GFC is developed and generalized [98–108]. Note these works include the GF vector calculus [98], the GFC the Riesz form [103], the scale-invariant GFC based Mellin convolution operators [104], the GFC with construction that is proposed by Al-Refai and Fernandez [106–108], the parametric GFC [105,107], and some others types.
- Papers devoted to the application of the Luchko GFC in various sciences [109–119].
- Works, in which the Kochubei, Hanyga and other forms of GFC are described [120–133]
- Articles, in which the Kochubei GFC is applied [134–142].

The GFC is a powerful tool that allows describing processes with general form of memory. In this paper, the GFC is used to construct mathematical models of economic growth processes with general memory. A simple economic growth model in which time is continuous is the model of natural growth [1–4]. This model uses integer-order derivatives and therefore cannot describe memory in economic dynamics. An economic model of natural growth in which a power-law type of memory is taken into account is proposed in [77,143,144]. Some generalizations with power memory are described in [77,145–148]. The exact solutions of these equations, which are fractional differential equations of arbitrary positive order α, are derived. For integer values with an order parameter ($\alpha = 1$), these equations give standard equations of natural growth without memory.

This article is the first to propose a self-consistent approach that makes it possible to describe economic growth processes with general memory, for which the power-law type is a special case. Natural growth models with power-law memory, which were suggested in 2016 [77,143,144], are special cases of the proposed natural growth models with general memory. We propose equations with general fractional integrals and derivatives to take into account the wide class of memory functions that are described by the kernels of the GF operators. Memory is described by pairs of memory functions that satisfy the Sonin and Luchko conditions. In the proposed equations, general fractional derivatives are used to take into account a general form of memory. Solutions of these equations, which describe general fractional natural growth models with memory, are derived.

Let us describe the structure of this article. Section 2 includes two preliminary subsections about standard models of natural growth without memory and basic definitions of general fractional integrals and derivatives in order to fix the notation. The third subsection is devoted to the derivation of the general fractional differential equations of the growth model with general memory. In Section 3, the general fractional equations of natural growth with general memory are proposed. The GF differential equations of the proposed model are solved. In Section 4, non-standard properties and examples of dynamical maps with general memory are described. In Section 5, an example of natural growth with power-law memory is described as a special case of the proposed general model. In Section 6, examples of growth with the general form of memory are described. In Section 7, a short conclusion is give.

2. Natural Growth Model with Memory

2.1. Natural Growth Model without Memory

The model of natural growth without memory and lag describes the dynamics of the value of output $Y(t)$ at time t. The standard model of natural growth uses the following assumptions.

The first assumption is that the market is unsaturated, which implies that all manufactured products are sold.

The second assumption is that the price $P(t) > 0$ is a constant since the volume of sales does not affect the price of the goods:

$$P(t) = P. \tag{1}$$

The third assumption is that the value $I(t)$ of the net investment is

$$I(t) = m\,(P\,Y(t) - C(t)), \tag{2}$$

where $P\,Y(t)$ is the income, $C(t)$ is the costs, and m is the share of profit ($0 < m < 1$) that is spent on the net investment (the rate of net investment).

The fourth assumption is that there is a linear dependence of the costs on the output:

$$C(t) = a\;Y(t) + b, \tag{3}$$

where the parameter a describes the marginal costs, and b is the costs that do not depend on the value of the output (the independent costs).

The fifth assumption is that the accelerator equation holds:

$$\frac{dY(t)}{dt} = \frac{1}{v}\,I(t), \tag{4}$$

where $1/v$ is the marginal productivity of capital, and v is the power of the accelerator [1], p.62. Using the fundamental theorem of calculus, Equation (4) can be written in the integral form

$$Y(t) = \frac{1}{v}\int_0^t I(\tau)\,d\tau + Y(0). \tag{5}$$

The equivalence of Equations (4) and (5) follows from the fundamental theorem of calculus, which connects the differentiation and integration of the functions. According to this theorem, the differentiation of Equation (5) gives Equation (4).

Substitution of (2) and (3) into (4) gives

$$\frac{dY(t)}{dt} - v^{-1}\,(P-a)\,Y(t) = -\,v^{-1}\,m\,b, \tag{6}$$

where the following quantities are used:
$Y(t)$ is the value of the output;
$P(t) = P$ is the price;
m is the share of the profit ($0 < m < 1$);
a is the marginal costs;
b is the independent costs that do not depend on the value of output;
v^{-1} is the marginal productivity of capital.

Differential Equation (6) describes natural growth without lag and memory.

As a result, the solution of Equation (6) is

$$Y(t) = \frac{b}{(P-a)}\Big(1 - \exp\{m\,v^{-1}(P-a)\,t\}\Big) + \exp\{m\,v^{-1}(P-a)\,t\}\,Y(0). \tag{7}$$

Solution (7) describes the natural growth of output without memory.

In the standard growth model, (6) is based on the accelerator Equation (4). Equations (4) and (6) contain only first-order derivatives. This leads to the fact that this model does not take into account the memory. One can state that Equation (6) neglects memory and lag. To take into account memory, we should use integral and integro-differential operators.

2.2. General Fractional Integrals and Derivatives

Let us define the set of kernel pairs that was proposed in [82].

Definition 1. *Let $M(t)$ and $K(t)$ be functions that satisfy the Sonin condition:*

$$(K * M)(t) = \int_0^t K(t-t')\, M(t')\, dt' = 1 \quad \textit{for all } x \in (0,\infty), \tag{8}$$

and the Luchko condition:

$$K(t),\, M(t) \in C_{-1,0}(0,\infty), \tag{9}$$

where

$$C_{-1,0}(0,\infty) = \{h(t) : h(t) = t^p\, g(t),\ t > 0,\ -1 < p < 0,\ g(t) \in C[0,\infty)\}. \tag{10}$$

Then, the set of $(M(t),\, K(t))$ is called the Luchko set and is denoted by $\mathcal{L}$.

Let us define general fractional operators of general fractional calculus (GFC) in the Luchko form proposed in [82,83].

Definition 2. *Let $f(t) \in C_{-1}(0,\infty)$ and $(M(t), K(t)) \in \mathcal{L}$.*

Then, the general fractional integral (GFI) is defined as

$$I^t_{(M)}[t']\, f(t') = (M * f)(t) = \int_0^t dt'\, M(t-t')\, f(t'), \tag{11}$$

where

$$C_{-1}(0,\infty) = \{h(t) : h(t) = t^p\, g(t),\ t > 0,\ p > -1,\ g(t) \in C[0,\infty)\}. \tag{12}$$

Definition 3. *Let $f(t) \in C^1_{-1}(0,\infty)$ and $(M(t), K(t)) \in \mathcal{L}$.*

Then, the general fractional derivative (GFD) of the Caputo type w is defined as

$$D^{t,*}_{(K)}[t']\, f(t') = (K * f^{(1)})(t) = \int_0^t dt'\, K(t-t')\, f^{(1)}(t'), \tag{13}$$

where

$$C^1_{-1}(0,\infty) = \{h(t) : h^{(1)}(t) \in C_{-1}[0,\infty)\}, \tag{14}$$

and $g^{(1)}(t) := dg(t)/dt$.

Remark 1. *If $(M(t), K(t)) \in \mathcal{L}$, then $(K(t), M(t)) \in \mathcal{L}$. Therefore, both $I^t_{(M)}[t']$, $D^{t,*}_{(K)}[t']$ and $I^t_{(K)}[t']$, $D^{t,*}_{(M)}[t']$ can be used as GFIs and GFDs [83].*

Remark 2. *Nonlocality in space is described by kernel pairs of GFIs and GFDs that belong to the Luchko set. One can see examples of kernel pairs $(M(t),\, K(t)) \in \mathcal{L}$ in Table 1 of [111], pp. 5–7; Table 1 of [112], p. 15, and [114], p.1 1; and Table 1 of [102], pp. 21–22, and [101], p. 10. Note that one can consider the kernel pairs $(M_{new} = \lambda^{-1} K(t), K_{new} = \lambda\, M(t))$, where $(M(t), K(t))$ are pairs of these tables. The kernels have the following dimensions: $[M(t)] = [1]$ and $[K(t)] = [t]^{-1}$, where $\lambda > 0$, $[\lambda] = [t]^{-1}$, and $t > 0$.*

Remark 3. *For the GFIs $I^x_{(M)}$, the semi-group property*

$$I^x_{(M_1)}[s]\, I^s_{(M_2)}[t]\, f(t) \;=\; I^x_{(M_1 * M_2)}[t]\, f(t). \tag{15}$$

is satisfied if the kernels $M_1(t)$ and $M_2(t)$ belong to the space $C_{-1,0}(0,\infty)$. This equality is proved as the index law in [82], p. 8.

In general, for the GFDs, we have the inequality

$$D^{x,*}_{(K_1)}[s]\, D^{s,*}_{(K_2)}[t]\, f(t) \;\neq\; D^{x,*}_{(K_1 * K_2)}[t]\, f(t), \tag{16}$$

if $K_1(t),\, K_2(t) \;\in\; C_{-1,0}(0,\infty)$.

The GFIs and GFDs are interconnected by the fundamental theorems of GFC (FT of GFC).

Theorem 1. *(First FT of GFC).*
Let $(M(t), K(t)) \;\in\; \mathcal{L}$ and $f(t) \in C_{-1,(K)}(0,\infty)$, where

$$C_{-1,(K)}(0,\infty) \;:=\; \{h(t) : \quad h(t) \;=\; I^t_{(K)}[t']g(t'), \quad g(t) \;\in\; C_{-1}(0,\infty)\}.$$

Then, the equality

$$D^{t,*}_{(K)}[t']I^{t'}_{(M)}[t'']\, f(t'') \;=\; f(t) \tag{17}$$

holds for all $t > 0$.

Proof. The proof of this theorem is given in [82] (see the proof of Theorem 3 in [82] or Theorem 1 in [83]). □

Theorem 2. *(Second FT of GFC).*
Let $f(t) \;\in\; C^1_{-1}(0,\infty)$ and $(M(t), K(t)) \;\in\; \mathcal{L}$.
Then, the equality

$$I^t_{(M)}[t']D^{t',*}_{(K)}[t'']\, f(t'') \;=\; f(t) \;-\; f(0) \tag{18}$$

holds for all $t > 0$.

Proof. The proof of this theorem is given in [82] (see the proof of Theorem 4 in [82] or Theorem 2 in [83]). □

2.3. Generalization of Growth Model using General Memory

In articles [143–145] and book [77], a standard natural growth model is generalized for the case of power-law memory. In this section, we propose a standard model of natural growth generalized for the case of a general form of memory function $M(t)$. To have a self-consistent mathematical theory of natural growth with memory, we should assume that there is a kernel $K(t)$, which is associated with the kernel $M(t)$, and the Sonin condition of its pairs are satisfied. In this case, we should use general fractional calculus (GFC). The fundamental theorems of GFC connect GF differentiation and GF integration and lead to self-consistency of a mathematical model of such processes with general memory.

To take into account memory, we assume that the value of output $Y(t)$ depends on the changes $I(\tau)$ in the past at $\tau \in (0,t)$ since economic agents can remember the previous changes of investments $I(t)$ and the impact of these changes on the value of output $Y(t)$.

To describe natural growth with memory, we propose the equation

$$Y(t) \;=\; \int_0^t M(t-\tau)\, I(\tau)\, d\tau \;+\; Y(0), \tag{19}$$

where $M(t)$ is the memory function. Equation (19) with $M(t) = 1$ takes the form (5).

Using GFIs, Equation (19) takes the form

$$Y(t) \,=\, \frac{1}{v}\, I^{t}_{(M)}[\tau]\, I(\tau) \,+\, Y(0), \tag{20}$$

where $I^{t}_{(M)}[\tau]$ is the general fractional integral. Equation (20) describes a multiplier with memory, and $\frac{1}{v}$ is a multiplier coefficient.

The action of the GFD $D^{s,*}_{(K)}[t]$ on Equation (20) gives

$$D^{s,*}_{(K)}[t]\, Y(t) \,=\, \frac{1}{v}\, D^{s,*}_{(K)}[t]\, I^{t}_{(M)}[\tau]\, I(\tau), \tag{21}$$

where we use

$$D^{s,*}_{(K)}[t]\ \mathrm{const} \,=\, 0.$$

Using the first fundamental theorem of GFC in the form

$$D^{s,*}_{(K)}[t]\, I^{t}_{(M)}[\tau]\, f(\tau) \,=\, f(s), \tag{22}$$

which holds for any function $I(t) \in C_{-1}(0, \infty)$, Equation (21) takes the form

$$D^{t,*}_{(K)}[\tau]\, Y(\tau) \,=\, \frac{1}{v}\, I(t). \tag{23}$$

Equations (2), (3) and (23), gives

$$D^{t,*}_{(K)}[\tau]\, Y(\tau) \,-\, v^{-1}\, m\, (P - a)\, Y(t) \,=\, -\, v^{-1}\, m\, b. \tag{24}$$

Equation (24) is the equation with GFD. The proposed model of growth that is based on Equation (24) is a model with a general form of memory.

Equation (24) $K(t) \,=\, h_{1-\alpha}(t)$ with $\alpha \in (0, 1)$ at the limit $\alpha \to 1-$ gives the standard Equation (6) of a model of growth without memory.

To obtain the solutions of Equation (24) of the natural growth model with memory, Equation (24) can be represented as

$$D^{t,*}_{(K)}[\tau]\, Y(\tau) \,=\, \Omega\, Y(t) \,+\, f(t) \tag{25}$$

with

$$\Omega \,=\, v^{-1}\, m\, (P - a). \tag{26}$$

It can be seen that Equation (24) has the form of (25) if

$$\Omega \,=\, \frac{m(P-a)}{v}, \quad f(t) \,=\, -\, \frac{mb}{v}. \tag{27}$$

where $Y(t)$ is the value of output, $P(t) \,=\, P$ is the price, m is the share of profit, a is the marginal costs, b is the independent costs, and v^{-1} is the marginal productivity of capital.

If the independent cost is equal to zero ($b \,=\, 0$), Equation (25) has the form

$$D^{t,*}_{(K)}[\tau]\, Y(\tau) \,=\, \Omega Y(t). \tag{28}$$

For the case when Ω is a positive number, Equation (28) describes the natural growth with a general form of memory. If Ω is a negative number, Equation (28) describes a decline with memory instead of growth with memory.

It is obvious that models of the behavior of economic agents must take into account that agents may have memory. The description of economic processes should take into account that the behavior of economic agents may depend on the history of previous changes to the economy [77]. For example, the demand may depend on all changes of prices over a finite time interval since the behavior of buyers can be determined by the presence of a memory of previous price changes [77]. A consistent theory of basic concepts and notions of economic dynamics and their mathematical implementations were proposed in book [77]. It is obvious that integer-order differential equations can be used only if all buyers have total amnesia. Because of this, the economic interpretation of memory is dependence of the behavior of economic agents and the economic processes themselves on the history of changes to this behavior and these processes in the past. Moreover, the influence of this history on current behavior and current processes may decrease over time, which leads to the need to take into account memory fading over time [15,77]. Mathematically, this influence is described by the kernels of integral and integro-differential operators. To describe such processes, it is not enough to use power-type memory. A self-consistent mathematical theory is needed that allows one to describe economic processes with general memory, whereas power-type memory is a special case of such general memory. Such a type of economic theory is suggested in this paper.

General memory is described by the kernel pairs $(M(t), K(t)) \in \mathcal{L}$, where $M(t)$ is the kernel of the GF integral and $K(t)$ is the kernel of the GF derivative that satisfy the fundamental theorems of GFC. Let us describe examples of general memory. The parameter λ is the inverse value of the characteristic time of the economic process ($\lambda\, t$ is a dimensionless quantity).

The power-law type of memory is described by

$$M_{PL}(t) = h_{\alpha}(\lambda t) = \frac{(\lambda\, t)^{\alpha-1}}{\Gamma(\alpha)}, \quad K_{PL}(t) = \lambda\, h_{1-\alpha}(\lambda t) = \frac{\lambda\, (\lambda\, t)^{-\alpha}}{\Gamma(1-\alpha)}, \tag{29}$$

where $\alpha \in (0, 1)$.

The gamma-distribution type of memory is described by

$$M_{GD}(t) = h_{\alpha,\lambda}(\lambda\, t) = \frac{(\lambda\, t)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda\, t}, \quad K_{GD}(t) = \lambda\, h_{1-\alpha,\lambda}(\lambda t) + \lambda\, \gamma_{1-\alpha}(\lambda t). \tag{30}$$

where $\gamma_{\alpha}(x) := \gamma(\alpha, x)/\Gamma(\alpha)$ and $\gamma(\alpha, x)$ is the incomplete gamma function, $\alpha \in (0, 1)$, and $\beta > 0$ [149].

The Mittag–Leffler type of memory is described by

$$M_{ML}(t) = (\lambda\, t)^{\beta-1}\, E_{\alpha,\beta}[-(\lambda\, t)^{\alpha}], \quad K_{ML}(t) = \lambda\, h_{\alpha-\beta,\lambda}(\lambda\, t) + \lambda\, h_{1-\beta}(\lambda\, t), \tag{31}$$

where $0 < \alpha \leq \beta - 1$, and $E_{\alpha,\beta}[z]$ are the two-parameters of the Mittag–Leffler function [42,150,151].

Other examples, which include the Bessel type memory and the confluent hypergeometric Kummer type memory, can be seen in [101,102,111,112,114]. Note that the dimensions of the kernels are $[M(t)] = [t]^0$ and $[K(t)] = [t]^{-1}$.

3. GF Equation of Natural Growth with Memory

3.1. Equation of Economic Growth with General Memory

In the description of economic processes, one can take into account memory (nonlocality in time). To take into account a general form of memory, we propose to use the general fractional derivatives and integrals instead of the standard integer-order derivatives and integrals with respect to time.

Let us consider the natural growth without memory that is described by the first-order differential equation

$$\frac{dY(t)}{dt} = \mathcal{Q}(t)\, Y(t). \tag{32}$$

Using the fundamental theorems of calculus, Equation (32) can be represented in the integral form

$$Y(t) - Y(0) = \int_0^t d\tau \, \Omega(\tau) \, Y(\tau). \tag{33}$$

The general form of memory can be taken into account by using a kernel $M(t)$ in the integral

$$Y(t) - Y(0) = \int_0^t d\tau \, M(t-\tau) \, \Omega(\tau) \, Y(\tau), \tag{34}$$

where $M(t)$ is a memory function. For $M(t) = 1$, Equation (34) takes the form (32) that describes economic processes without memory.

Remark 4. *It should be emphasized that not all kernels $M(t)$ can describe memory (non-locality in time). If integral Equation (34) can be written as an integer-order differential equation (or a finite system of such equations), then the equation cannot describe processes with memory in time. As an example of "local" kernel, one can consider $M(t) = h_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$, with $\alpha = n \in \mathbb{N}$. As a "local" kernel, one can consider the PDF of the exponential distribution and the Erlang distribution [152,153].*

As an example of a "nonlocal" kernel, we consider the kernels of the Luchko set $\mathcal{L}$. In this case, Equation (34) can be written as

$$Y(t) - Y(0) = I^t_{(M)}[\tau] \, \Omega(\tau) \, Y(\tau), \tag{35}$$

where $I^t_{(M)}[\tau]$ are the GFIs; that is,

$$I^t_{(M)} : \quad C_{-1}(0,\infty) \to C_{-1}(0,\infty). \tag{36}$$

Using the first fundamental theorem of GFC, the action of the GFD $D^{t,*}_{(K)}[\tau]$ is such that $(M(t), K(t)) \in \mathcal{L}$, Equation (35) takes the form

$$D^{t,*}_{(K)}[\tau] \, Y(\tau) = \Omega(t) \, Y(t). \tag{37}$$

Equation (37) describes economic processes with a general form of memory, where the memory is described by kernel pairs belonging to the Luchko set $\mathcal{L}$.

Theorem 3. *Let $Y(t) \in C^1_{-1}(0,\infty)$ and $M(t) \in C_{-1,0}(0,\infty)$.*
Then, the GFI equation

$$Y(t) - Y(0) = \int_0^t d\tau \, M(t-t') \, \Omega(t') \, Y(t') \tag{38}$$

can be represented as the GFD equation

$$D^{t,*}_{(K)}[\tau] \, Y(\tau) = \Omega(t) \, Y(t), \tag{39}$$

if there exists $K(t) \in C_{-1,0}(0,\infty)$ such that $(M(t), K(t)) \in \mathcal{L}$.

In subsequent sections, only the case of $\Omega(t) = \Omega = \text{const}$ will be considered.

3.2. Solutions of Equations of Growth with General Memory

The solutions of Equation (39) with $\Omega(t) = \Omega = \text{const}$ can be derived by using Luchko operational calculus [84] (see also [89,96,107]). To describe the solution of Equation (39), we should define the Luchko functions to use Theorem 5.1 of [84].

Definition 4. *Let $G(t) \in C_{-1,0}(0,\infty)$, and $G^{*,k}(t)$ is the convolution k-power:*

$$M^{*,k}(t) \;:=\; (M_1 \,*\, \ldots \,*\, M_k)(t), \tag{40}$$

where $G_j(t) \;=\; G(t)$ for all $j = 1, \ldots, j$, and $t \in (0,\infty)$.

Then, the first Luchko function is

$$\mathbb{F}(G,\Omega,t) \;=\; \sum_{k=1}^{\infty} G^{*,k}(t)\,\Omega^{k-1}. \tag{41}$$

An examples of function (41) is the following [84]:

Example 1. *If*

$$M(t) \;=\; h_{\alpha}(t) \;=\; \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad M^{*,j}(t) \;=\; h_{j\alpha}(t), \tag{42}$$

then

$$\mathbb{F}(M,\Omega,t) \;=\; t^{\alpha-1}\,E_{\alpha,\alpha}(\Omega\,t^{\alpha}), \tag{43}$$

where

$$E_{\alpha,\beta}(z) \;:=\; \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} \tag{44}$$

is the two-parameter Mittag–Leffler function [150] with $\alpha > 0$ and $\beta \in \mathbb{R}$ (or $\beta \in \mathbb{C}$).

Example 2. *If*

$$M(t) = h_{\alpha,\beta}(t) = h_{\alpha}(t)\,e^{-\beta t}, \quad M^{*,j}(t) = h_{j\alpha,\beta}(t), \tag{45}$$

then

$$\mathbb{F}(M,\Omega,t) \;=\; e^{-\beta t}\,t^{\alpha-1}\,E_{\alpha,\alpha}(\Omega\,t^{\alpha}). \tag{46}$$

Example 3. *If*

$$M(t) = h_{1-\beta+\alpha}(t) \;+\; h_{1-\beta}(t), \tag{47}$$

where $0 < \alpha < \beta < 1$, then

$$\mathbb{F}(M,\Omega,t) \;=\; (\Omega t)^{-1} E_{(1-\beta,1-\beta+\alpha),0}(\Omega t^{1-\beta}, \Omega t^{1-\beta+\alpha}), \tag{48}$$

where

$$E_{(\alpha_1,\ldots,\alpha_m),\beta}(z_1,\ldots,z_m) \;:=\; \sum_{j=0}^{\infty} \sum_{l_1+\cdots+l_m=j} \frac{j!}{\prod_{i=1}^{m} l_i!} \frac{\prod_{i=1}^{m} z_i^{l_i}}{\Gamma(\sum_{i=1}^{m} \alpha_i l_i+\beta)} \tag{49}$$

is a multinomial Mittag–Leffler function [42,154].

The second Luchko function has been defined in [84].

Definition 5. *Let $(M(t),K(t)) \in \mathcal{L}$.*

Then, the second Luchko function is

$$\mathbb{L}(G,\Omega,t) \;:=\; \int_0^t ds\, K(t-s)\,\mathbb{F}(G,\Omega,s) \;=\; I^t_{(K)}[s]\,\mathbb{F}(G,\Omega,s), \tag{50}$$

where $\mathbb{F}(G,\Omega,t)$ is the first Luchko function.

The solution of Equation (37) is described by the following Luchko theorem.

Theorem 4. *Let $Y(t) \in C^1_{-1}(0,\infty)$ and the pair $(M(t), K(t))$ belong to the Luchko set $\mathcal{L}$. Then, the initial value problem*

$$D^{t,*}_{(K)}[\tau]\, Y(\tau) \;=\; \mathcal{Q}\, Y(t), \quad Y(0) \;=\; Y_0 \tag{51}$$

has the unique solution

$$Y(t) \;=\; \mathcal{U}^{(M)}_t Y_0, \tag{52}$$

where

$$\mathcal{U}^{(M)}_t \;=\; \mathbb{L}(M, \mathcal{Q}, t) \;=\; \int_0^t d\tau\, K(t-\tau)\, \mathbb{F}(M, \mathcal{Q}, \tau) \;=\; I^t_{(K)}[\tau]\, \mathbb{F}(M, \mathcal{Q}, \tau). \tag{53}$$

This theorem is proved similar to Theorem 5.1 in [84], p. 366.

Remark 5. *Note that the dynamical map $\mathcal{U}^{(M)}_t$ is independent on $K(t)$ since the Sonin condition*

$$(K * M)(t) \;=\; \{1\} \tag{54}$$

holds for all $t \in (0,\infty)$. Using Equation (54), we have

$$\mathcal{U}^{(M)}_t \;=\; (K * \mathbb{F})(t) \;=\; L_I \;+\; \left(\{1\} * \sum_{k=1}^{\infty} \left(M^{*,k}\, \mathcal{Q}^k \right) \right)(t), \tag{55}$$

where $\mathcal{Q}^0 = L_I$ is the unit function ($L_I\, Y = Y$ for all Y).

Therefore, we have

$$\mathcal{U}^{(M)}_t \;=\; \int_0^t K(t-\tau)\, \mathbb{F}(M, \mathcal{Q}, \tau) d\tau \;=\; L_I \;+\; \int_0^t \Big(\sum_{j=1}^{\infty} M^{*,j}(\tau)\, \mathcal{Q}^j \Big) d\tau. \tag{56}$$

In the next section, we describe some properties and examples of the maps $\mathcal{U}^{(M)}_t$ of economic processes with a general form of memory.

4. Non-Standard Properties and Examples of Dynamical Maps with Memory

4.1. Properties of Economic Dynamical Maps with Memory

The concept of dynamical maps provides a suitable mathematical representation for evolutions of processes and dynamics of systems. In quantum physics, dynamical maps are powerful generalizations of unitary evolutions of closed Hamiltonian systems. The evolution of open and non-Hamiltonian systems are described by dynamical maps that form the one-parameter Markovian semigroup (for examples, see [155,156]). In ordinary differential equations, dynamical maps are described by one-parameter diffeomorphism groups and phase flows (for examples, see Sections 1.3 and 1.4 in [157,158]).

Theorem 4 and Equation (53) mean that dynamical maps $\mathcal{U}^{(M)}_t$ are described by the second Luchko function. In the framework of the general fractional dynamics (GFDynamics) proposed in [109] and its application to economic processes, dynamical maps should describe properties of the economic processes for all forms of memory functions.

Therefore, let us give some properties of the dynamical maps $\mathcal{U}^{(M)}_t$ for general memory.

In our case, the dynamical map $\mathcal{U}^{(M)}_t$, $t \geqslant 0$ is the operator

$$\mathcal{U}^{(M)}_t : \quad Y(0) \;\to\; Y(t), \tag{57}$$

which describes the GF economic dynamics with general memory, where $\mathcal{Q}$ can be considered as a generator of the one-parameter groupoid of maps $\mathcal{U}_t^{(M)}$ on the space of functions $Y(t)$:

$$D^{t,*}_{(K)}[\tau]\,\mathcal{U}_\tau^{(M)} \;=\; \mathcal{Q}\,\mathcal{U}_t^{(M)}. \tag{58}$$

The set $\{\,\mathcal{U}_t^{(M)}\,|\,t \geqslant 0\}$ of these map does not form a one-parameter group. The set of these maps form a one-parameter dynamical groupoid [110,155,159]. The following property characterizes this set:

$$\lim_{t\to 0+} \mathcal{U}_t^{(M)} \;=\; L_I, \tag{59}$$

where L_I is an identity ($L_I\,Y \;=\; Y$).

For $M(t) = h_\alpha(t)$ and $K(t) = h_{1-\alpha}(t)$, with $0 < \alpha \leq 1$, the map is

$$\mathcal{U}_s^{(M)} \;=\; E_\alpha[s^\alpha\,\mathcal{Q}], \tag{60}$$

where $E_\alpha[z]$ is the Mittag–Leffler function [150]. For $\alpha \;=\; 1$, we have

$$\mathcal{U}_s^{(M)} \;=\; E_1[s\,\mathcal{Q}] \;=\; \exp\{s\,\mathcal{Q}\} \;=\; \mathcal{U}_s. \tag{61}$$

The map $\mathcal{U}_s$ forms a semigroup such that

$$\mathcal{U}_t\,\mathcal{U}_s \;=\; \mathcal{U}_{t+s}, \quad (t,s > 0), \quad \mathcal{U}_0 \;=\; L_I, \tag{62}$$

since

$$\exp\{t\,\mathcal{Q}\}\;\exp\{s\,\mathcal{Q}\} \;=\; \exp\{(t+s)\,\mathcal{Q}\}.$$

For power-law memory, which is described by the kernels $M(t) = h_\alpha(t)$ and $K(t) = h_{1-\alpha}(t)$ with $0 < \alpha \leq 1$, the semigroup property [160–162] is violated:

$$E_\alpha[t^\alpha\,\mathcal{Q}]\,E_\alpha[s^\alpha\,\mathcal{Q}] \;\neq\; E_\alpha[(t+s)^\alpha\,\mathcal{Q}]. \tag{63}$$

Therefore, the semigroup property is not satisfied in economic dynamics with memory:

$$\mathcal{U}_t^{(h_\alpha)}\,\mathcal{U}_s^{(h_\alpha)} \;\neq\; \mathcal{U}_t^{(v)}, \quad (t,s > 0) \tag{64}$$

for $\alpha \in (0,1)$.

The same inequality is satisfied for dynamical maps with a general form of memory. As a result, we have that the inequality

$$\mathcal{U}_t^{(M)}\,\mathcal{U}_s^{(M)} \;\neq\; \mathcal{U}_t^{(M)}, \quad (t,s > 0) \tag{65}$$

holds for all types of memory functions $M(t)$ of the kernel pairs $(M(t),K(t)) \;\in\; \mathcal{L}$.

As a result, the dynamical maps $\mathcal{U}_t^{(M)}$ with a general form of memory cannot form a semigroup. The set of these dynamical maps with general memory form a one-parameter dynamical groupoid only. Therefore, one can state that inequality (65) is one of the characteristic properties of dynamical maps with memory. The maps $\mathcal{U}_t^{(M)}$ describe economic processes of natural growth with a general form of memory. This memory means that the present value of $Y(t) \;=\; \mathcal{U}_t^{(M)} Y_0$ depends on all past values of $Y(\tau)$ for $\tau \;\in\; [0,t]$.

4.2. Examples of Economic Dynamical Maps with Memory

Let us give examples of the solutions for GFD equations, which are derived by the Luchko operational calculus [84], and dynamical maps.

Example 4. *For the kernel pairs,*

$$M(t) \,=\, h_{\alpha}(t) \,=\, \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad K(t) \,=\, h_{1-\alpha}(t), \tag{66}$$

where $t \in (0,\infty)0$, $0 < \alpha < 1$. *Then,*

$$\mathcal{U}_t^{(M)} \,=\, L_I \,+\, \{1\} * \sum_{j=1}^{\infty} h_{j\alpha}(t)\, \mathcal{Q}^j \,=\, L_I + \sum_{j=1}^{\infty} h_{j\alpha+1}(t)\, \mathcal{Q}^j \,=\, E_{\alpha}(t^{\alpha}\,\mathcal{Q}), \tag{67}$$

where the Mittag–Leffler function $E_{\alpha}(z) = E_{\alpha,1}(z)$ *[150]. Equation* (51) *has the solution*

$$Y(t) \,=\, E_{\alpha}(t^{\alpha}\,\mathcal{Q})\, Y_0. \tag{68}$$

Example 5. *For*

$$M(t) \,=\, h_{\alpha,\beta}(t) \,=\, \frac{t^{\alpha-1}}{\Gamma(\alpha)}\, e^{-\beta t}, \quad K(t) \,=\, h_{1-\alpha,\beta}(t) \,+\, \frac{\beta^{\alpha}}{\Gamma(1-\alpha)}\,\gamma(1-\alpha, \beta t), \tag{69}$$

where $t \in (0,\infty)0$, $0 < \alpha < 1)$, $\beta > 0$, *and* $\gamma(\beta, t)$ *is the incomplete gamma function:*

$$\gamma(\beta, t) \,=\, \int_0^t \tau^{\beta-1} e^{-\tau}\, d\tau. \tag{70}$$

The GFD is

$$D^{t,*}_{(K)}[s]\, Y(s) \,=\, \int_0^t \left(e^{-\beta s} h_{1-\alpha}(s) \,+\, \gamma(1-\alpha, \beta\, s)\, \frac{\beta^{\alpha}}{\Gamma(1-\alpha)} \right) Y^{(1)}(t-s)\, ds. \tag{71}$$

The dynamical map $\mathcal{U}_t^{(M)}$ *is written as*

$$\mathcal{U}_t^{(M)} \,=\, e^{-\beta t}\, E_{\alpha}(t^{\alpha}\,\mathcal{Q}) \,+\, \beta \int_0^t d\tau\, e^{-\beta \tau} E_{\alpha}(\tau^{\alpha}\,\mathcal{Q}), \tag{72}$$

where we use

$$(h_{\alpha,\beta} \,*\, h_{\gamma,\beta}) \,=\, h_{\alpha+\gamma,\beta}.$$

For (69)*, Equation* (51) *has the solution*

$$Y(t) \,=\, \left(E_{\alpha}(t^{\alpha}\,\mathcal{Q})\, e^{-\beta t} \,+\, \beta \int_0^t e^{-\beta \tau} E_{\alpha}(\tau^{\alpha}\,\mathcal{Q})\, d\tau \right) Y_0. \tag{73}$$

Example 6. *For the kernel pairs*

$$M(t) \,=\, h_{1+\alpha-\beta}(t) \,+\, h_{1-\beta}(t), \quad K(t) \,=\, t^{\beta-1}\, E_{\alpha,\beta}(-t^{\alpha}), \tag{74}$$

where $0 < \alpha < \beta < 1$. *In this case,*

$$D^{t,*}_{(K)}[s]\, Y(s) \,=\, \int_0^t ds\, s^{\beta-1}\, E_{\alpha,\beta}(-s^{\alpha})\, Y^{(1)}(t-s). \tag{75}$$

Then, the natural growth is described by the map

$$\mathcal{U}_t^{(M)} \,=\, L_I + \{1\} * \sum_{j=1}^{\infty} \mathcal{Q}^j \sum_{i=0}^{j} \binom{j}{i} h_{i\alpha+j(1-\beta)}(t) \,=\,$$

$$E_{(1-\beta, 1-\beta+\alpha),1}(t^{1-\beta}\,\mathcal{Q}, t^{1-\beta+\alpha}\,\mathcal{Q}), \tag{76}$$

where $E_{(1-\beta,1-\beta+\alpha),1}$ is a binomial Mittag–Leffler function [154]. Equation (51) has the solution

$$Y(t) \;=\; E_{(1-\beta,1-\beta+\alpha),1}(t^{1-\beta}\Omega, t^{1-\beta+\alpha}\,\Omega)\,Y_0. \tag{77}$$

These examples of equations and their solutions describe general fractional economic dynamics with memory.

5. Example of Natural Growth with Power-Law Memory

Let us consider the case of the kernel pair $M(t) = h_\alpha(t)$ and $K(t) = h_{1-\alpha}(t)$ that describes the power-law memory. Substituting expressions (2) and (3) into Equation (23), we obtain

$$D^{t,*}_{\{h_{1-\alpha}\}}[\tau]\,Y(\tau) \;-\; \frac{m(P-a)}{v}\,Y(t) \;=\; -\,\frac{mb}{v}. \tag{78}$$

Equation (78) is the fractional differential equation with the Caputo fractional derivative of the order $\alpha > 0$. For non-integer values of $\alpha > 0$, this equation describes natural growth with the power-law memory. For integer values of α, Equation (78) describes the process without memory. For $\alpha = 1$, Equation (78) gives Equation (6) of the standard model of natural growth without memory.

Equation (78) can be generalized from $\alpha \in (0,1)$ to the case $\alpha \in (0,\infty)$ in the form of the fractional differential equation

$$D^{t,*}_{\{h_{n-\alpha}\}}[\tau]\,Y(\tau) \;-\; \Omega Y(t) \;=\; f(t) \tag{79}$$

with initial conditions $Y^{(k)}(0) = c_k$, $(k = 0,\dots,n-1)$, where $n-1 < \alpha \leq n$, and $f(t)$ is a real-valued function. For non-integer values of $\alpha > 0$, Equation (79) describes processes with power-law memory. For $\alpha \in (0,1)$, Equation (78) can be represented in the form (79) with

$$\Omega \;=\; \frac{m(P-a)}{v}, \quad f(t) \;=\; -\,\frac{mb}{v}, \tag{80}$$

where $Y(t)$ is the value of output, $P(t) = P$ is the price, m is the share of profit, a is the marginal costs, b is the independent costs, and v^{-1} is the marginal productivity of capital.

The solution of Equation (78) is

$$Y(t) \;=\; \sum_{k=0}^{n-1} Y^{(k)}(0) t^k E_{\alpha,k+1}\left[\frac{m(P-a)}{v}t^\alpha\right] -$$

$$\frac{b}{(P-a)}\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left[\frac{m(P-a)}{v}(t-\tau)^\alpha\right]d\tau, \tag{81}$$

where $n-1 < \alpha \leq n$, $E_{\alpha,\beta}[z]$ is the two-parameter Mittag–Leffler function

$$E_{\alpha,\beta}[z] \;=\; \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \tag{82}$$

where $z, \beta \in \mathbb{C}$ and $\alpha > 0$. Note that $E_{1,1}[z] \;=\; \exp(z)$.

Using Equation 4.4.4 of [150], p. 51, to calculate the integral in Equation (81), solution (81) takes the form

$$Y(t) \;=\; \frac{b}{(P-a)}\left(1 - E_{\alpha,1}\left[v^{-1}\,m\,(P-a)\,t^\alpha\right]\right) +$$

$$\sum_{k=0}^{n-1} Y^{(k)}(0)\,t^k\,E_{\alpha,k+1}\left[v^{-1}\,m\,(P-a)\,t^\alpha\right], \tag{83}$$

where $n-1<\alpha\le n$, and $Y^{(k)}(0)$ are the values of the derivatives of the function $Y(t)$ at $t=0$. Solution (83) with non-integer values of $\alpha>0$ describes economic dynamics with power-law memory.

Natural growth models with the power-law memory, which were proposed in [77,143,144] with $\alpha\in(0,1)$, are a special case of the proposed models with a general form of memory.

6. Examples of Growth with a General Form of Memory

Let us consider a dynamic intersectoral model (DISM) with general memory for two sectors. The national income $Y(t)=(Y_k(t))$ is connected with the gross product $X(t)=(X_k(t))$ by the equation $Y(t)=(E-A)\,X(t)$. Equation of a DISM with general memory is

$$D^{t,*}_{(K)}[\tau]\,Y(\tau)\;=\;\mathfrak{Q}\,Y(t), \tag{84}$$

where

$$\mathfrak{Q}\;=\;(E\,-\,A)\,B^{-1}. \tag{85}$$

Here, E is a unit diagonal matrix of nth order. The matrix $A=(a_{ij})$ describes the direct material costs (DMC), where a_{ij} describes the DMC of the ith sector ($i=1,\ldots,n$) for the production of a unit of output for the jth sector ($j=1,\ldots,n$). The matrix $B=(b_{ij})$ describes the incremental capital intensity of production, where b_{ij} describes the expenses of production for the ith sector to increase the production of the jth sector. Note that dynamic intersectoral models with power-law memory have been described in [77,145,146].

For two sectoral models with values $Y_1(t)$ and $Y_2(t)$, one can consider the economic process with general memory that is described by the equations

$$D^{t,*}_{(K)}[\tau]\,Y_1(\tau)\;=\;(\theta-\zeta)\,Y_1(t)\;+\;w^{-1}\,Y_2(t), \tag{86}$$

$$D^{t,*}_{(K)}[\tau]\,Y_2(\tau)\;=\;-\,w\,\omega^2\,Y_1(t)\;-\;(\theta+\zeta)\,Y_2(t), \tag{87}$$

where the pair $(M(t),K(t))\in\mathcal{L}$, and $D^{t,*}_{(K)}$ is the GFD. The parameters in Equations (86) and (87) are chosen for the convenience of obtaining a solution. There parameters are expressed through the coefficients a_{ij} and b_{ij} by the equation

$$\theta\,-\,\zeta\;=\;Q_{11}\;=\;(\det B)^{-1}\Big((1\,-\,a_{11})\,b_{22}\;+\;a_{12}\,b_{21}\Big), \tag{88}$$

$$\theta\,+\,\zeta\;=\;-\,Q_{22}\;=\;-\,(\det B)^{-1}\Big((1\,-\,a_{22})\,b_{11}\;+\;a_{12}\,b_{21}\Big), \tag{89}$$

$$w^{-1}\;=\;Q_{12}\;=\;(\det B)^{-1}\Big(-(1\,-\,a_{11})\,b_{12}\;-\;a_{12}\,b_{11}\Big), \tag{90}$$

$$w\,\omega^2\;=\;-\,Q_{12}\;=\;-\,(\det B)^{-1}\Big(-(1\,-\,a_{22})\,b_{21}\;-\;a_{21}\,b_{22}\Big). \tag{91}$$

Equations (86) and (87) describe the exactly solvable model of economic processes with general memory. Let us derive the exact solutions of Equations (86) and (87).

Equations (86) and (87) can be written as

$$D^{t,*}_{(K)}[s]\,Y(s)\;=\;\mathfrak{Q}\,Y(t), \tag{92}$$

where

$$Y(t)\;=\;\begin{pmatrix} Y_1(t)\\ Y_2(t)\end{pmatrix},\quad \mathfrak{Q}\;=\;\begin{pmatrix}\theta-\zeta & w^{-1}\\ -w\,\omega^2 & -\theta-\zeta\end{pmatrix}. \tag{93}$$

The following corollary directly follows from Theorem 4.

Corollary 1. *Let a kernel pair $M(t), K(t) \in C_{-1,0}(0,\infty)$ belong to the Luchko set $\mathcal{L}$. Then, the initial value problem*

$$D^{t,*}_{(K)}[s]\, Y(s) \;=\; \mathcal{Q}\, Y(t), \quad Y(t=0) \;=\; Y(0), \tag{94}$$

where $Y(t)$ and $\mathcal{Q}$ are defined by (93), has the solution

$$Y(t) \;=\; \mathcal{U}^{(M)}_t\, Y(0) \tag{95}$$

with the dynamical map

$$\mathcal{U}^{(M)}_t \;=\; \mathbb{L}(M, \mathcal{Q}, t) \;=\; \sum_{j=1}^{\infty} \Big(K * M^{*,j}\Big)(t)\, \mathcal{Q}^{j-1}, \tag{96}$$

where $\mathbb{L}(M, \mathcal{Q}, t)$ is the second Luchko function and has the matrix argument $\mathcal{Q}$.

An exact solution of Equation (94) and the dynamical map (96) can be written through the special functions

$$\mathrm{Sh}_{(M)}[\zeta, \chi, t] \;:=\; \frac{1}{2}\Big(\mathbb{L}(M, (-\zeta+\chi), t) \;-\; \mathbb{L}(M, (-\zeta-\chi), t)\Big), \tag{97}$$

$$\mathrm{Ch}_{(M)}[\zeta, \chi, t] \;:=\; \frac{1}{2}\Big(\mathbb{L}(M, (-\zeta+\chi), t) \;+\; \mathbb{L}(M, (-\zeta-\chi), t)\Big), \tag{98}$$

where $\mathbb{L}(M, (-\zeta \pm \chi), t)$ is the first Luchko function (see Definition 4) and depends on the memory function $M(t)$. The functions $\mathrm{Sh}_{(M)}[\zeta, \chi, t]$ and $\mathrm{Ch}_{(M)}[\zeta, \chi, t]$ can be interpreted as GF analogs of the hyperbolic sine "sinh" and hyperbolic cosine "cosh" since

$$\lim_{\alpha \to 1-} \mathrm{Sh}_{(h_\alpha)}[\zeta, \chi, t] \;=\; e^{-\zeta t} \sinh(\chi t), \quad \lim_{\alpha \to 1-} \mathrm{Ch}_{(h_\alpha)}[\zeta, \chi, t] \;=\; e^{-\zeta t} \cosh(\chi t). \tag{99}$$

Let us prove the following theorem.

Theorem 5. *Let $M(t), K(t) \in \mathcal{L}$.*

Then, the equation

$$D^{t,*}_{(K)}[s]\, Y(s) \;=\; \mathcal{Q}\, Y(t), \quad Y(t=0) \;=\; Y(0), \tag{100}$$

where $Y(t)$ and $\mathcal{Q}$ are defined by (93), has solution (95) with the dynamical map

$$\mathcal{U}^{(M)}_t \;=\; \begin{pmatrix} Ch_{(M)}[\zeta, \chi, t] + (\theta/\chi)\, Sh_{(M)}[\zeta, \chi, t] & (w^{-1}\chi)\, Sh_{(M)}[\zeta, \chi, t] \\ -(w\, \omega^2/\chi)\, Sh_{(M)}[\zeta, \chi, t] & Ch_{(M)}[\zeta, \chi, t] - (\theta/\chi)\, Sh_{(M)}[\zeta, \chi, t] \end{pmatrix}, \tag{101}$$

where $Sh_{(M)}[\zeta, \chi, t]$ and $Ch_{(M)}[\zeta, \chi, t]$ are defined by Equations (97) and (98), respectively.

Equations (86) and (87) have the solutions

$$Y_1(t) \;=\; \Big(Ch_{(M)}[\zeta, \chi, t] \;+\; \frac{\theta}{\chi}\, Sh_{(M)}[\zeta, \chi, t]\Big) Y_1(0) \;+\; \frac{1}{w\chi}\, Sh_{(M)}[\zeta, \chi, t] Y_2(0), \tag{102}$$

$$Y_2(t) \;=\; -\frac{w\, \omega^2}{\chi}\, Sh_{(M)}[\zeta, \chi, t] Y_1(0) + \Big(Ch_{(M)}[\zeta, \chi, t] - \frac{\theta}{\chi}\, Sh_{(M)}[\zeta, \chi, t]\Big) Y_1(0), \tag{103}$$

where $Sh_{(M)}[\zeta, \chi, t]$ and $Ch_{(M)}[\zeta, \chi, t]$ are defined by (97) and (98), respectively, and χ is a complex parameter such that $\chi^2 = \theta^2 - \omega^2$.

Proof. To obtain an exact expression of the solution, we represent the matrix $\mathcal{Q}$ in the form

$$\mathcal{Q} \,=\, A\,B\,A^{-1}, \tag{104}$$

where

$$B \,:=\, \begin{pmatrix} -\zeta-\chi & 0 \\ 0 & -\zeta+\chi \end{pmatrix}, \tag{105}$$

$$A \,:=\, \begin{pmatrix} -(\theta-\chi)\,c_{-}^{-1} & -(\theta+\chi)\,c_{+}^{-1} \\ m\,\omega^2\,c_{-}^{-1} & m\,\omega^2\,c_{+}^{-1} \end{pmatrix}, \tag{106}$$

and

$$c_{\pm} \,=\, \sqrt{|\theta\pm\chi|^2+(w\,\omega^2)^2}, \quad \chi \,=\, \sqrt{\theta^2-\omega^2}. \tag{107}$$

Using (104), the map $\mathcal{U}_t^{(M)}$ is given as

$$\mathcal{U}_t^{(M)} \,=\, \mathbb{L}(M,\mathcal{Q},t) \,=\, \sum_{k=1}^{\infty}\Big(K * M^{*,k}\Big)(t)\,\Big(A\,B\,A^{-1}\Big)^{k-1} \,=$$

$$\sum_{k=1}^{\infty}\Big(K * M^{*,k}\Big)(t)\,A\,B^{k-1}\,A^{-1} \,=\, A\,\mathbb{L}(M,B,t)\,A^{-1}. \tag{108}$$

Therefore, we have

$$\mathcal{U}_t^{(M)} \,=\, A\,\mathbb{L}(M,B,t)\,A^{-1}. \tag{109}$$

Substitution of (105) and (106) into Equation (109) gives

$$\mathcal{U}_t^{(M)} \,=\, \begin{pmatrix} \mathrm{Ch}_{(M)}[\zeta,\chi,t]+(\theta/\chi)\,\mathrm{Sh}_{(M)}[\zeta,\chi,t] & (w^{-1}\,\chi)\,\mathrm{Sh}_{(M)}[\zeta,\chi,t] \\ -(w\,\omega^2/\chi)\,\mathrm{Sh}_{(M)}[\zeta,\chi,t] & \mathrm{Ch}_{(M)}[\zeta,\chi,t]-(\theta/\chi)\,\mathrm{Sh}_{(M)}[\zeta,\chi,t] \end{pmatrix}, \tag{110}$$

where we use the functions (97) and (98).
□

Remark 6. *For $M(t)=h_\alpha(t)$, we have*

$$\mathcal{U}_t^{(h_\alpha)} \,=\, A\,E_\alpha[t^\alpha B]\,A^{-1}. \tag{111}$$

For $\alpha=1$, map (111) is given by the standard expression

$$\mathcal{U}_t^{(h_\alpha)} \,=\, \mathcal{U}_t \,=\, A\,e^{t\,B}\,A^{-1}. \tag{112}$$

Theorem 5 describes economic processes with a general form of memory.

Let us describe some examples that describe processes with various forms of memory.

Example 7. *For $M(t) = h_\alpha(t)$, the solution of Equations (86) and (87) is described by Equations (102) and (103), where*

$$Sh_{(M)}[\zeta,\chi,t] \,=\, S_\alpha[\zeta,\chi,t] \,:=\, \frac{1}{2}\Big(E_{\alpha,1}[(-\zeta+\chi)t^\alpha]-E_{\alpha,1}[(-\zeta-\chi)t^\alpha]\Big), \tag{113}$$

$$Ch_{(M)}[\zeta,\chi,t] \,=\, C_\alpha[\zeta,\chi,t] \,:=\, \frac{1}{2}\Big(E_{\alpha,1}((-\zeta+\chi)t^\alpha)\,+\,E_{\alpha,1}((-\zeta-\chi)t^\alpha)\Big). \tag{114}$$

For the case with $\alpha=1$, the equations describe economic processes without memory ($\alpha=1$), since $E_1[z]=\exp(z)$. If $\omega^2 > \theta^2$, then

$$\chi \,=\, i\,\Omega \,=\, i\,\sqrt{\omega^2-\theta^2} \tag{115}$$

and

$$\sinh(i\,\Omega t) \,=\, i\,\sin(\Omega t), \quad \cosh(i\,\Omega t) \,=\, \cos(\Omega t). \tag{116}$$

Example 8. *For*

$$M(t) \,=\, h_\alpha(t)\, e^{-\beta t} \,=\, \frac{t^{\alpha-1}}{\Gamma(\alpha)}\, e^{-\beta t}, \tag{117}$$

where $t > 0$, $\alpha \in (0,1)$, and $\beta > 0$, the solution of Equations (86) and (87) is described by Equations (102) and (103), where

$$Sh_{(M)}[\zeta,\chi,t] \,= e^{-\beta t}\, S_\alpha[\zeta,\chi,t] + \beta \int_0^t d\tau\, e^{-\beta\tau} S_\alpha[\beta,\chi,\tau], \tag{118}$$

$$Ch_{(M)}[\zeta,\chi,t] \,=\, e^{-\beta t}\, C_\alpha[\zeta,\chi,t] \,+\, \beta \int_0^t d\tau\, e^{-\beta\tau} C_\alpha[\beta,\chi,\tau], \tag{119}$$

where $S_\alpha[\zeta,\chi,t]$ and $C_\alpha[\zeta,\chi,t]$ are defined by (113) and (114), respectively.

Example 9. *For*

$$M(t) \,=\, h_{1+\alpha-\beta}(t) \,+\, h_{1-\beta}(t), \tag{120}$$

where $0 < \alpha < \beta < 1$, the solution of Equations (86) and (87) is described by Equations (102) and (103), where

$$\begin{aligned} Sh_{(M)}[\zeta,\chi,t] \,=\, \frac{1}{2}\Big(& E_{(1-\beta,1-\beta+\alpha),0}((-\zeta+\chi)t^{1-\beta},(-\zeta+\chi)t^{1-\beta+\alpha}) - \\ & E_{(1-\beta,1-\beta+\alpha),0}((-\zeta-\chi)t^{1-\beta},(-\zeta-\chi)t^{1-\beta+\alpha})\Big), \end{aligned} \tag{121}$$

$$\begin{aligned} Ch_{(M)}[\zeta,\chi,t] \,=\, \frac{1}{2}\Big(& E_{(1-\beta,1-\beta+\alpha),0}((-\zeta+\chi)t^{1-\beta},(-\zeta+\chi)t^{1-\beta+\alpha}) + \\ & E_{(1-\beta,1-\beta+\alpha),0}((-\zeta-\chi)t^{1-\beta},(-\zeta-\chi)t^{1-\beta+\alpha})\Big), \end{aligned} \tag{122}$$

where $E_{(1-\beta,1-\beta+\alpha),0}$ is defined in (49).

7. Conclusions

In this paper, we proposed to use general fractional calculus (GFC) in the Luchko form to describe economic processes with a general form of memory. Before this article, no one had ever proposed a self-consistent description of economic processes for such a wide class of memory functions. The power-type of memory, which is usually used, is a special case of general memory. The self-consistency of the description is ensured by the fulfillment of the Sonin and Luchko conditions and by the fundamental theorems of GFC. General memory is described by the pair $(M(t),\, K(t))$ of memory functions for which these conditions are satisfied. The existence of memory (nonlocality in time) in the process means that the behavior of a value $Y(t)$ depends on the history of previous changes of $Y(s)$ with $s < t$. The pairs of memory functions $(M(t),\, K(t))$ describe the measure of influence of past states on the current state.

Using general fractional calculus, we proposed simple economic models of natural growth with general memory. In these models, nonlocality in time is described by a pair of operator kernels: namely, the kernel $M(t)$ of a general fractional integral and the kernel $K(t)$ of a general fractional derivative. These kernels should satisfy the Sonin condition, which allows us to formulate self-consistent mathematical models of processes with a general form of memory. In this paper, general fractional differential equations that describe an exactly solvable model of economic natural growth with general memory are proposed. Exact solutions for the equations of exactly solvable models of natural growth with general memory are derived using Luchko operational calculus [84,89,96]. Various examples of solutions to these equations for different types of memory functions are offered in this work.

The suggested approach to economic dynamics with memory allows researchers to consider economic processes with a general form of a memory function apart from models previously described in [76,77] for power-law type memory. Using general fractional calculus, all models and concepts can be generalized from power-law type memory to a general form of memory.

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