Computing Lyapunov exponents of switching systems

Nicola Guglielmi* and Vladimir Protasov†

*DISIM, Università dell'Aquila and Gran Sasso Science Institute I-67100 L'Aquila, Italy
†Department of Mechanics and Mathematics, Moscow State University, Vorobyovy Gory, Moscow, Russia

Abstract. We discuss a new approach for constructing polytope Lyapunov functions for continuous-time linear switching systems. The method we propose allows to decide the uniform stability of a switching system and to compute the Lyapunov exponent with an arbitrary precision. The method relies on the discretization of the system and provides - for any given discretization stepsize - a lower and an upper bound for the Lyapunov exponent. The efficiency of the new method is illustrated by numerical examples. For a more extensive discussion we remand the reader to [8].

Keywords: Linear switching systems, Lyapunov exponent, polytope Lyapunov funcions, joint spectral radius, iterative method.
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PROBLEM AND BACKGROUND

We consider the following class of linear switching system (LSS):

$$\begin{cases} \dot{x}(t) \;=\; A(t)\,x(t)\,; \qquad A(t) \in \mathscr{A}\,,\; t \geq 0 \\ x(0) \;=\; x_0 \in \mathbf{R}^d. \end{cases} \tag{1}$$

Here $A(t) \in \mathbf{R}^{d,d}$ is assumed to be a summable function that takes values on a given compact set of matrices $\mathscr{A}$. The *upper Lyapunov exponent* $\sigma(\mathscr{A})$ is the infimum of numbers α such that $\|x(t)\| \leq C\mathrm{e}^{\alpha t}$ for every trajectory of (1). The system is said uniformly stable if $\|x(t)\| \to 0$ as $t \to +\infty$ for every trajectory of (1). If $\sigma(\mathscr{A}) < 0$, then the system is stable, and, conversely, the stability implies that $\sigma(\mathscr{A}) \leq 0$. There is a wide bibliography concerning the computation of $\sigma(\mathscr{A})$ and several approaches based on the construction of Lyapunov functions (see e.g. [11, 2, 3]) have been proposed in the literature. Here we make use of a polytope based approach which provides an arbitrarily accurate approximation of $\sigma(\mathscr{A})$ when $\mathscr{A}$ is either finite or essentially finite (i.e. there exists a finite subset $\mathscr{A}'$ of $\mathscr{A}$ such that $\sigma(\mathscr{A}') = \sigma(\mathscr{A})$, as is the case - for example - if $\mathscr{A}$ is the convex hull of $\mathscr{A}'$).

Optimal norms

In order to deal with the upper Lyapunov exponent we make use of the following key instrument.

Definition 1 *A norm $\|\cdot\|$ is called extremal for a set $\mathscr{A}$ if for every trajectory of (1) we have $\|x(t)\| \leq \mathrm{e}^{\sigma(\mathscr{A})t}\|x(0)\|\,,\; t \geq 0$. An extremal norm is called invariant if for every $x_0 \in \mathbf{R}^d$ there exists a trajectory $\bar{x}(t)$ with $\bar{x}(0) = x_0$ such that $\|x(t)\| = \mathrm{e}^{\sigma(\mathscr{A})t}\,\|x_0\|\,,\; t \geq 0$.*

Note that for an extremal norm the function $\mathrm{e}^{-\sigma(\mathscr{A})t}\,\|x(t)\|$ is non-increasing in t on every trajectory. For an invariant norm this function is identically constant on some trajectory, and for every point $x_0 \in \mathbf{R}^d$ there is such a trajectory starting in it. In particular, for $\sigma(\mathscr{A}) = 0$ we have the following result.

Corollary 1 *If $\sigma(\mathscr{A}) = 0$ a norm is extremal for $\mathscr{A}$ if and only if it is non-increasing in t on every trajectory of (1). An extremal norm is invariant if and only if for any $x_0 \in \mathbf{R}^d$ there exists a trajectory $\bar{x}(t)$ with $\bar{x}(0) = x_0$, on which this norm is identically constant.*

If we take a unit ball B of that norm, we see that a norm is extremal if and only if every trajectory starting on the unit sphere ∂B never leaves the ball B. This norm is invariant if for each point of the unit sphere there exists a trajectory starting at this point that remains on the sphere. A set of operators $\mathscr{A}$ is called *irreducible* if these operators do not share a nontrivial common invariant subspace. N.Barabanov [1] proved that an irreducible set of operators possesses an invariant norm, which is a very deep and important result in terms of computability of $\sigma(\mathscr{A})$.

METHODOLOGY AND RESULTS

We take into consideration here a finite family $\mathscr{A} = \{A_1, A_2, \ldots, A_m\}$ of matrices and think of the restriction of the function $A(t)$ to the space of piecewise constant functions on the discrete grid $\{t_j = j\Delta t\}_{j\geq 0}$. This transforms the problem of computing the upper Lyapunov exponent into that of computing the joint spectral radius (for an extended analysis see [10]) of the family of matrices $\mathscr{B}_{\Delta t} = \{B_1, \ldots, B_m\} := \{e^{A_1 \Delta t}, \ldots, e^{A_m \Delta t}\}$.

Joint spectral radius

Let $\mathscr{B} = \{B_1, \ldots, B_m\}$ and $\|\cdot\|$ a given norm on $\mathbf{R}^d$ and also denote the corresponding induced $d \times d$-matrix norm defined by $\|B\| = \max\limits_{\|x\|=1} \|Bx\|$. Let $\mathscr{I} = \{1, \ldots, m\}$. Then, for $k = 1, 2, \ldots$, consider

$$\Sigma_k(\mathscr{B}) = \{B_{i_k} \cdots B_{i_1} \ \mid \ i_1, \ldots, i_k \in \mathscr{I}\}$$

of all products of *length* k and the number

$$\hat{\rho}_k(\mathscr{B}) = \max_{P \in \Sigma_k(\mathscr{B})} \|P\|^{1/k}.$$

Definition 2 (joint spectral radius [13]) *The number*

$$\hat{\rho}(\mathscr{B}) = \limsup_{k\to\infty} \hat{\rho}_k(\mathscr{B})$$

is said the joint spectral radius (j.s.r.) of the family $\mathscr{B}$.

It represents the maximal rate of growth of a sequence of products $\{P_k\}_{k>0}$, with $P_k \in \Sigma_k(\mathscr{B})$ so that all sequences vanish asymptotically as $k \to \infty$ if and only if $\hat{\rho}(\mathscr{B}) < 1$. Analogously, let $\rho(\cdot)$ denote the spectral radius of a $d \times d$-matrix. For each positive integer k, consider the number

$$\bar{\rho}_k(\mathscr{B}) = \sup_{P \in \Sigma_k(\mathscr{B})} \rho(P)^{1/k}.$$

Definition 3 (generalized spectral radius (see [5])) *The number*

$$\hat{\rho}(\mathscr{B}) = \limsup_{k\to\infty} \bar{\rho}_k(\mathscr{B})$$

is said to be the generalized spectral radius (g.s.r.) of the family $\mathscr{B}$.

In their paper [5], Daubechies and Lagarias also proved that

$$\bar{\rho}_k(\mathscr{B}) \leq \bar{\rho}(\mathscr{B}) \leq \hat{\rho}(\mathscr{B}) \leq \hat{\rho}_k(\mathscr{B}) \quad \text{for all } k \geq 1. \tag{2}$$

The fundamental equality $\hat{\rho}(\mathscr{B}) = \bar{\rho}(\mathscr{B})$ has been proved later by Berger and Wang [4]. Consequently we can simply denote as $\rho(\mathscr{B})$ the *spectral radius* of $\mathscr{B}$. An important characterization of the spectral radius $\rho(\mathscr{B})$ of a matrix family is the generalization of Gelfand formula. In order to state this characterization, we define the *norm of the family* $\mathscr{B} = \{B_i\}_{i\in\mathscr{I}}$ as $\|\mathscr{B}\| = \hat{\rho}_1(\mathscr{B}) = \max_{i\in\mathscr{I}} \|B_i\|$.

Proposition 1 (see [13, 6]) *The spectral radius of a bounded family* $\mathscr{B}$ *of* $d \times d$*-matrices is characterized by*

$$\rho(\mathscr{B}) = \inf_{\|\cdot\| \in \mathrm{Op}} \|\mathscr{B}\|.$$

where Op *denotes the set of operator norms.*

In order to establish whether the infimum in (3) is a minimum, we give the following definition.

Definition 4 (Extremal norm) *We say that a norm* $\|\cdot\|_*$ *satisfying* $\|\mathscr{B}\|_* = \rho(\mathscr{B})$ *is extremal for the family* $\mathscr{B}$.

The main idea of the method for the j.s.r. computation by a polytope norm is that of finding a *spectrum maximizing product* (s.m.p.), i.e., a product P of matrices from $\mathscr{B}$ of length k for which the value $[\rho(P)]^{1/k}$ is maximal among all products of matrices from $\mathscr{B}$. The algorithm in [7] (see also [9]) follows (for a real set of vectors X we denote by $\mathrm{absco}(X) = \mathrm{co}(\{X, -X\})$). Algorithm 1 includes indeed stopping criteria to detect whether the given product P is actually not an s.m.p.; when the algorithm halts it means that the polytope $\mathscr{P}$ is mapped by the family $\mathring{\mathscr{B}}$ into itself. Here we assume for simplicity that the leading eigenvalue of the s.m.p. is real.

Data: $\mathscr{B} = \{B_1, \ldots, B_m\}$
begin
  Preprocessing: find a product P of length $k \geq 1$ s.t. $\rho(P)^{1/k}$ is maximal among $\bigcup_{\ell \leq \bar{k}} \Sigma_\ell(\mathscr{B})$ with $\bar{k}$ fixed
  Set $R := \rho(P)^{1/k}$ and $\mathring{\mathscr{B}} := R^{-1}\mathscr{B}$
  Compute v_0, real leading eigenvector of P (for an extension to complex eigenpairs see [7])
  Set $V_0 := \{v_0\}$. Set $i = 0$
  while $\text{span}(V_i) \neq \mathbf{R}^d$ **do**
    $V_{i+1} := V_i \cup \mathring{\mathscr{B}} V_i$
    Set $i = i + 1$
  end
  $\mathscr{P}^{(i)} = \text{absco}(V_i)$
  while $\mathring{\mathscr{B}} V_i \not\subseteq \mathscr{P}^{(i)}$ **do**
    Set $i = i + 1$
    Let V_i a system of vertices of $\text{absco}(V_{i-1} \cup \mathring{\mathscr{B}} V_{i-1})$. Set $\mathscr{P}^{(i)} = \text{absco}(V_i)$
  end
  Return $\mathscr{P} := \mathscr{P}^{(i)}$ (extremal polytope)
end

Algorithm 1: Basic polytope algorithm

Bilateral bounds

The key idea relies on the simple equality (for real η) $\sigma\left(\mathscr{A} + \eta\, I\right) \; = \; \sigma\left(\mathscr{A}\right) + \eta$.

We name $\mathscr{P}_{\Delta t}$ the polytope computed by Algorithm 1 and $V_{\Delta t}$ its vertices. An obvious lower bound for $\sigma(\mathscr{A})$ is

$$\sigma_{\Delta t} = \log\left(\rho\left(\mathscr{B}_{\Delta t}\right)\right)/\Delta t$$

that is the upper Lyapunov exponent restricted to functions which are piecewise constant on every mesh interval $[j\Delta t, (j+1)\Delta t)]$ (j nonnegative). In order to obtain an upper bound it is convenient to consider the shifted family

$$\widehat{\mathscr{A}_{\alpha,\Delta t}} \quad := \quad \{\widehat{A}_i\}_{i=1}^m \quad \text{with} \quad \widehat{A}_i = A_i - (\sigma_{\Delta t} + \alpha) I$$

with $\alpha \geq 0$, whose upper Lyapunov exponent is just $\sigma(\mathscr{A}) - \sigma_{\Delta t} - \alpha$. Obviously we can infer that $\sigma(\widehat{\mathscr{A}_{0,\Delta t}}) \geq 0$ but for sufficiently large α we have $\sigma(\widehat{\mathscr{A}_{\alpha,\Delta t}}) \leq 0$. Let $v \in \partial(\mathscr{P}_{\Delta t})$; if the vectorfield $\widehat{A}_i v$ is oriented inside $\mathscr{P}_{\Delta t}$ for all v and i then $\mathscr{P}_{\Delta t}$ is positively invariant and every trajectory is bounded so that $\sigma(\widehat{\mathscr{A}_{\alpha,\Delta t}}) \leq 0$.

Exploiting this property, by minimizing w.r.t. α,

$$\alpha^*_{\Delta t} \longrightarrow \min_{\alpha \geq 0} \mathscr{P}_{\Delta t} \text{ is invariant for } \widehat{\mathscr{A}_{\alpha,\Delta t}}$$

we obtain the upper bound $\sigma(\mathscr{A}) \leq \sigma_{\Delta t} + \alpha^*_{\Delta t} := \gamma_{\Delta t}$. The following main result is proved in [8].

Theorem 1 *For every compact irreducible family $\mathscr{A}$ of matrices, there is a constant C such that*

$$\gamma_{\Delta t} \; - \; \sigma_{\Delta t} \quad \leq \quad C\,\Delta t\,, \qquad \Delta t > 0\,,$$

where $\gamma_{\Delta t}, \sigma_{\Delta t}$ are the bounds computed with the chosen dwell time Δt.

Thus the algorithm localizes the Lyapunov exponent $\sigma(\mathscr{A})$ on the segment $[\,\sigma_{\Delta t}\,,\,\gamma_{\Delta t}\,]$ of length at most $C\,\Delta t$.

Illustrative examples

Example 1. Let $\mathscr{A} = \{A_1, A_2\}$ with $A_1 = \log(B_1)$ and $A_2 = \log(B_2)$, where

$$B_1 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}, \quad B_2 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & 0 \end{pmatrix}.$$

Setting $\Delta t = 1$ we apply Algorithm 1. As a result we obtain $\sigma_{\Delta t} = -0.725149\ldots$ and the polytope norm in Figure 1.

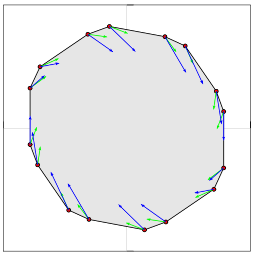

FIGURE 1. In red the vertices of the polytope computed by Algorithm 1; in green the vectors $(A_1 - (\sigma_{\Delta t} + \alpha^*_{\Delta t})I)v$ and in blue the vectors $(A_2 - (\sigma_{\Delta t} + \alpha^*_{\Delta t})I)v$, for all vertices v of $\mathscr{P}_{\Delta t}$ and $\alpha^*_{\Delta t} = 0.433445\ldots$. One of the vectorfields is aligned with one of the sides of the polytope which means that if $\alpha < \alpha^*_{\Delta t}$ then the polytope is not anymore invariant for the shifted family $\widehat{\mathscr{A}}_{\alpha,1}$.

Applying the shifting technique explained in the previous section we obtain the bilateral estimate

$$-0.725149\ldots \le \sigma(\mathscr{A}) \le -0.291704\ldots$$

This estimate allows to state that the system is uniformly stable, that is, for any control law the solution of (1) vanishes.

Example 2. Let $\mathscr{A} = \{A_1, A_2\}$ with

$$A_1 = \begin{pmatrix} -0.0822 & 0.0349 & -0.1182 \\ 0.0953 & -0.0897 & -0.1719 \\ 0.0787 & 0.0223 & -0.2781 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.1391 & 0.1397 & -0.0916 \\ 0.0338 & -0.1769 & -0.0707 \\ 0.7417 & 0.3028 & -0.4621 \end{pmatrix}.$$

In Table 1 we report the obtained results. Using a dwell time $\Delta t = 1/8$ we get an accuracy smaller than 10^{-2}.

TABLE 1. Approximation $\sigma(\mathscr{A})$ of *Example 2*.

Δt	$\sigma_{\Delta t}$	$\gamma_{\Delta t}$	$\alpha^*_{\Delta t}$	s.m.p.
$1/2$	-0.0470	-0.0148	0.0322	$B_1^{27}B_2^{29}$
$1/4$	-0.0470	-0.0243	0.0227	$B_1^{55}B_2^{58}$
$1/8$	-0.0470	-0.0374	0.0096	$B_1^{109}B_2^{117}$

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