# Chapter 3

## Variational Version of Henstock type Integral and Application in Harmonic Analysis

V. Skvortsov *

*Lomonosov Moscow State University, Moscow Center of Fundamental and Applied Mathematics*
*Email : vaskvor2000@yahoo.com*

A variational version of a Henstock type integral with respect to an abstract derivation basis in a topological measure space is defined for the case of Banach space-valued integrands. It is shown that this integral recovers a primitive from its derivative which is defined with respect to the same basis.

As an example of an application of this theory in harmonic analysis, a derivation bases and the respective Henstock type integrals on a zero-dimensional group are considered. It is shown that the variational integral on such a group solves the problem of recovering, by generalized Fourier formulas, the Banach space-valued coefficients of a series with respect to characters of this group.

**Keywords and phrases:** Henstock type integrals, variational Henstock integral, zero-dimensional group.

AMS Subject Classification No(2020): 58A30, 26A39, 46B04.

### 1. Introduction

A Riemann type integral, which solves the problem of recovering a primitive from its derivative and covers the Lebesgue integral, was introduced by Jaroslav Kurzweil in the late 1950s and independently by Ralph Henstock in the early 60s. For a good introduction to the theory of this integral and the history of its creation, including the reason why this integral is usually referred to as Henstock integral, see.[3]

One of the way to generalize the construction of this integral and to

*Author.

apply it in various fields of analysis, is to consider Riemann sums in the definition of the integral with respect to derivation bases more general than a basis constituted by all intervals on the real line. In Subsection 2.1 of this chapter we define an abstract derivation basis $\mathcal{B}$ in a topological measure space, generalizing some partial cases known in the theory of Henstock integral (see,[6] and[13 –16]). We consider a Henstock type integral with respect to this bases, $H_{\mathcal{B}}$-integral, which integrates Banach space-valued functions. In this Banach space-valued case $H_{\mathcal{B}}$-integral being defined with respect to the usual interval basis, is a generalization of Bochner integral.

An important role in this theory is played by a notion of variational equivalence. Using this notion, we can obtain another, so called variational version of Henstock type integral with respect to the basis $\mathcal{B}$. Subsection 2.2 is devoted to this issue. For the full interval basis on the real line in the Banach space-valued case, such an integral was defined in.[12] While those two definitions are equivalent in the real-valued case, variational integral is strictly included into $H_{\mathcal{B}}$-integral for the Banach space-valued functions.

The advantage of the variational version of an integral with respect to any basis is that it reveals a direct connection of the concept of an integral with a derivative with respect to the same basis. In Subsection 2.3 a theorem on recovering a function from its derivative by the variational integral is proved. As for the problem of differentiability almost everywhere of the indefinite integral, in classical real-valued case it is solved in fact for the variational version of the integral. In the Banach space-valued case theorem on differentiability almost everywhere of the indefinite variational integral holds true for a wide class of bases, including a basis considered in Section 3.

As an example of application of this theory in harmonic analysis we consider in Section 3 a derivation bases and the respective integrals defined on a zero-dimensional group. Typical examples of such a group are Cantor Dyadic group and the group of p-adic integers.

We show that a problem of recovering, by generalized Fourier formulas, the coefficients of a series with respect to characters of such a group can be reduced to the one of recovering the primitive from its derivative with respect to a special basis defined on the group, which in turn can be solved by the variational integral with respect to the same basis (see Subsection 3.2).

As we consider Banach space-valued coefficients of a series, we need to mention cases of strong and weak convergence. In the last case a Pettis type variation integral is used to solve the coefficient problem.

## 2. Henstock type integrals with respect to a basis

### 2.1. *Derivational basis and Henstock integral with respect to the basis*

A *derivation basis* (or simply a *basis*) $\mathcal{B}$ in a measure space $(X, \mathcal{M}, \mu)$ is a non-empty family of non-empty subsets $\beta$ of the product space $\mathcal{I} \times X$, where $\mathcal{I}$ is a family of measurable subsets of $X$ of positive measure $\mu$ called *generalized intervals* or *$\mathcal{B}$-intervals* and having property

(a) *For every $\beta_1, \beta_2 \in \mathcal{B}$ there exists $\beta \in \mathcal{B}$ such that $\beta \subset \beta_1 \cap \beta_2$.*

So each basis is a directed set with the order given by "reversed" inclusion. We shall refer to the elements $\beta \in \mathcal{B}$ as *basis sets*. In this paper we shall suppose that all the pairs $(I, x)$ constituting each $\beta$ are such that $x \in I$, although it is not the case in the general theory (see,[126]).

For a set $E \subset X$ and $\beta \in \mathcal{B}$ we write

$$\beta(E) := \{(I, x) \in \beta : I \subset E\} \text{ and } \beta[E] := \{(I, x) \in \beta : x \in E\}.$$

We suppose that the basis $\mathcal{B}$ *ignores no point*, i.e., $\beta[\{x\}] \neq \emptyset$ for any point $x \in X$ and for any $\beta \in \mathcal{B}$. We assume also that the basis $\mathcal{B}$ has a *local character* by which we mean that for any family of basis sets $\{\beta_\tau\}$, $\beta_\tau \in \mathcal{B}$ and for any pairwise disjoint sets $E_\tau$ there exists $\beta \in \mathcal{B}$ such that $\beta[\bigcup_\tau E_\tau] \subset \bigcup_\tau \beta_\tau[E_\tau]$.

Assuming that $X$ is a topological space we shall suppose that $\mathcal{B}$ is a *Vitali basis* by which we mean that for any $x$ and for any neighborhood $U(x)$ of $x$ there exists $\beta_x \in \mathcal{B}$ such that $I \subset U(x)$ for each pair $(I, x) \in \beta_x$.

A *$\beta$-partition* is a finite collection $\pi$ of elements of $\beta$, where the distinct elements $(I', x')$ and $(I'', x'')$ in $\pi$ have $I'$ and $I''$ nonoverlapping, i.e., $\mu(I' \cap I'') = 0$. Let $L \in \mathcal{I}$. If $\pi \subset \beta(L)$ then $\pi$ is called $\beta$-partition *in $L$,* if $\bigcup_{(I,x)\in\pi} I = L$ then $\pi$ is called $\beta$-partition *of* $L$ and is denoted by $\pi(L)$.

We say that a basis $\mathcal{B}$ has the *partitioning property* if the following conditions hold: $(i)$ for each finite collection $I_0, I_1, ..., I_n$ of $\mathcal{B}$-intervals with $I_1, ...I_n \subset I_0$ the difference $I_0 \setminus \bigcup_{i=1}^n I_i$ can be expressed as a finite union of pairwise non-overlapping $\mathcal{B}$-intervals; $(ii)$ for each $\mathcal{B}$-interval $I$ and for any $\beta \in \mathcal{B}$ there exists a $\beta$-partition of $I$.

Let $P_\beta(L)$ denote a set of $\beta$-partitions of a fixed $\mathcal{B}$-interval $L$. Using the partitioning property and property (a) of basis $\mathcal{B}$, it is not difficult to see that *family $\{P_\beta(L)\}_{\beta\in\mathcal{B}}$ is a filter base.* Then for functions defined on $\beta$-partitions $\pi \in P_\beta(L)$ with values in some metric space we can consider *a limit with respect to this filter base*, and denote it $\lim_{\mathcal{B}} F(\pi)$. In this term Henstock integral with respect to basis $\mathcal{B}$ of a function $\Phi : \mathcal{I} \times X \to Y$, where Y is a Banach space, is defined as follows:

**Definition 1.** A function $\Phi : \mathcal{I} \times X \to Y$, is said to be *Kurzweil-Henstock integrable with respect to basis* $\mathcal{B}$ (or $H_{\mathcal{B}}$*-integrable*) on $L \in \mathcal{I}$, with $H_{\mathcal{B}}$*-integral value* $A \in Y$, if there exists limit

$$\lim_{\mathcal{B}} \sum_{(I,x)\in\pi(L)} \Phi(I,x) = A.$$

We denote the integral value $A$ by $(H_{\mathcal{B}}) \int_L \Phi$.

It is easy to check that the set of all $H_{\mathcal{B}}$-tntegrable functions on a fixed $\mathcal{B}$-interval constitutes a linear space.

We note that if $\Phi$ is $H_{\mathcal{B}}$-integrable on $L$ then it is $H_{\mathcal{B}}$-integrable also on any $\mathcal{B}$-interval $I \subset L$. It can be easily proved that the $\mathcal{B}$-interval function $F : I \mapsto (H_{\mathcal{B}}) \int_I \Phi$ is additive on $\mathcal{I}$ and we call it the *indefinite* $H_{\mathcal{B}}$*-integral* of $\Phi$.

In particular case $\Phi(I,x) = f(x)\mu(I)$, where $f : L \to Y$, we obtain $H_{\mathcal{B}}$*-integral of a function* $f$ *on* $L$ *with respect to measure* $\mu$. In this case $H_{\mathcal{B}}$-integral is a generalization of Bochner integral. This can be checked in the same way as it is done in the classical case of the basis constituted by usual intervals on the real line (see[8]).

We consider also a Pettis type version of $H_{\mathcal{B}}$-integral. It is natural to denote it by $HP_{\mathcal{B}}$-integral.

**Definition 2.** A function $f : L \to Y$ is *Henstock-Pettis integrable with respect to basis* $\mathcal{B}$ (or $HP_{\mathcal{B}}$*-integrable*) on $L \in \mathcal{I}$ if $y^*(f)$ is $H_{\mathcal{B}}$-integrable on each $\mathcal{B}$-interval $I \subset L$, for each functional $y^* \in Y^*$, and there exists $A_I \in X$ such that

$$y^*(A_I) = (H_{\mathcal{B}}) \int_I y^*(f)$$

for each $y^*$. $A_I$ is the value of the *indefinite* $HP_{\mathcal{B}}$ *integral* on $I$ and we write

$$A_I = (HP_{\mathcal{B}}) \int_I f.$$

### 2.2. *Variational equivalence and variational Henstock type integral*

In the same terms of the limit with respect to the filter base which was introduced above, we can define a notion of variational equivalence which is in fact an analogue of Kolmogorov notion of differential equivalence (see[4]).

**Definition 3.** (see[6]) Two functions $\Phi_1 : \mathcal{I} \times X \to Y$ and $\Phi_2 : \mathcal{I} \times X \to Y$ are said to be *variationally equivalent on a $\mathcal{B}$-interval* $L$ if

$$\lim_{\mathcal{B}} \sum_{(I,x)\in\pi(L)} \Big\| \Phi_1(I,x) - \Phi_2(I,x) \Big\| = 0$$

or, what is the same,

$$(H_{\mathcal{B}}) \int_L \Big\| \Phi_1(I,x) - \Phi_2(I,x) \Big\| = 0.$$

Variational equivalence can be connected also with another notion, which plays an important role in the Henstock integration theory. Namely, with the notion of variational measure. A standard definition of variational measure with respect to basis $\mathcal{B}$, generated by a function $\Phi : \mathcal{I} \times X \to Y$, on a fixed $\mathcal{B}$-interval $L$ is this. First we define a $\beta$-variation on a set $E \subset L$:

$$Var(E, \Phi, \beta) := \sup_{\pi\subset\beta[E]} \sum ||\Phi(I,x)||.$$

Then variational measure of a set $E \subset L$ is defined by

$$V_\Phi(E) = V(E, \Phi, \mathcal{B}) := \inf_{\beta\in\mathcal{B}} Var(E, \Phi, \beta).$$

Following the proof given in[15] for the interval bases in $\mathbb{R}$ it is possible to show that the extended real-valued set function $V_\Phi(\cdot)$ is an outer measure and a metric outer measure in the case of a metric space $X$ (in the last case the definition of Vitali basis should be used).

Variational measure, generated by a function $\Phi$, of a set $E \subset L$ can be also defined as $H_{\mathcal{B}}$-integral over $L$ of a function defined by

$$\Phi_1(I,x) := \begin{cases} \|\Phi(I,x)\|, & x \in E, \\ 0, & x \in L \setminus E. \end{cases}$$

With this definition we have to admit that $H_{\mathcal{B}}$-integral of nonnegative function can have an infinite value.

Using the notion of variational equivalence, we can obtain another, so called variational version of Henstock type integral with respect to basis $\mathcal{B}$. For the full interval basis on the real line such an integral was defined in.[12]

**Definition 4.** A function $f : L \to Y$, where $L \in \mathcal{I}$ and $Y$ is a Banach space, is said to be *$VH_{\mathcal{B}}$-integrable on* $L$, if there exists an *additive $\mathcal{B}$-interval function* $F : \mathcal{I} \to Y$ such that the function $\Phi_1(I,x) = F(I)$ for all $x \in I$, is variationally equivalent to the function $\Phi_2(I,x) = f(x)\mu(I)$. In this case $F$is *the indefinite $VH_{\mathcal{B}}$-integral* of $f$, in particular $(VH_{\mathcal{B}}) \int_L f = F(L)$.

It is easy to prove that $VH_{\mathcal{B}}$-integrable function on $L$ is also $H_{\mathcal{B}}$-integrable and the integral values coincide. Indeed, let $F$ be the indefinite $VH_{\mathcal{B}}$-integral. Using Definitions 3, 4 and the definition of the limit with respect to basis $\mathcal{B}$ we can state, that for any $\varepsilon > 0$ there exits $\beta \in \mathcal{B}$ such that for any $\beta$-partition $\pi(L)$ we have

$$\sum_{(I,x)\in\pi(L)} \big\| f(x)\mu(I) - F(I) \big\| < \varepsilon.$$

Using additivity of $VH_{\mathcal{B}}$-integral we obtain from the above estimate that for any $\beta$-partition $\pi(L)$ where $\beta \in \mathcal{B}$ is chosen above, we have

$$\Big\| \sum_{(I,x)\in\pi} f(x)\mu(I) - F(L) \Big\| = \Big\| \sum_{(I,x)\in\pi} f(x)\mu(I) - \sum_{(I,x)\in\pi} F(I) \Big\| \leq \sum_{(I,x)\in\pi} \big\| f(x)\mu(I) - F(I) \big\| < \varepsilon.$$

This means that

$$\lim_{\mathcal{B}} \sum_{(I,x)\in\pi(L)} f(x)|I| = F(L),$$

i. e., $f$ is $H_{\mathcal{B}}$-integrable on $L$ with $F(L)$ being its $H_{\mathcal{B}}$-integral.

A result in the opposite direction is known, for a real valued functions and for usual interval basis on an interval of the real line, as Saks-Henstock lemma (see[3]). It can easily be generalized for the basis considered here. Note that the version of this Lemma was used and proved by Kolmogorov a long time ago in his paper.[4] So it would be fair to call this result as Kolmogorov-Henstock lemma.

Hence in the real valued case the $H_{\mathcal{B}}$-integral and the $VH_{\mathcal{B}}$-integral are equivalent. Many properties of the $H_{\mathcal{B}}$-integral are based on this equivalence, i.e., thay are proved in fact for variational version of integral (see[3]).

However, this equivalence of two integrals fails to be true in the Banach valued case. It is proved in[12] (see also[5]) that in the case of basis of usual intervals on the real line the $VH_{\mathcal{B}}$-integral is equivalent to the $H_{\mathcal{B}}$-integral if and only if the range space is of finite dimension. It is likely that this result can be extended to the case of our abstract basis.

If we define a variational version of Henstock-Pettis integral then, due to the above result on the real valued case, it would be equivalent to Henstock-Pettis integral (see Definition 2).

It is easy to check that a function which is equal to zero almost everywhere on $L \in \mathcal{I}$, is $VH_{\mathcal{B}}$-integrable (and also $H_{\mathcal{B}}$-integrable) on $L$ with integral value zero. This implies that $H_{\mathcal{B}}$-integrability of a function and the value of the $H_{\mathcal{B}}$-integral does not depend on values of the function on a set of measure zero. This justifies the following extension of Definitions 1 and 4 to the case of functions defined only almost everywhere on $L$.

**Definition 5.** A function $f$ defined almost everywhere on $L \in \mathcal{I}$ is said to be $H_{\mathcal{B}}$*-integrable* (or $VH_{\mathcal{B}}$*-integrable*) on $L$, with integral value $A$, if the function

$$f_1(g) := \begin{cases} f(g), & \text{where } f \text{ is defined,} \\ 0, & \text{otherwise} \end{cases}$$

is $H_{\mathcal{B}}$-integrable (respectively, $VH_{\mathcal{B}}$-integrable) on $L$ to $A$ in the sense of Definition 1 (or 4).

### 2.3. *Recovering the primitive and problem of differentiation*

The advantage of the variational version of an integral is that it reveals a direct connection of the concept of an integral with a derivative.

$\mathcal{B}$*-derivative* of a Banach valued function $F : \mathcal{I} \to Y$ at a point $x$ is defined as a limit

$$D_{\mathcal{B}}F(x) := \lim_{\mathcal{B}} \frac{F(I)}{\mu(I)},$$

if the limit exists. In other words, $A \in Y$ is a value of $\mathcal{B}$-derivative $D_{\mathcal{B}}F(x)$ if for any $\varepsilon > 0$ there exists $\beta$ such that for all $(I, x) \in \beta[\{x\}]$

$$\left\| \frac{F(I)}{\mu(I)} - A \right\| < \varepsilon.$$

We define also a *weak* $\mathcal{B}$*-derivative* of $F$ at $x$ as an element $wD_{\mathcal{B}}F(x) \in Y$ such that for any $y^* \in Y^*$

$$\lim_{\mathcal{B}} \frac{y^*(F(I)}{\mu(I)} = y^*(wD_{\mathcal{B}}F(x)).$$

In this case we say that $F$ is *weakly* $\mathcal{B}$*-differentiable* at $x$.

The following statement on recovering a function from its $\mathcal{B}$-derivative holds.

**Theorem 1.** *Let an additive function $F : \mathcal{I} \to Y$ be $\mathcal{B}$-differentiable everywhere on $L \in \mathcal{I}$, outside a set $E \subset L$ such that $V_F(E) = 0$. Then the function*

$$f(x) := \begin{cases} D_{\mathcal{B}}F(x), & \text{if it exists,} \\ 0, & \text{if } x \in E, \end{cases}$$

*is $VH_{\mathcal{B}}$-integrable on $L$ and $F$ is its indefinite $VH_{\mathcal{B}}$-integral.*

**Proof.** Fix $\varepsilon > 0$ and according to definition of variational measure find $\beta$ such that for any $\beta[E]$-partition $\pi_1$ we have $\sum_{\pi_1} ||F(I)|| < \frac{\varepsilon}{2}$.

For each point $x$ at which $F$ is $\mathcal{B}$-differentiable find $\beta_x$ such that for $(I, x) \in \beta_x[\{x\}]$

$$\left\|F(I) - f(x)\mu(I)\right\| < \varepsilon \frac{\mu(I)}{2\mu(L)}.$$

In this way according to property (l) we define $\beta$ on $L$. Then for any $\beta$-partition $\pi(L)$ of $L$ we get

$$\sum_{(I,x)\in\pi} \left\|F(I)) - f(x)\mu(I))\right\| \leq \sum_{(I,x)\in\pi,\, x\notin E} ||f(x)\mu(I) - F(I)|| +$$
$$+ \sum_{(I,x)\in\pi_1,} ||f(x)\mu(I) - F(I)|| \leq \frac{\varepsilon}{2\mu(L)} \sum_{(I,x)\in\pi,\, x\notin E} \mu(I) + \frac{\varepsilon}{2} \leq \varepsilon.$$

Thus $F$ is the indefinite $VH_{\mathcal{B}}$-integral of $f$. In particular

$$F(L) = (VH_{\mathcal{B}}) \int_L f.$$

□

Note that the condition $V_F(E) = 0$ put on the exceptional set $E$ is in fact necessary in the case $\mu(E) = 0$. Moreover, the following theorem giving a descriptive characterization of the indefinite $VH_{\mathcal{B}}$-integral holds.

**Theorem 2.** *An additive $\mathcal{B}$-differentiable almost everywhere on $L \in \mathcal{I}$ function $F : \mathcal{I} \to X$ is the indefinite $VH_{\mathcal{B}}$-integral of its derivative if and only if the variational measure, generated by $F$, with respect to basis $\mathcal{B}$, is absolutely continuous with respect to $\mu$.*

**Proof.** The sufficiency follows from Theorem 1. Conversely, let $F$ be the indefinite $VH_{\mathcal{B}}$-integral of its derivative $f$. Take any set $E \subset L$, $\mu(E) = 0$. As $f$ is integrable we can assume that $f(x) = 0$ on $E$. According to Definitions 4 and 3

$$(H_{\mathcal{B}}) \int_L \Big\| f(x)|I| - F(I) \Big\| = 0.$$

Especially, $(H_{\mathcal{B}}) \int_E ||f(x)|I| - F(I)|| = (H_{\mathcal{B}}) \int_E ||F(I)|| = 0$. This means that $V_F(E) = 0$ and we obtain the absolute continuity of $V_F$.

□

We formulate also a weak version of Theorem 1:

**Theorem 3.** *Let an additive function $F : \mathcal{I} \to X$ be $w\mathcal{B}$-differentiable on $L$ outside a set $E$ such that $V_{y^*F}(E) = 0$ for any $y^* \in Y^*$. Then the function*

$$f(x) := \begin{cases} wD_{\mathcal{B}}F(x), & \text{if it exists,} \\ 0, & \text{if } x \in E \end{cases}$$

*is $HP_{\mathcal{B}}$-integrable on $L$ and $F$ is its indefinite $HP_{\mathcal{B}}$-integral.*

The problem of differentiability almost everywhere of the indefinite $H_{\mathcal{B}}$-integral in classical real valued case is also proved in fact for the variational version of integral. In the Banach space-valued case theorem on differentiability almost everywhere of the indefinite $VH_{\mathcal{B}}$-integral holds true for a wide class of bases (see an example in the next section). It is possible that validity of this result in the case of an abstruct basis, we consider here, could depend on some additional requirements for the basis

As for the indefinite $H_{\mathcal{B}}$-integral, we shall see in the next section that, at least in the case of many particular bases, classical interval basis on the real line including, for any Banach space of infinite dimension it is possible to construct an example of an $H_{\mathcal{B}}$-integrable function with the indefinite $H_{\mathcal{B}}$-integral $\mathcal{B}$-differentiable nowhere.

## 3. Variational Integral in Harmonic Analisys

Henstock type integrals with respect to various bases are especially useful in the problem of recovering, by generalized Fourier formulas, vector-valued coefficients of orthogonal series from their sums. A choice of a particular basis depends on the orthogonal system we are dealing with.

In classical harmonic analysis (i.e, on one-dimensional torus) various kind of symmetric bases are useful and an integral with respect to the so-called approximate symmetric basis (see[16]) solves the problem of recovering the coefficients in this case

In the case of Haar and Walsh systems (see,[2 14]), considered on the unterval $[0, 1]$ the Dyadic basis is used in which a family $\mathcal{I}$ of $\mathcal{B}$-intervals is constituted by the family of dyadic intervals

$$\left[\frac{j}{2^n}, \frac{j+1}{2^n}\right], \quad 0 \le j \le 2^n - 1, \quad n = 0, 1, 2, \ldots .$$

### 3.1. *Henstock type integrals on a zero-dimensional group*

As a very important particular example of application of the above theory in harmonic analysis we consider here in more detail a derivation bases and respective derivatives and integrals defined on a zero-dimensional group with second countability axiom. Typical examples of such a group are Cantor Dyadic group and the group of p-adic integers. We consider here the case of abelian group although in the part related to the construction of basis and integral this is not essential. Non-abelian case, the problem of recovering the coefficients including, is considered in.[11]

It is known (see[1]) that with our assumption a topology in such a group can be given by a chain of subgroups

$$G = G_0 \supset G_1 \supset G_2 ... \supset G_n \supset ... \tag{1}$$

with $G = \bigcup_{n=0}^{+\infty} G_n$ and $\{0\} = \bigcap_{n=0}^{+\infty} G_n$. The subgroups $G_n$ are clopen sets with respect to this topology. As $G$ is compact, the factor group $G_0/G_n$ for each $n$ is finite. Let its order be $m_n$. We denote by $K_n$ any coset of the subgroup $G_n$. For any $g \in G$ we denote by $K_n(g)$ the coset of the subgroup $G_n$ which contains the element $g$, i.e.,

$$K_n(g) = g + G_n. \tag{2}$$

For each $g \in G$ the sequence $\{K_n(g)\}$ is decreasing and $\{g\} = \bigcap_n K_n(g)$.

We denote by $\lambda$ the normalized Haar measure on the group $G$. We can make this measure to be complete by including all the subsets of the sets of measure zero into the class of measurable sets.

Since $\lambda(G_0) = 1$ and $\lambda$ is translation invariant then

$$\lambda(G_n) = \lambda(K_n) = \frac{1}{m_n} \tag{3}$$

for all cosets $K_n$, $n \geq 0$.

The family of all $K_n$ for all $n \in \mathbb{N}$ is a semiring of sets which forms the family $\mathcal{I}$ of $\mathcal{B}$-intervals of a basis $\mathcal{B}$ on $G$. For any function $\nu : G \to \mathbb{N}$, we define the basis set

$$\beta_\nu := \{(I, g) : g \in G, I = K_n(g), n \geq \nu(g)\}.$$

Then our derivation basis $\mathcal{B}$ is the family $\{\beta_\nu\}_\nu$ where $\nu$ runs over the set of all natural-valued functions on $G$. This basis has all the properties described in Section 1. But in this case they are not postulated but are easily checked.

If integrals $H_\mathcal{B}$, $VH_\mathcal{B}$ and $HP_\mathcal{B}$ are defined with respect to the basis on the group $G$ described above, we denote them as $H_G$-integral, $VH_G$-integral and $HP_G$-integral, respectively. Definition of the $\mathcal{B}$-derivative is reduced in this basis to the ordinary limit

$$D_G F(x) = \lim_{n\to\infty} \frac{F(K_n(g))}{\lambda(K_n(g))}.$$

A particular case of Theorem 1, in which condition $V_F(E) = 0$ are ensured by requirement that difference ratio is bounded, is formulated in the following way:

**Theorem 4.** *Let an additive function $F : \mathcal{I} \to Y$ be $\mathcal{B}$-differentiable everywhere on $G$ outside of a set $E$ with $\mu(E) = 0$, and*

$$\overline{\lim}_{n\to\infty} \frac{||F(K_n(g))||}{\mu(K_n(g))} < \infty \tag{4}$$

*everywhere on $E$. Then the function*

$$f(x) := \begin{cases} D_{\mathcal{B}}F(x), & \text{if it exists,} \\ 0, & \text{if } x \in E, \end{cases}$$

*is $VH_{\mathcal{B}}$-integrable on $G$ and $F$ is its indefinite $VH_{\mathcal{B}}$-integral.*

In the weak version of Theorem 4 we need not use variational type integral by the reason mentioned in the previous Section.

**Theorem 5.** *Let an additive function $F : \mathcal{I} \to X$ be weakly $\mathcal{B}$-differentiable everywhere on $G$ outside of a set $E$ with $\mu(E) = 0$, and for any $x^* \in X^*$*

$$\overline{\lim}_{n\to\infty} \frac{|x^*F(K_n(g))||}{\mu(K_n(g))} < \infty$$

*everywhere on $E$. Then the function*

$$f(x) := \begin{cases} wD_{\mathcal{B}}F(x), & \text{if it exists,} \\ 0, & \text{if } x \in E, \end{cases}$$

*is $HP_{\mathcal{B}}$-integrable on $G$ and $F$ is its indefinite $HP_{\mathcal{B}}$-integral.*

It was proved in[13] that in the scalar-valued case the indefinite $H_G$-integral of any $H_G$-integrable function is $G$-differentiable everywhere on $G$ and $D_G F(g) = f(g)$ almost everywhere. In a similar way this property can be proved in a case when a range of a function is of finite dimension, and for $VH_G$-integral it is true in the case of any Banach space:

**Theorem 6.** *If a function $f : G \to X$ is $VH_G$-integrable on $G$ then the indefinite $VH_G$-integral $F(K) = (VH)_G \int_K f$ as an additive function on the set of all $\mathcal{B}$-intervals is $G$-differentiable almost everywhere on $G$ and*

$$D_G F(g) = f(g) \quad \text{a.e. on} \quad L. \tag{5}$$

The proof follows the line of the argument in [13, Theorem 3.1] for the scalar case provided one replaces a reference to Kolmogorov-Henstock lemma by a reference to variational equivalence of functions $F(I)$ and $f(g)\lambda(I)$ (see Definitions 4 and 3).

The next theorem, proved in,[10] shows that this result can not be extended to the case of $H_{\mathcal{B}}$-integral.

**Theorem 7.** *For any infinite-dimensional Banach space $Y$, there exists a $H_{\mathcal{B}}$-integrable on $G$ function $f : G \to Y$ with the indefinite $H_G$-integral which is $G$-differentiable nowhere on $G$.*

### 3.2. *Application to harmonic analysis on the group G*

Let $\Gamma$ denotes the dual group of $G$, i.e., the group of characters of the group $G$. It is known (see[1]) that under assumption imposed on $G$ the group $\Gamma$ is a discrete abelian group (with respect to the point-wise multiplication of characters) and it can be represented as a sum of increasing chain of finite subgroups

$$\Gamma_0 \subset \Gamma_1 \subset \Gamma_2 \subset ... \subset \Gamma_n \subset ... \tag{6}$$

where $\Gamma_0 = \{\gamma_0\}$ with $\gamma_0(g) = 1$ for all $g \in G$. For each $n \in \mathbb{N}$ the group $\Gamma_n$ is the annulator of $G_n$, i.e.,

$$\Gamma_n := \{\gamma \in \Gamma : \gamma(g) = 1 \text{ for all } g \in G_n\}.$$

The factor groups $\Gamma_{n+1}/\Gamma_n$ and $G_n/G_{n+1}$ are isomorphic (see[1]) and so they are of the same finite order for each $n \in \mathbb{N}$.

It is easy to check that if $\gamma \in \Gamma_n$ then $\gamma$ is constant on each coset $K_n$ of $G_n$, and if $\gamma \in \Gamma \setminus \Gamma_n$ then $\int_{K_n} \gamma d\mu = 0$ for each coset $K_n$.

This implies that the characters $\gamma$ constitute a countable orthonormal system on $G$ with respect to normalized measure $\lambda$, and we can consider a series

$$\sum_{\gamma \in \Gamma} a_\gamma \gamma \tag{7}$$

with respect to this system. We define the convergence of this series at a point $g$ as the convergence of its partial sums

$$S_n(g) := \sum_{\gamma \in \Gamma_n} a_\gamma \gamma(g) \tag{8}$$

when $n$ tends to infinity. If coefficients $a_\gamma$ are Banach-valued we can consider strong and weak convergence of this series.

We associate with the series (7) a function $F$ defined on each coset $K_n$ by

$$F(K_n) := \int_{K_n} S_n(g) d\mu \tag{9}$$

where $S_n$ are partial sums given by (8). Similar to the scalar case (see[13]) it is easy to check that $F$ is an additive function on the family $\mathcal{I}$ of all $\mathcal{B}$-intervals. As it was in the case of Haar and Walsh series (see[9] and[7]) we call this function a *quasi-measure* associated with the series (7).

The properties of the characters, described above, imply that the sum $S_n$, defined by (8), is constant on each $K_n$. Then by (4) we have

$$S_n(g) = \frac{F(K_n(g))}{\mu(K_n(g))}. \tag{10}$$

It follows directly from this equality that if the series (7) converges at some point $g \in G$ to a value $f(g)$ then the associated quasi-measure $F$ is $\mathcal{B}$-differentiable at $g$ and $D_{\mathcal{B}}F(g) = f(g)$. The same is true for the weak convergence.

The following statement is essential for establishing that a given series with respect to characters is the Fourier series in the sense of some general integral.

**Theorem 8.** *Let some integration process $\mathcal{A}$ be given which produces an integral additive on $\mathcal{I}$. Let a function $F$ defined on $\mathcal{I}$ be the quasi-measure associated with the series (7). Then this series is the Fourier series of an $\mathcal{A}$-integrable function $f$ if and only if $F(K) = (\mathcal{A}) \int_K f$ for any $K \in \mathcal{I}$.*

In view of (10) and Theorem 8, in order to solve the coefficient problem, it is enough to show that the quasi-measure associated with the series (7) is the indefinite integral of its derivative (strong or weak, respectively).

By this we reduce the problem of recovering the coefficients to the one of recovering the primitive and we can use a corresponding theorem on primitives in Subsection 2.1.

**Theorem 9.** *Suppose that the partial sums* (8) *of the series* (7) *converge to a function $f$ everywhere on $G$ outside of a set $E$ with $\mu(E) = 0$, and*

$$\overline{\lim}_{n\to\infty} ||S_n(g)|| < \infty$$

*everywhere on $E$. Then $f$ is $VH_G$-integrable and* (7) *is the $VH_G$-Fourier series of $f$.*

In the same way, using Theorem 8 for the case of $HP_{\mathcal{B}}$-integral we get

**Theorem 10.** *Suppose that the partial sums* (8) *of the series* (7) *converge weakly to a function $f$ everywhere on $G$ outside of a set $E$ with $\mu(E) = 0$, and for any $y^* \in Y^*$*

$$\overline{\lim}_{n\to\infty}|y^* S_n(g)|| < \infty$$

*everywhere on $E$. Then $f$ is $HP_G$-integrable and* (7) *is the $HP_G$-Fourier series of $f$.*

Note that the above theorem cover in particular the case of the convergence of the series (7) everywhere on $G$.

It is remarkable that in Theorems 9 and 10 there is no need to suppose that the sum of the series is integrable in the respective sense. The integrability is an implication of the convergence. However if we assume appriori that the sum is integrable in the sense of Lebesgue (or Bochner in the Banach space-valued case) then we obtain an analogue of the classical du Bois Reymond – Vallee Poissin theorem.

**Theorem 11.** *Suppose that the partial sums* (8) *of the series* (7) *converge (converge weakly) to a Lebesgue (Bochner) integrable function $f$ everywhere on $G$ except a countable set. Then the series* (7) *is the Lebesgue–Fourier series (resp. Bochner–Fourier series) of $f$.*

A proof is reduced to checking that any Lebesgue (Bochner) integrable function is also $VH_G$-integrable and the integrals coincide what can be done following the line of the proof in the case of the usual interval basis on the real line.

Now we consider the problem of convergence of Fourier series in the sense of $VH_G$-integral and $H_{\mathcal{B}}$-integral. The partial sums $S_n(f,g)$ of Fourier series, with respect to the system $\Gamma$, of a function $f : G \to Y$ integrable in the sense of these integrals can be represented, according to Theorem 8 and formula (10), as

$$S_n(f,g) = \frac{1}{\mu(K_n(g))}\int_{K_n(g)} f. \tag{11}$$

From this equality together with differentiability property of the indefinite $VH_{\mathcal{B}}$-integral (see Theorem 6) follows

**Theorem 12.** *The partial sums $S_n(f,g)$ of the $VH_{\mathcal{B}}$-Fourier series of a $VH_{\mathcal{B}}$-integrable on $G$ function $f$ are convergent to $f$ almost everywhere on $G$.*

At the same time such a theorem fails to be true for $H_{\mathcal{B}}$-Fourier series. Indeed, we get from Theorem 7:

**Theorem 13.** *For any infinite-dimensional Banach space $Y$ there exists $H_G$-integrable function $f : G \to Y$ such that partial sums of its $H_G$-Fourier series with respect to the system $\Gamma$ diverge everywhere.*

It is interesting to note that a rate of growth of these partial sums can not be made arbitrary large for the whole class of infinite-dimensional Banach spaces. For some particular spaces such a rate have some restriction. For example it can be deduced from[9] that for some class of infinite-dimensional Banach spaces for a Pettis-integrable function $f$ taking values in such a space, sums of its Fourier series with respect to the Walsh system, which is the systems of characters of a particular case of a zero-dimensional group, satisfy the relation $||S_n(f, g)|| = o(2^{\frac{1}{2}n})$.

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